

More on the Direct Sum of Parafree Lie Algebras

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Abstract

In this paper, we show that the finite direct sum of parafree Lie algebras is a parafree Lie algebra.

Keywords: Parafree, Lie algebra, direct sum

1. Introduction and Preliminaries

In 2005, Baumslag [1] gave some properties of parafree groups. In 2006, Baumslag and Cleary [2] gave some properties of one-relator parafree groups. In [5], Baur introduced the parafree Lie algebras. Recently, Velioglu [7] gave some properties of parafree Lie algebras. In this paper, we show that the finite direct sum of parafree Lie algebras is a parafree Lie algebra.

Let L and H be Lie algebras over a field K . We list the notations and definitions as follows.

$$\gamma_1(L) = L,$$

$$\gamma_{n+1}(L) = [\gamma_n(L), L]$$

for all positive integer n .

We say that L is nilpotent of degree n if

$$\gamma_n(L) = \{0\}.$$

We say that L is residually nilpotent if

$$\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\}.$$

We say that L and H have the same lower central sequence if

$$L / \gamma_n(L) \cong H / \gamma_n(H)$$

for all positive integer n .

We say that L is parafree over a set X if L is residually nilpotent, and L has the same lower central sequence as a free Lie algebra generated by the set X .

2. Main Results

Theorem 2.1. Let L_1, L_2, \dots, L_m be parafree Lie algebras and $L = L_1 \oplus L_2 \oplus \dots \oplus L_m$. Then L is parafree.

Proof. First, we will show that L is residually nilpotent. We have

$$\begin{aligned}\gamma_1(L) &= L \\ &= L_1 \oplus L_2 \oplus \dots \oplus L_m \\ &= \gamma_1(L_1) \oplus \gamma_1(L_2) \oplus \dots \oplus \gamma_1(L_m).\end{aligned}$$

Assume that

$$\gamma_k(L) = \gamma_k(L_1) \oplus \gamma_k(L_2) \oplus \dots \oplus \gamma_k(L_m)$$

where k is a positive integer.

Then

$$\begin{aligned}\gamma_{k+1}(L) &= [\gamma_k(L), L] \\ &= [\gamma_k(L_1) \oplus \gamma_k(L_2) \oplus \dots \oplus \gamma_k(L_m), L] \\ &= [\gamma_k(L_1) \oplus \gamma_k(L_2) \oplus \dots \oplus \gamma_k(L_m), L_1 \oplus L_2 \oplus \dots \oplus L_m].\end{aligned}$$

Then

$$\gamma_{k+1}(L_1) \cap \gamma_{k+1}(L_2) \cap \dots \cap \gamma_{k+1}(L_m) = \{0\}$$

and

$$\begin{aligned}\gamma_{k+1}(L) &= [\gamma_k(L_1) + \gamma_k(L_2) + \dots + \gamma_k(L_m), L_1 + L_2 + \dots + L_m] \\ &= [\gamma_k(L_1), L_1] + [\gamma_k(L_2), L_2] + \dots + [\gamma_k(L_m), L_m] \\ &= \gamma_{k+1}(L_1) + \gamma_{k+1}(L_2) + \dots + \gamma_{k+1}(L_m).\end{aligned}$$

By the mathematical induction,

$$\gamma_n(L) = \gamma_n(L_1) \oplus \gamma_n(L_2) \oplus \dots \oplus \gamma_n(L_m).$$

for all positive integer n .

Then

$$\begin{aligned}\bigcap_{n=1}^{\infty} \gamma_n(L) &= \bigcap_{n=1}^{\infty} (\gamma_n(L_1) \oplus \gamma_n(L_2) \oplus \dots \oplus \gamma_n(L_m)) \\ &= \left(\bigcap_{n=1}^{\infty} \gamma_n(L_1) \right) \oplus \left(\bigcap_{n=1}^{\infty} \gamma_n(L_2) \right) \oplus \dots \oplus \left(\bigcap_{n=1}^{\infty} \gamma_n(L_m) \right).\end{aligned}$$

Since L_1, L_2, \dots, L_m are parafree Lie algebras, it follows that

$$\bigcap_{n=i}^{\infty} \gamma_n(L_i) = \{0\}$$

for all positive integer i .

Then

$$\begin{aligned} \bigcap_{n=1}^{\infty} \gamma_n(L) &= \{0\} \oplus \{0\} \oplus \dots \oplus \{0\} \\ &= \{0\}. \end{aligned}$$

Thus, L is residually nilpotent.

Next, we will show that L has the same lower central sequence as a free Liealgebra. Since L_1, L_2, \dots, L_m are parafree Lie algebras then there exist free Liealgebras H_1, H_2, \dots, H_m such that

$$L_i / \gamma_n(L_i) \cong H_i / \gamma_n(H_i)$$

for all positive integers i and n .

Let

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_m.$$

Then

$$\begin{aligned} L / \gamma_n(L) &\cong (L_1 / \gamma_n(L_1)) \oplus (L_2 / \gamma_n(L_2)) \oplus \dots \oplus (L_m / \gamma_n(L_m)) \\ &\cong (H_1 / \gamma_n(H_1)) \oplus (H_2 / \gamma_n(H_2)) \oplus \dots \oplus (H_m / \gamma_n(H_m)) \\ &\cong H / \gamma_n(H) \end{aligned}$$

for all positive integer n .

Thus, L and H have the same lower central sequence.

Hence, L is parafree.

Corollary 2.2. [7] Let L_1 and L_2 be parafree Lie algebras and $L = L_1 \oplus L_2$. Then L is parafree.

Corollary 2.3. Let L_1, L_2, \dots, L_m be parafree Lie algebras. Then the free product of L_1, L_2, \dots, L_m is parafree.

3. References

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