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# On the Controllability of impulsive neutral functional evolution integro-differential systems with an infinite delay

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**Abstract**

In this manuscript, we consider the controllability of a certain class of impulsive neutral evolution integrodifferential equations in Banach spaces. Sufficient conditions for controllability results are obtained by using the Hausdorff measure of non compactness and Mönch fixed point theorem under the assumption of noncompactness of the evolution system.

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## 1. Introduction

In the past ten years, the mathematical descriptions of many hybrid dynamical systems have an impulsive behavior due to abrupt changes at certain instants during the evolution process. Recent development in the theory of impulsive differential equations and inclusions has been object interest because of its wide applications in medical domains, industry, information science, system and control, communication security and space techniques see for instance ([17, 39]). These processes tend to be more suitably modeled by impulsive systems which allow for discontinuities in the evolution of the state. For more details about this theory and its applications, we refer to the monographs of Bainov and Simeonov [1], Lakshmikantham et al. [23] and Samoilenko and Perestyuk [32].

On the other hand, controllability is one of the fundamental and important concepts in mathematical control theory. This is the qualitative and quantitative property of dynamical control systems and is of particular importance in control theory. Controllability for differential systems in Banach spaces under the assumption of compactness and noncompactness of the operator semigroups has been studied by many authors [2, 3, 5, 7, 10, 11, 12, 9, 13, 18, 19, 24, 26, 27, 28, 30, 31, 34, 35] by using various fixed point theorems. Recently, Guo et al. [13] derived the sufficient conditions for the controllability of the following class of impulsive evolution inclusions with nonlocal conditions by using Mönch fixed point theorem, under the assumption of noncompactness of the semigroup generated by the evolution system.

Very recently, by using the same fixed point theorem, Ji et al. [18] extended the controllability results of Guo et al. [13] into the impulsive evolution differential systems and the evolution system generated by $A(t)$ equicontinuous.

To the best of our knowledge, till now no work reported on Controllability of impulsive neutral functional evolution integrodifferential systems with an infinite delay has been an untreated topic in the literature, and this fact is the main aim of the present work. Motivated by the above mentioned works [13, 18, 40], in this paper, we establish the sufficient conditions for controllability of the impulsive neutral evolution integrodifferential equations with infinite delay of the form:

$$\frac{d}{dt}\Big[x(t) - g\Big(t, x_t, \int_0^t a(t,s,x_s)ds\Big)\Big]$$

$$= A(t)x(t) + f\Big(t, x_t, \int_0^t e(t,s,x_s)ds\Big) + Bu(t),$$

$$t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\dots,m, \tag{1.1}$$

$$\Delta x|_{t=t_k} = I_k(x_{t_k}), k = 1,2,\dots,m, \tag{1.2}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.3}$$

where $\{A(t)\}_{t\in J}$ is a family of linear operators in a Banach space $X$ generating an evolution operator $U : \Delta = \{(t,s) \in [0,b] \times [0,b] : 0 \leq s \leq t \leq b\} \to \mathcal{L}(X)$, here $\mathcal{L}(X)$ is Banach space of all bounded linear operators in $X$; the history $x_t : (-\infty, 0] \to$

$X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $g, f : J \times \mathcal{B} \times X \to X$ and $a, e : J \times J \times \mathcal{B} \to X$ are appropriate functions; the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are given and $I_k : \mathcal{B} \to X, k = 1, 2, \ldots, m$, are given impulsive functions; the control function $u(\cdot)$ is considered in the space $L^2(J, V)$, where $V$ is a Banach space of control and $B : V \to X$ is a bounded linear operator.

## 2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. We denote $C([0, b], X)$ the space of all $X$-valued functions on $[0, b]$ with norm $\|x\| = \sup\{\|x(t)\| : t \in [0, b]\}$ and by $L^1([0, b], X)$ the space of $X$-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\|dt$.

To describe appropriately our problems, we say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if $u$ is piecewise continuous and left continuous on $(\sigma, \tau]$. By the symbol $\mathcal{PC}([\sigma, \tau]; X)$, we denote the space of normalized piecewise continuous function from $[\sigma, \tau]$ into $X$. In particular, we denote the space $\mathcal{PC}$ formed by all function $u : [0, b] \to X$ such that $u$ is continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exists, for all $k = 1, 2, \ldots, m$. It is easy to see that $\mathcal{PC}$ is a Banach space with the norm $\|x\|_{\mathcal{PC}} = \sup_{s \in [0,b]} \|x(s)\|$.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [14] and it is appropriate to treat retarded impulsive differential equations.

**Definition 2.1. [14]** Let $\mathcal{B}$ be a linear space of functions mapping from $(-\infty, 0]$ into $X$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

$(A)$ If $x : (-\infty, \sigma + b] \to X, b > 0$, such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma, \sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b]; X)$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

- $(i)$ $x_t \in \mathcal{B}$,
- $(ii)$ $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
- $(iii)$ $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K$ is continuous, $M$ is locally bounded and $H, K, M$ are independent of $x(\cdot)$.

$(B)$ The space $\mathcal{B}$ is complete.

For the family of linear operators $\{A(t) : t \in J\}$, we assume the following hypotheses.

$(A1)$ The domain $D(A(t))$ of $A(t)$ is dense in $X$ and independent of $t$

$(A2)$ For each $t \in J$, resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all $\lambda$ with Re $\lambda \leq 0$ and there exists a constant $M > 0$ such that $\|R(\lambda : A(t))\| \leq M(|\lambda| + 1)^{-1}$.

$(A3)$ There exists constants $L > 0$ and $0 < \mu \leq 1$ such that $\|A(t) - A(s)A^{-1}(\tau)\| \leq L|t - s|^{\mu}$ for $t, s, \tau \in J$.

Under the assumptions (A1)-(A3), the family $\{A(t) : t \in J\}$ generates an unique evolution system $\{U(t, s) : 0 \leq s \leq t \leq b$ satisfying:

$(a)$ There exists a positive constant $M_0$ such that $\|U(t, s)\| \leq M_0$ for $0 \leq s \leq t \leq b$.

$(b)$ For every $v \in D(A(t))$ and $t \in J$, $U(t, s)v$ is differential with respect to $s$ on $0 \leq s \leq t \leq b$ and $\frac{\partial}{\partial s}U(t, s) = -U(t, s)A(s)v$.

**Definition 2.2.** A two parameter family of bounded linear operators $U(t, s), 0 \leq s \leq t \leq b$ on $X$ is called an evolution system if the following two conditions are satisfied:

(i) $U(s, s) = I, U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq b$;

(ii) $(t, s) \to U(t, s)$ is strongly continuous on $\Delta$,i.e. for each $x \in X$, the function $(t, s) \in \Delta \to U(t, s)x$ is continuous.

More details about evolution system can be found in Pazy [29].

**Lemma 2.3. [4]** Let $E^+$ be the positive cone of an order Banach space $(E, \leq)$. A function $\Phi$ defined on the set of all bounded subsets of the Banach space $X$ with values in $E^+$ is called a measure of noncompactness(MNC) on $X$ if $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subseteq X$, where $\overline{co}\Omega$ stands for the closed convex hull of $\Omega$.

The MNC $\Phi$ is said:

(1) Monotone if for all bounded subsets $\Omega_1, \Omega_2$ of $X$ we have :$(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;

(2) Nonsingular if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X, \Omega \subset X$;

(3) Regular if $\Phi(\Omega) = 0$ if and only if $\Omega$ is relatively compact in $X$.

One of the most examples of MNC is the noncompactness measure of Hausdorff $\beta$ defined on each bounded subset $\Omega$ of $X$ by

$$\beta(\Omega) = \inf\{\epsilon > 0; \Omega \text{ can be covered by a finite number of balls of radii smaller than } \epsilon\}$$

It is well known that MNC $\beta$ enjoys the above properties and other properties see [4, 20]: For all bounded subsets $\Omega, \Omega_1, \Omega_2$ of $X$.

(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$,where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;

(5) $\beta(\Omega_1 \cup \Omega_2) \le \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;

(6) $\beta(\lambda\Omega) \le |\lambda|\beta(\Omega)$ for any $\lambda \in \mathbb{R}$;

(7) If the map $Q : D(Q) \subseteq X \to Z$ is Lipschitz continuous with constant $k$, then $\beta_Z(Q\Omega) \le k\beta(\Omega)$ for any bounded subset $\Omega \subseteq D(Q)$, where $Z$ is a Banach space.

**Lemma 2.4. [4]** If $W \subset C([a,b], X)$ is bounded equicontinuous, then $\beta(W(t))$ is continuous for $t \in [a,b]$ and

$$\beta(W) = \sup\{\beta(W(t)), t \in [a,b]\} \; where \; W(t) = \{x(t) : x \in W\} \subseteq X.$$

**Lemma 2.5. [20]** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,b], \mathbb{R}^+)$. Assume that there exist $\mu, \eta \in L^1([0,b], \mathbb{R}^+)$ satisfying $\sup_{n\ge 1} \|f_n(t)\| \le \mu(t)$ and $\beta(\{f_n(t)\}_{n=1}^{\infty}) \le \eta(t)$ a.e.$t \in [0,b]$, then for all $t \in [0,b]$, we have

$$\beta\left(\left\{\int_0^t U(t,s) f_n(s) ds : n \ge 1\right\}\right) \le 2M_0 \int_0^t \eta(s) ds.$$

The following fixed-point theorem, a nonlinear alternative of Mönch type, plays a key role in our proof of controllability of the system (1.1)-(1.3).

**Lemma 2.6. ([25],Theorem 2.2)** Let $D$ be a closed convex subset of a Banach space $X$ and $0 \in D$.Assume that $F : D \to X$ is a continuous map which satisfies Mönch's condition, that is ($M \subseteq D$ *is countable*, $M \subseteq \overline{co}(\{0\} \cup F(M)) \Rightarrow \overline{M}$ *is compact*. Then $F$ has a fixed point in $D$.

## 3. Controllability results

In this section, we present and prove the controllability results for the system (1.1)-(1.3). First, we give the mild solution of the problem (1.1)-(1.3).

**Definition 3.1.** A function $x : (-\infty, b] \to X$ is a mild solution of the initial value problem (1.1)-(1.3) if $x_0 = \varphi \in \mathcal{B}$, $x(\cdot)|_J \in \mathcal{PC}$ and

$$\begin{aligned} x(t) &= U(t,0)[\varphi(0) - g(0,\varphi,0)] + g\Big(t, x_t, \int_0^t a(t,s,x_s)ds\Big) \\ &+ \int_0^t U(t,s)A(s)g\Big(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau\Big)ds \\ &+ \int_0^t U(t,s)\Big[f\Big(s, x_s, \int_0^s e(s,\tau,x_\tau)d\tau\Big) + Bu(s)\Big]ds \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(x_{t_k}), \quad t \in J. \end{aligned}$$

**Definition 3.2.** The system $(1.1)-(1.3)$ is said to be controllable on the interval $J$ if for every initial function $\varphi \in \mathcal{B}$ and $x_1 \in X$, there exists a control $u \in L^2(J, V)$ such that the mild solution $x(\cdot)$ of $(1.1)-(1.3)$ satisfies $x(b) = x_1$.

We assume the following hypotheses:

(H1) The evolution system $\{U(t,s)_{(t,s)\in\Delta}\}$ generated by the family of linear operators $\{A(t)\}_{t\in J}$ is equicontinuous. i.e. $(t,s) \to \{U(t,s)x : x \in E\}$ is equicontinuous for $t > 0$ and for all bounded subsets $E$.

(H2) The function $f : J \times \mathcal{B} \times X \to X$ satisfies the following conditions:

(i) For each $t \in J$, the function $f(t,\cdot,\cdot) : \mathcal{B} \times X \to X$ is continuous and for each $(\varphi, x) \in \mathcal{B} \times X$, the function $f(\cdot, \varphi, x) : J \to X$ is strongly measurable.

(ii) For each positive number $r$, there exists a integrable function $\alpha_r \in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathcal{B}}\le r} \|f(t,\varphi,x)\| \le \alpha_r(t), \quad \text{for a.e. } t \in J,$$

$$\text{and} \lim_{r\to\infty} \inf \int_0^b \frac{\alpha_r(t)}{r} dt = \delta < +\infty.$$

(iii) There exists an integrable function $\eta : J \to [0, +\infty)$ such that

$$\beta(f(t, D_1, D_2)) \le \eta(t)\Big[\sup_{-\infty<\theta\le 0} \beta(D_1(\theta)) + \beta(D_2)\Big]$$

for a.e. $s, t \in J$, and any bounded subset $D_1 \subset \mathcal{PC}([-\infty, 0]; X)$ and $D_2 \subset X$,

$$\beta\left(\int_0^t e(t,s,D_1)ds\right) \le L_2 \sup_{-\infty<\theta\le 0} \beta(D_1(\theta)),\ t \in J,\ L_2 = \sup_{s\in J}\int_0^b e(s)ds,$$

where $D_1(\theta) = \{v(\theta) : v \in D_1\}$ and $\beta$ is the Hausdorff MNC.

(H3) The linear operator $W : L^2(J, V) \to X$ is defined by

$$Wu = \int_0^b U(b,s)Bu(s)ds,$$

such that

(i) $W$ has an invertible operator $W^{-1}$ which take the values in $L^2(J, V)/KerW$, and there exists positive constants $M_1, M_2$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

(ii) There is $K_W \in L^1(J, R^+)$ such that, for every bounded set $Q \subset X$,

$$\beta(W^{-1}Q)(t) \leq K_W(t)\beta(Q).$$

(H4) There exist a positive constant $M_3 > 0$ such that $\|A(t)A^{-1}(0)\| \leq M_3$ for $t \in J$.

(H5) The function $g : J \times \mathcal{B} \times X \to X$ is continuous and there exist positive constants $L_0, L_1, C_1, C_2, C_3, C_4$ such that

(i) $\|A(0)g(t, \varphi_1, x_1) - A(0)g(t, \varphi_2, x_2)\| \leq L_0[\|\varphi_1 - \varphi_2\|_{\mathcal{B}} + \|x_1 - x_2\|], \forall\, t \in J$,
$\varphi_1, \varphi_2 \in \mathcal{B}, \quad x_1, x_2 \in X$;

(ii) $\|A(0)g(t, \varphi, x)\| \leq L_0\|\varphi\|_{\mathcal{B}} + C_1\|x\|_{\mathcal{B}} + C_2$, where $C_1 = L_0L_1,\ C_2 = L_0C_4 + C_3$,
$C_1' = L_0 + C_1,\ C_5 = L_0(1 + L_1), \quad C_3 = \|A(0)g(0, 0, 0)\|, \quad t \in J, \quad \varphi \in \mathcal{B}, \quad x \in X$;

(iii) $\left\| \int_0^t [a(t, s, x) - a(t, s, y)]ds \right\| \leq L_1 \|x - y\|_{\mathcal{B}}$, for $t, s \in J, \quad x, y \in \mathcal{B}$ and $\left\| \int_0^t a(t, s, x)ds \right\| \leq L_1 \|x\|_{\mathcal{B}} + C_4$, where $C_4 = \left\| \int_0^t a(0, 0, 0)ds \right\|$.

(H6) (i) There exist positive constants $\gamma_k$ such that
$\|I_k(\varphi_1) - I_k(\varphi_2)\| \leq \gamma_k \|\varphi_1 - \varphi_2\|_{\mathcal{B}}, \qquad \forall\, \varphi_1, \varphi_2 \in \mathcal{B}$.

(ii) There exist a continuous nondecreasing functions $L_k : [0, +\infty) \to (0, +\infty)$ such that
$\|I_k(\varphi)\| \leq L_k(\|\varphi\|_{\mathcal{B}}), \varphi \in \mathcal{B}$ and $\lim_{\rho\to\infty} \inf \frac{L_k(\rho)}{\rho} = \lambda_k < +\infty$, where $\sum_{k=1}^{m} \lambda_k = \lambda$.

(H7) The following estimation holds true: $N + \overline{N} < 1$, where

$$N = K_b(1 + M_0M_1M_2b)\left( \|A^{-1}(0)\|C_1 + M_0M_3bC_1 + M_0 \sum_{k=1}^{m} \gamma_k \right)$$

and

$$\overline{N} = (2M_0 + 4M_0^2M_1\|K_W\|_{L^1})(1 + L_1)\|\eta\|_{L^1}.$$

**Remark 3.3.** From (A3), we obtain $\|A(t)A^{-1}(0)\| \leq L|b|^{\mu} + 1$. Thus we can choose a positive constant $M_3 = L|b|^{\mu} + 1$ satisfying (H4).

**Theorem 3.4.** Assume that the hypotheses (H1)-(H7) are satisfied. Then the system (1.1)-(1.3) is controllable on $J$ provided that

$$K_b(1+M_0M_1M_2b)\Big[C_1(\|A^{-1}(0)\|+M_0M_3b)+M_0(\delta+\lambda)\Big]<1. \tag{3.1}$$

*Proof.* Using the hypothesis (H3) for an arbitrary function $x:(-\infty,b]\to X$, define the control

$$\begin{aligned} u_x(t)=W^{-1}\Bigg[& x_1-U(b,0)[\varphi(0)-g(0,\varphi,0)]-g\left(b,x_b,\int_0^b a(b,s,x_s)ds\right)\\ &-\int_0^b U(b,s)A(s)g\left(s,x_s,\int_0^s a(s,\tau,x_\tau)d\tau\right)ds\\ &-\int_0^t U(b,s)f\left(s,x_s,\int_0^s e(s,\tau,x_\tau)d\tau\right)ds-\sum_{k=1}^m U(b,t_k)I_k(x_{t_k})\Bigg](t). \end{aligned}$$

We shall now show that using this control $u_x(\cdot)$ function, the operator defined by

$$\Phi x(t)=\begin{cases}\varphi(t), & t\in[-\infty,0],\\ U(t,0)[\varphi(0)-g(0,\varphi,0)]+g\left(t,x_t,\int_0^t a(t,s,x_s)ds\right)\\ +\int_0^t U(t,s)A(s)g\left(s,x_s,\int_0^s a(s,\tau,x_\tau)d\tau\right)ds\\ +\int_0^t U(t,s)\left[f\left(s,x_s,\int_0^s e(s,\tau,x_\tau)d\tau\right)+Bu_x(s)\right]ds\\ +\sum_{0<t_k<t} U(t,t_k)I_k(x_{t_k}), \quad t\in J\end{cases}$$

has a fixed point. This fixed point is then a solution of (1.1)-(1.3). Clearly, $\Phi x(b)=x_1$, which implies that the system (1.1)-(1.3) is controllable.

Suppose that $x(t)=y(t)+z(t), t\in(-\infty,b]$, where $y:(-\infty,0]\to X$ be a function defined by $y_0=\varphi$ and $y(t)=U(t,0)\varphi(0)$ on $J$. Then by the axioms of phase space, it is easy to see that $\|z_t+y_t\|_{\mathcal{B}}\le(K_bM_bH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|z\|_t$, where $\|z\|_t=\sup_{0\le s\le t}=\|z(s)\|$, $K_b=\sup_{0\le t\le b}K(t)$, $M_b=\sup_{0\le t\le b}M(t)$.

Define $S(b)=\{z:(-\infty,b]\to X \text{such that } z_0,\ z|_J\in\mathcal{PC}\}$ be a space endowed with the supremum norm $\|.\|_b$. Then $(S(b),\|.\|_b)$ is a Banach space. Let $\Gamma:S(b)\to S(b)$

be the operator defined by

$$(\Gamma z)(t) = \begin{cases} 0, & t \in [-\infty, 0], \\ -U(t,0)g(0,\varphi,0) + g\left(t, z_t + y_t, \int_0^t a(t,s,z_s + y_s)ds\right) \\ + \int_0^t U(t,s)A(s)g\left(s, z_s + y_s, \int_0^s a(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\ + \int_0^t U(t,s)\left[f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right) + Bu_z(s)\right]ds \\ + \sum_{0<t_k<t} U(t,t_k)I_k(z_{t_k} + y_{t_k}), & t \in J. \end{cases}$$

where $u_z(.) \in L^2(J, V)$,

$$\begin{aligned} u_z(t) = W^{-1}\Bigg[ & x_1 - U(b,0)[\varphi(0) - g(0,\varphi,0)] - g\left(b, z_b + y_b, \int_0^b a(b,s,z_s + y_s)ds\right) \\ & - \int_0^b U(b,s)A(s)g\left(s, z_s + y_s, \int_0^s a(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\ & - \int_0^b U(b,s)f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\ & - \sum_{0<t_k<t} U(t,t_k)I_k(z_{t_k} + y_{t_k})\Bigg](t). \end{aligned}$$

Clearly, $\Gamma$ is well defined and with values in $S(b)$. It is easy to see that if $z$ is a fixed point of $\Gamma$, then $y + z$ is a fixed point of $\Phi$. So our aim is to find a fixed point of $\Gamma$. Set $B_q = \{z \in S(b) : \|z\|_b \le q\}$ for some $q > 0$. Clearly, $B_q$ is a nonempty, closed, convex and bounded set in $S(b)$. Then for any $z \in B_q$,

$$\|z_t + y_t\|_{\mathcal{B}} \le (K_b M_b H + M_b)\|\varphi\|_{\mathcal{B}} + K_b\, q = q'. \tag{3.2}$$

For better readability, we break the proof into sequence of steps.

**Step 1.** There exists $q \ge 1$ such that $\Gamma(B_q) \subseteq B_q$. Suppose the contrary. Then for each positive integer $q$, there exists $z \in B_q$ such that $\|(\Gamma z)(t)\| > q$ for some $t \in J$. It follows from the hypotheses (H1)-(H6) and (3.2) that

$$\begin{aligned} q &< \|(\Gamma z)(t)\| \\ &\le M_0\|A^{-1}(0)\|(L_0\|\varphi\|_{\mathcal{B}} + C_2) + \|A^{-1}(0)\|(C_1'\, q' + C_2) + M_0 M_3 b(C_1'\, q' + C_2) \\ &+ M_0\int_0^t \alpha_{q'}(s)ds + M_0\sum_{k=1}^m L_k(q') + M_0 M_1 b\|u_z\|_{L^2}, \end{aligned} \tag{3.3}$$

where

$$\|u_z\|_{L^2} \leq M_2\Big[\|x_1\| + M_0[\|\varphi(0)\| + \|A^{-1}(0)\|(L_0\|\varphi\|_{\mathcal{B}} + C_2)] + \|A^{-1}(0)\|(C_1'\, q' + C_2) + M_0M_3b(C_1'\, q' + C_2) + M_0\int_0^t \alpha_{q'}(s)ds + M_0\sum_{k=1}^{m} L_k(q')\Big]. \tag{3.4}$$

Hence by using (3.4) in (3.3), we have

$$q \leq \overline{L} + (1 + M_0M_1M_2b)\Big[C_1\, q'(\|A^{-1}(0)\| + M_0M_3b) + M_0\int_0^t \alpha_{q'}(s)ds + M_0\sum_{k=1}^{m} L_k(q')\Big], \tag{3.5}$$

where $\overline{L}$ is independent of $q$.

Noting that $q' = (K_bM_bH + M_b)\|\varphi\|_{\mathcal{B}} + K_bq \to +\infty$ as $q \to +\infty$, we obtain by hypotheses (H2) and (H6),

$$\lim_{q\to+\infty} \inf\left(\frac{\int_0^b \alpha_{q'}(s)ds}{q}\right) = \lim_{q\to+\infty} \inf\left(\frac{\int_0^b \alpha_{q'}(s)ds}{q'}.\frac{q'}{q}\right) = \delta K_b,$$
$$\lim_{q\to+\infty} \inf\left(\frac{\sum_{k=1}^{m} L_k(q')}{q}\right) = \lim_{q\to+\infty} \inf\left(\frac{\sum_{k=1}^{m} L_k(q')}{q'}.\frac{q'}{q}\right) = \lambda K_b.$$

Dividing both sides of (3.5) by $q$ and employing the above two equalities, we have that

$$1 \leq K_b(1 + M_0M_1M_2b)[C_1(\|A^{-1}(0)\| + M_0M_3b) + M_0(\delta + \lambda)].$$

This contradicts (3.1). Thus, there exists $q \geq 1$ such that $\Gamma(B_q) \subseteq B_q$.

**Step 2.** $\Gamma : S(b) \to S(b)$ is continuous.

Let $(z^n)_{n\in N}$ be a sequence in $S(b)$ such that $z^n \to z$ in $S(b)$. Then by hypotheses (H2),(H5)-(H6), we can prove that

$$f\left(s, z_s^n + y_s, \int_0^s e(s,\tau,z_\tau^n + y_\tau)d\tau\right) \to f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right),$$

$$g\left(s, z_s^n + y_s, \int_0^s a(s,\tau,z_\tau^n + y_\tau)d\tau\right) \to g\left(s, z_s + y_s, \int_0^s a(s,\tau,z_\tau + y_\tau)d\tau\right)$$

and $I_k(z_{t_k}^n + y_{t_k}) \to I_k(z_{t_k} + y_{t_k})$ uniformly on $J$.

Then by hypotheses (H2) and (H5) with dominated convergence theorem, we conclude that

$$\int_0^t U(t,s)f\left(s, z_s^n + y_s, \int_0^s e(s,\tau,z_\tau^n + y_\tau)d\tau\right)ds$$
$$\to \int_0^t U(t,s)f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right)ds,$$

and

$$\int_0^t U(t,s)A(s)g\left(s, z_s^n + y_s, \int_0^s e(s,\tau,z_\tau^n + y_\tau)d\tau\right)ds$$
$$\to \int_0^t U(t,s)A(s)g\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right)ds, \quad as \ \ n \to \infty.$$

which implies together with the continuity of the operators $B$, $W^{-1}$ that, we have $\|\Gamma z^n - \Gamma z\| \to 0$, as $n \to \infty$. Hence $\Gamma$ is continuous on $S(b)$.

**Step 3.** The Mönch condition holds:

To prove this, we decompose $\Gamma$ in the form $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\begin{aligned}
&(\Gamma_1 z)(t) \\
&= -U(t,0)g(0,\varphi,0) + g\left(t, z_t + y_t, \int_0^t a(t,s,z_s + y_s)ds\right) \\
&\quad + \int_0^t U(t,s)A(s)g\left(s, z_s + y_s, \int_0^s a(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\
&\quad + \sum_{0<t_k<t} U(t,t_k)I_k(z_{t_k} + y_{t_k}) \\
&\quad + \int_0^t U(t,\zeta)BW^{-1}\Big[x_1 - U(b,0)[\varphi(0) - g(0,\varphi,0)] \\
&\quad - g\left(s, z_s + y_s, \int_0^b a(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\
&\quad - \int_0^b U(b,s)A(s)g\left(s, z_s + y_s, \int_0^s a(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\
&\quad - \sum_{0<t_k<t} U(b,t_k)I_k(z_{t_k} + y_{t_k})\Big](\zeta)d\zeta, \ t \in J.
\end{aligned}$$

and

$$\begin{aligned}(\Gamma_2 z)(t) &= \int_0^t U(t,s) f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right)ds \\ &\quad - \int_0^t U(t,\zeta) B W^{-1} \\ &\quad \times \left[\int_0^b U(b,s) f\left(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau\right)ds\right](\zeta)d\zeta,\ t \in J.\end{aligned}$$

Firstly, we prove that $\Gamma_1$ is Lipschitz continuous.

Take $z_1, z_2 \in S(b)$. Then by the axioms of phase space and hypotheses (H5)-(H6), we get that

$$\begin{aligned}&\|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\| \\ &= \|A^{-1}(0)\| C_5 \|z_{1t} - z_{2t}\|_{\mathcal{B}} + M_0 M_3 b C_5 \|z_{1s} - z_{2s}\|_{\mathcal{B}} + M_0 \sum_{k=1}^{m} \gamma_k \|z_{1t_k} - z_{2t_k}\|_{\mathcal{B}} \\ &+ M_0 M_1 M_2 b \Bigg[ \|A^{-1}(0)\| C_5 \|z_{1b} - z_{2b}\|_{\mathcal{B}} + M_0 M_3 b C_5 \|z_{1s} \\ &\quad - z_{2s}\|_{\mathcal{B}} + M_0 \sum_{k=1}^{m} \gamma_k \|z_{1t_k} - z_{2t_k}\|_{\mathcal{B}} \Bigg] \\ &\le K_b (1 + M_0 M_1 M_2 b) \left( \|A^{-1}(0)\| C_5 + M_0 M_3 b C_5 + M_0 \sum_{k=1}^{m} \gamma_k \right) \|z_1 - z_2\|_b.\end{aligned}$$

$$\text{That is, } \|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\|_b \le N \|z_1 - z_2\|_b, \tag{3.6}$$

where $N = K_b (1 + M_0 M_1 M_2 b) \left( \|A^{-1}(0)\| C_5 + M_0 M_3 b C_5 + M_0 \sum_{k=1}^{m} \gamma_k \right)$.

Hence $\Gamma_1$ is Lipschitz continuous with Lipschitz constant $N$.

Next we prove that, $\Gamma_2$ maps $B_q$ into an equicontinuous family on $J$. Indeed let $t_1, t_2 \in J$, $0 < t_1 < t_2$. Then for arbitrary $z \in B_q$, we have

$$\begin{aligned}&\|(\Gamma_2 z)(t_2) - (\Gamma_2 z)(t_1)\| \\ &\le \int_0^{t_1} \|[U(t_2,s) - U(t_1,s)] f(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau)\| ds \\ &\quad + \int_{t_1}^{t_2} \|U(t_2,s) f(s, z_s + y_s, \int_0^s e(s,\tau,z_\tau + y_\tau)d\tau)\| ds \\ &\quad + \int_0^{t_1} \|[U(t_2,\zeta) - U(t_1,\zeta)] B W^{-1} \Big[ \int_0^b U(b,s)\end{aligned}$$

$$f\left(s, z_s + y_s, \int_0^s e(s, \tau, z_\tau + y_\tau)d\tau\right) ds\Big](\zeta)\|d\zeta$$
$$+ \int_{t_1}^{t_2} \|U(t_2, \zeta)BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s + y_s, \int_0^s e(s, \tau, z_\tau + y_\tau)d\tau)ds\Big](\zeta)\|d\zeta.$$

Let $Y(\zeta) = BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s + y_s, \int_0^s e(s, \tau, z_\tau + y_\tau)d\tau)ds\Big](\zeta)d\zeta$, then

$$\begin{aligned}\|(\Gamma_2 z)(t_2) - (\Gamma_2 z)(t_1)\| \leq & \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\|\alpha_{q'}(s)ds + \int_{t_1}^{t_2} \|U(t_2, s)\|\alpha_{q'}(s)ds \\ & + \int_0^{t_1} \|U(t_2, \zeta) - U(t_1, \zeta)\|\|Y(\zeta)\|d\zeta + \int_{t_1}^{t_2} \|U(t_2, \zeta)\|\|Y(\zeta)\|d\zeta.\end{aligned} \tag{3.7}$$

By the equicontinuity property of $\{U(t, s) : (t, s) \in \Delta\}$ and the absolute continuity of the Lebesgue integral, we can see that the right hand side of (3.7) tends to zero and independent of $z$ as $t_2 \to t_1$. Hence, $\Gamma_2(B_q)$ is equicontinuous on $J$.

To prove the Mönch condition, let $W \subseteq B_q$ is countable and $W \subseteq \overline{co}(\{0\} \cup \Gamma(W))$. We shall show that $\beta(W) = 0$. Without loss of generality, we may suppose that $W = \{z^n\}_{n\in N}$. Then by the hypothesis (H2)-(H3) and lemma 2.3, we have

$$\beta(\Gamma_2 W(t)) = \beta(\{\Gamma_2 z^n(t)\}_{n=1}^{\infty})$$
$$\leq \beta\Big(\Big\{\int_0^t U(t, s)f(s, z_s^n + y_s, \int_0^s e(s, \tau, z_\tau^n + y_\tau)d\tau)ds\Big\}_{n=1}^{\infty}\Big)$$
$$+ \beta(\{\int_0^t U(t, \zeta)BW^{-1}\Big[\int_0^b U(b, s)f(s, z_s^n + y_s, \int_0^s e(s, \tau, z_\tau^n + y_\tau)d\tau)ds\Big](\zeta)d\zeta\Big\}_{n=1}^{\infty}\Big)$$
$$\leq 2M_0 \int_0^b \eta(s)\Big[\sup_{-\infty\leq\theta\leq 0} \beta(z^n(s + \theta) + y(s + \theta)) + \beta(\int_0^s e(s, \tau, z_\tau + y_\tau)d\tau)\Big]ds$$
$$+ 2M_0 M_1 \int_0^b \beta\Big(W^{-1}\Big[\Big\{\int_0^b U(b, s)f(s, z_s^n + y_s, \int_0^s e(s, \tau, z_\tau^n + y_\tau)d\tau)ds\Big\}_{n=1}^{\infty}\Big](\zeta)\Big)d\zeta$$
$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} (1 + L_2)\beta(Z_\tau^n) + 2M_0 M_1(\int_0^b K_W(\zeta)d\zeta)$$
$$\times \beta\Big(\Big\{\int_0^b U(b, s)f(s, z_s^n + y_s, \int_0^s e(s, \tau, z_\tau^n + y_\tau)d\tau)ds\Big\}_{n=1}^{\infty}\Big),$$
$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} (1 + L_2)\beta(W(\tau)) + 4M_0^2 M_1(\int_0^b K_W(\zeta)d\zeta)$$

$$(\times)\int_0^b \eta(s)ds \sup_{0\le\tau\le s}(1+L_2)\beta(W(\tau)),$$
$$= (2M_0 + 4M_0^2 M_1\|K_W\|_{L^1})(1+L_2)\|\eta\|_{L^1} \sup_{0\le\tau\le s}\beta(W(\tau))$$
$$\text{That is } \beta(\Gamma_2 W(t)) \le \overline{N} \sup_{0\le\tau\le s}\beta(W(\tau)), \tag{3.8}$$

where $\overline{N} = (2M_0 + 4M_0^2 M_1\|K_W\|_{L^1})(1+L_2)\|\eta\|_{L^1}$.

Since $\Gamma_2$ maps $B_q$ into an equicontinuous family on $J$, $\Gamma_2(W)$ is equicontinuous on $J$ and so $W$ is equicontinuous on $J$. Then by Lemma 2.2, taking supremum on both sides of (3.8) over $J$, we have

$$\beta(\Gamma_2 W) \le \overline{N}\beta(W). \tag{3.9}$$

By the property (7) of Lemma 2.1,

$$\beta(\Gamma_1 W) \le N\beta(W). \tag{3.10}$$
$$\text{Hence} \quad \beta(\Gamma(W)) = \beta(\Gamma_1(W)) + \beta(\Gamma_2(W)) \le (N+\overline{N})\beta(W)$$

From the Mönch condition, we get that

$$\beta(W) \le \beta(\overline{co}(\{0\}\cup\Gamma(W)) = \beta(\Gamma(W)) \le (N+\overline{N})\beta(W) = 0.$$

By (H7), $N+\overline{N} < 1$, which implies that $\beta(W) = 0$. In the view of Lemma 2.4, i.e., Mönch fixed point theorem, we conclude that $\Gamma$ has a fixed point $z$ in $W$. Then $x = y+z$ is a fixed point of $\Phi$ and thus the system (1.1)-(1.3) is controllable on $[0, b]$. ■

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