

λ –Pseudo Starlike Functions With A Bounded Positive Real Part

T. Janani¹ and G. Murugusundaramoorthy^{2*}

Corresponding Author, ^{1,2}School of Advanced Sciences,
VIT University, Vellore 632014, Tamilnadu, India
E-mail: janani.t@vit.ac.in; gmsmoorthy@yahoo.com.

Abstract

By considering a λ –pseudo starlike function in the open unit disk \mathbb{U} which maps \mathbb{U} onto the strip domain w with $\alpha < R(w) < \beta$, we estimate bounds of coefficients and solve Fekete-Szegő problem for functions in this class.

2010 Mathematics Subject Classification. 30C45, 30C50.

Key Words and Phrases: Analytic function, Univalent function, Starlike function, λ –pseudo starlike function, Subordination, Coefficient estimate, Fekete-Szegő problem.

1. Introduction

Let \mathcal{A} denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}) \quad (1.1)$$

in the open unit disk

$$\mathbb{U} = \{z: z \in \mathbb{C}: |z| < 1\}.$$

Let \mathcal{S} be the subclass of \mathcal{A} consisting of all univalent functions and normalized by the condition $f(0) = 0 = f'(0) - 1$ in \mathbb{U} . A function $f \in \mathcal{A}$ is said to be starlike of order α ($0 \leq \alpha < 1$) in \mathbb{U} if it satisfies

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha.$$

This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [11]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f \in \mathcal{A}$ which satisfy

$$\Re \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{U})$$

for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi et. al [16].

Let $P(z)$ and $Q(z)$ be analytic in \mathbb{U} . Then the function $P(z)$ is said to subordinate to $Q(z)$ in \mathbb{U} written by

$$P(z) < Q(z) \quad (z \in \mathbb{U}), \quad (1.2)$$

if there exists a function $w(z)$ which is analytic in \mathbb{U} with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $P(z) = Q(w(z))$ ($z \in \mathbb{U}$). From the definition of the subordinations, it is easy to show that the subordination (1.2) implies that

$$P(0) = Q(0) \quad \text{and} \quad P(\mathbb{U}) \subset Q(\mathbb{U}). \quad (1.3)$$

In particular, if $Q(z)$ is univalent in \mathbb{U} , then the subordination (1.2) is equivalent to the condition (1.3).

Remark 1.1 Let $P(z)$ and $Q(z)$ be analytic in \mathbb{U} . Then the subordination (1.2) implies that

$$|P'(0)| \leq |Q'(0)| \quad (1.4)$$

and $|P'(0)| = |Q'(0)|$ if and only if $P(z) = Q(xz)$ for some real numbers x with $|x| = 1$ [5].

Several authors have discussed various subfamilies of the well-known Bazilevič functions $\mathcal{B}(\alpha)$ satisfying the geometric condition:

$$\Re \left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \right) > 0$$

where α is greater than 1 ($\alpha \in \mathbb{R}$). (see, for details, [4]; see also [1, 2, 7, 10, 15]) from various viewpoints such as the perspective of convexity, inclusion theorems, radii of starlikeness and convexity, boundary rotational problems, subordination relationships and so on. The class includes the starlike and bounded turning functions as the cases $\alpha = 0$ and $\alpha = 1$ respectively. Further the study has been extended by defining a subclass Bazilevič functions of type α order β if and only if

$$\Re \left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \right) > \beta$$

denoted by $\mathcal{B}(\alpha, \beta)$.

More recently Babalola [3] defined a new subclass λ -pseudo starlike function of order β ($0 \leq \beta < 1$) satisfying the analytic condition

$$\Re \left(\frac{z(f'(z))^\lambda}{f(z)} \right) > \beta \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \quad (1.5)$$

and denoted by $\mathcal{L}_\lambda(\beta)$.

Remark 1.2

1. Throughout this work, all powers shall mean principal determinations only.
2. If $\lambda = 1$, we have the class of starlike functions of order β , which in this context, are 1 – *pseudo* –starlike functions of order β .
3. If $\beta = 0$, we simply write \mathcal{L}_λ instead of $\mathcal{L}_\lambda(0)$.

Further, for $\lambda = 2$ we note that functions in $\mathcal{L}_2(\beta)$ are defined by

$$\Re \left(f'(z) \frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{U})$$

which is a product combination of geometric expressions for bounded turning and starlike functions. Babalola [3] remarked that though for $\lambda > 1$, these classes of λ –pseudo starlike functions clone the analytic representation of starlike functions, it is not yet known the possibility of any inclusion relations between them. We recall the following Lemmas which are relevant for our study.

Lemma 1.3 [3] If z is a complex number having positive real part, then for any real number t such that $t \in [0,1]$, we have $\Re(z^t) \geq (\Re(z))^t$.

Let \mathcal{P} be the family of all functions p analytic in \mathbb{U} for which $\Re(p(z)) > 0, p(0) = 1$ and

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \quad (z \in \mathbb{U}). \tag{1.6}$$

Lemma 1.4 [3] Let $p \in \mathcal{P}$ and $\Re(p(z)) > \beta$. If $q(z) = [p(z)]^t$ and $t \in [0,1]$, then $q(0) = 1$ and $\Re(q(z)) > \beta^t$.

Recently, Kuroki and Owa [8] also defined the following class of starlike functions with bounded positive real part.

Definition 1.5 [8] Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < R \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{U}). \tag{1.7}$$

Kuroki and Owa [8] also defined an analytic function

$$\mathcal{S}_{\alpha,\beta}(z): \mathbb{U} \rightarrow \mathbb{C}$$

by

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right) \tag{1.8}$$

which maps \mathbb{U} onto a convex domain w with $\alpha < R(w) < \beta$, conformally. Further the study has been extended to the class of convex functions [14](see also [13]).

Motivated by the works of Kuroki and Owa [8] and Sim and Kwon[13, 14], there has been interest to study the class of analytic functions with bounded positive real part, in this paper we made an attempt to study the new subclass λ –pseudo starlike function in the open unit disk \mathbb{U} which maps \mathbb{U} onto the strip domain w with $\alpha <$

$R(w) < \beta$. Further we estimate bounds of coefficients and solve Fekete-Szegö problem for functions in this functional class.

2 Functions With Bounded Positive Real Part

Definition 2.1 Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{L}_\lambda(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < R\left(\frac{z(f'(z))^\lambda}{f(z)}\right) < \beta \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}). \quad (2.1)$$

Remark 2.2 $f \in \mathcal{L}_\lambda(\alpha, \beta)$ if and only if f satisfies each of the following two subordination relations:

$$\frac{z(f'(z))^\lambda}{f(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R})$$

and

$$\frac{z(f'(z))^\lambda}{f(z)} < \frac{1 - (1 - 2\beta)z}{1 + z} \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}).$$

By taking $\lambda = 2$, we define the following new subclass $\mathcal{G}(\alpha, \beta)$ of \mathcal{S} .

Example 2.3 Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{L}_2(\alpha, \beta) \equiv \mathcal{G}(\alpha, \beta)$ if f satisfies the following inequality

$$\alpha < R\left(f'(z) \frac{zf'(z)}{f(z)}\right) < \beta \quad (z \in \mathbb{U}). \quad (2.2)$$

Remark 2.4 By specializing the parameter $\lambda = 1$, we get the definition of the subclass $\mathcal{L}_1(\alpha, \beta) \equiv \mathcal{S}(\alpha, \beta)$ of \mathcal{S} satisfying the condition (1.7) studied by Kuroki and Owa [8].

In our present study we give some example for $f \in \mathcal{L}_\lambda(\alpha, \beta)$ as follows:

Example 2.5 Let us consider the function $f(z)$ given by

$$f(z) = \begin{cases} \left[\frac{\lambda-1}{\lambda} \int_0^z \left\{ \frac{1}{t} + \frac{\beta-\alpha}{\pi t} i \log \left(\frac{1-e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} t}}{1-e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} t}} \right) \right\}^\lambda dt \right]^{\frac{1}{\lambda-1}}, & \lambda > 1 \\ \left[z \exp \left[\frac{\beta-\alpha}{\pi} i \int_0^z \frac{1}{t} \log \left(\frac{1-e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} t}}{1-e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} t}} \right) dt \right] \right], & \lambda = 1 \end{cases} \quad (2.3)$$

with $0 \leq \alpha < 1 < \beta$. Then we have

$$\frac{z(f'(z))^\lambda}{f(z)} = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right) = \mathcal{S}_{\alpha,\beta}(z) \quad (z \in \mathbb{U}).$$

Hence, the function $f(z)$ given by (2.3) satisfies the inequality (2.1) which implies that $f \in \mathcal{L}_\lambda(\alpha, \beta)$.

We state the following example for $f \in \mathcal{G}(\alpha, \beta)$:

Example 2.6 Let us consider the function $f(z)$ given by

$$f(z) = \left[\frac{1}{2} \int_0^z \left\{ \frac{1}{t} + \frac{\beta - \alpha}{\pi t} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} t}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} t}} \right) \right\}^{\frac{1}{2}} dt \right]^2 \tag{2.4}$$

with $0 \leq \alpha < 1 < \beta$. Then we have

$$f'(z) \frac{zf'(z)}{f(z)} = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right) = \mathcal{S}_{\alpha,\beta}(z) \quad (z \in \mathbb{U}).$$

Hence, the function $f(z)$ given by (2.4) satisfies the inequality (2.2) which implies that $f \in \mathcal{G}(\alpha, \beta)$.

Using the function $\mathcal{S}_{\alpha,\beta}(z)$ defined by (1.8), we state a necessary and sufficient condition for $f \in \mathcal{L}_\lambda(\alpha, \beta)$ and $f \in \mathcal{G}(\alpha, \beta)$ in the following lemma:

Lemma 2.7 Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1 < \beta$, then $f \in \mathcal{L}_\lambda(\alpha, \beta)$ if and only if

$$\frac{z(f'(z))^\lambda}{f(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right), \quad z \in \mathbb{U} \tag{2.5}$$

and

$f \in \mathcal{G}(\alpha, \beta)$ if and only if

$$f'(z) \frac{zf'(z)}{f(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right), \quad z \in \mathbb{U}. \tag{2.6}$$

For our discussions, (unless otherwise stated) we let

$$\mathcal{S}_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{\frac{i\pi(1-\alpha)}{\beta-\alpha} z}}{1 - e^{-\frac{i\pi(1-\alpha)}{\beta-\alpha} z}} \right) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta-\alpha} \quad (n = 1, 2, 3, \dots). \quad (2.7)$$

In order to prove our main results we recall the following lemma due to Miller and Mocanu [9].

Lemma 2.8 [9] Let Ω be a set in the complex plane \mathbb{C} and let b be a complex number such that $\Re(b) > 0$. Suppose that a function $\psi: \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies the condition $\psi(i\rho, \sigma; z) \notin \Omega$ for all real $\rho, \sigma \leq \frac{-|b-i\rho|^2}{2\Re(b)}$ and all $z \in \mathbb{U}$. If the function $p(z)$ defined by $p(z) = b + b_1z + b_2z^2 + \dots$ is analytic in \mathbb{U} and if

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then

$$\Re(p(z)) > 0, \quad (z \in \mathbb{U}).$$

Theorem 2.9 Let $f \in \mathcal{A}$, $\frac{1}{2} \leq \alpha < 1$, $\lambda \geq 1$ and $\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha$ in \mathbb{U} . Then

$$\Re\left(\frac{f(z)}{z}\right) > \gamma(\alpha), \quad (z \in \mathbb{U}), \quad (2.8)$$

where $\alpha = \frac{(3\gamma-1)^\lambda}{\gamma 2^\lambda}$.

Proof. Let $p(z)$ be defined by

$$p(z) = \frac{1}{1-\gamma} \left(\frac{f(z)}{z} - \gamma \right).$$

Then $p(z)$ is analytic in \mathbb{U} , $p(0) = 1$ and

$$f(z) = (1-\gamma)zp(z) + \gamma z$$

$$\Rightarrow \frac{z(f'(z))^\lambda}{f(z)} = \frac{[(1-\gamma)(zp'(z) + p(z)) + \gamma]^\lambda}{(1-\gamma)p(z) + \gamma} = \psi(p(z), zp'(z)),$$

where

$$\psi(r, s) = \frac{[(1-\gamma)(s+r) + \gamma]^\lambda}{(1-\gamma)r + \gamma}. \quad (2.9)$$

By using the lemma (2.8) and proceeding similar to the proof of Theorem 2 by Sim and Kwon [13], we get,

$$\Re(\psi(i\rho, \sigma)) \leq \frac{(3\gamma-1)^\lambda}{\gamma 2^\lambda} = \alpha.$$

Thus the proof is completed.

Theorem 2.10 Let $f \in \mathcal{A}$, $1 < \beta < \frac{3}{2}$, $\lambda \geq 1$ and $\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) < \beta$ in \mathbb{U} . Then

$$\Re\left(\frac{f(z)}{z}\right) < \delta(\beta), \quad (z \in \mathbb{U}), \quad (2.10)$$

where $\beta = \frac{(3\delta-1)^\lambda}{\delta 2^\lambda}$.

Proof. Let $p(z)$ be defined by

$$p(z) = \frac{1}{1-\delta} \left(\frac{f(z)}{z} - \delta \right).$$

Then $p(z)$ is analytic in \mathbb{U} , $p(0) = 1$ and

$$f(z) = (1-\delta)zp(z) + \delta z$$

$$\Rightarrow \frac{z(f'(z))^\lambda}{f(z)} = \frac{[(1-\delta)(zp'(z) + p(z)) + \delta]^\lambda}{(1-\delta)p(z) + \delta} = \psi(p(z), zp'(z)),$$

where $\psi(r, s)$ is defined by (2.9). Also by using the lemma (2.8) and proceeding similar to the proof of Theorem 3 by Sim and Kwon [13], we get,

$$\Re(\psi(i\rho, \sigma)) \leq \frac{(3\delta-1)^\lambda}{\delta 2^\lambda} = \beta.$$

Thus the proof is completed.

Theorem 2.11 Let $f \in \mathcal{A}$, $\frac{1}{2} \leq \alpha < 1 < \beta < \frac{3}{2}$, $\lambda \geq 1$ and $\alpha < R\left(\frac{z(f'(z))^\lambda}{f(z)}\right) < \beta$ in \mathbb{U} . Then

$$\gamma(\alpha) < R\left(\frac{f(z)}{z}\right) < \gamma(\beta), \quad (z \in \mathbb{U}),$$

where $\gamma(\alpha)$ and $\gamma(\beta)$ is given by (2.8) and (2.10).

3. Some Coefficient Problems

Using the subordination (2.5), we find sharp bounds on the second and third coefficients for $f \in \mathcal{L}_\lambda(\alpha, \beta)$, making use of the following lemma due to Rogosinski [12].

Lemma 3.1 Let $P(z) = \sum_{n=1}^\infty A_n z^n$ and $Q(z) = \sum_{n=1}^\infty B_n z^n$ be analytic in \mathbb{U} .

If $P(z) \prec Q(z)$ ($z \in \mathbb{U}$), then

$$\sum_{k=1}^m |A_k|^2 \leq \sum_{k=1}^m |B_k|^2 \quad (m = 1, 2, \dots).$$

Theorem 3.2 If the function $f \in \mathcal{L}_\lambda(\alpha, \beta)$, then

$$|a_2| \leq \frac{2(\beta - \alpha)}{\pi(2\lambda - 1)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{2(\beta - \alpha)}{\pi(3\lambda - 1)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left[\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} - \frac{2(2\lambda^2 - 4\lambda + 1)(\beta - \alpha)}{\pi(2\lambda - 1)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right].$$

Moreover, the equality holds in either inequality if and only if

$$f(z) = \begin{cases} \left[\frac{\lambda - 1}{\lambda} \int_0^z \left\{ \frac{\mathcal{S}_{\alpha, \beta}(e^{i\theta t})}{t} \right\}^{\frac{1}{\lambda}} dt \right]^{\frac{\lambda}{\lambda-1}}, & \lambda > 1 \\ z \exp \left[\int_0^z \frac{\mathcal{S}_{\alpha, \beta}(e^{i\theta t}) - 1}{t} dt \right], & \lambda = 1 \end{cases}$$

for some real number θ ($0 \leq \theta < 2\pi$), where $\mathcal{S}_{\alpha, \beta}(z)$ is defined by (1.8).

Proof. Let $P(z) = \frac{z(f'(z))^\lambda}{f(z)}$

$$= 1 + (2\lambda - 1)a_2z + [(3\lambda - 1)a_3 + (2\lambda^2 - 4\lambda + 1)a_2^2]z^2 + \quad (3.1)$$

and

$$Q(z) = \mathcal{S}_{\alpha, \beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad (3.2)$$

where B_n is as in (2.7).

Now by applying Lemma 3.1 and using Remark 1.1, we obtain the desired estimates of $|a_2|$ and $|a_3|$.

By taking $\lambda = 2$ we state the following result without proof.

Theorem 3.3 If the function $f \in \mathcal{G}(\alpha, \beta)$, then

$$|a_2| \leq \frac{2(\beta - \alpha)}{3\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$|a_3| \leq \frac{2(\beta - \alpha)}{5\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \left[\cos \frac{\pi(1 - \alpha)}{\beta - \alpha} - \frac{2(\beta - \alpha)}{9\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right].$$

Moreover, the equality holds in either inequality if and only if

$$f(z) = \left[\frac{1}{2} \int_0^z \left\{ \frac{\mathcal{S}_{\alpha, \beta}(e^{i\theta t})}{t} \right\}^{\frac{1}{2}} dt \right]^2,$$

for some real number θ ($0 \leq \theta < 2\pi$), where $\mathcal{S}_{\alpha, \beta}(z)$ is defined by (1.8).

4. Fekete-Szegő Problem

Making use of the following lemma we shall solve the Fekete-Szegő problem for $f \in \mathcal{L}_\lambda(\alpha, \beta)$.

Lemma 4.1 (Keogh and Merkers [6]) Let $h(z) = 1 + h_1z + h_2z^2 + \dots$ be a function with positive real part in \mathbb{U} . Then for any complex number ν ,

$$|h_2 - \nu h_1^2| \leq 2 \max\{1, |1 - 2\nu|\}.$$

Theorem 4.2 Let $0 \leq \alpha < 1 < \beta$ and let the function f given by (1.1) be in the class $\mathcal{L}_\lambda(\alpha, \beta)$. Then for any complex number μ ,

$$|a_3 - \mu a_2^2| \leq \frac{2(\beta - \alpha)}{\pi(3\lambda - 1)} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \\ \times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} - \frac{2[2\lambda^2 + (3\mu - 4)\lambda + (1 - \mu)](\beta - \alpha)}{\pi(2\lambda - 1)^2} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

Proof. Let $P(z) = \frac{z(f'(z))^\lambda}{f(z)}$. Then, since $f \in \mathcal{L}_\lambda(\alpha, \beta)$, we have $P(z) \prec Q(z)$, where $Q(z)$ is given by (3.2). Let

$$h(z) = \frac{1 + Q^{-1}(P(z))}{1 - Q^{-1}(P(z))} = 1 + h_1z + h_2z^2 + \dots \quad (z \in \mathbb{U}).$$

Then h is analytic and has positive real part in the open disk \mathbb{U} . We also have

$$P(z) = Q\left(\frac{h(z)-1}{h(z)+1}\right) \quad (z \in \mathbb{U}). \tag{4.1}$$

We find from the equation (4.1) that

$$a_2 = \frac{B_1 h_1}{2(2\lambda - 1)}$$

$$a_3 = \frac{1}{(3\lambda - 1)} \left[\frac{B_2 h_1^2}{4} - \frac{B_1 h_1^2}{4} + \frac{B_1 h_2}{2} - \frac{(2\lambda^2 - 4\lambda + 1)B_1^2 h_1^2}{4(2\lambda - 1)^2} \right]$$

which imply that

$$a_3 - \mu a_2^2 = \frac{B_1}{2(3\lambda - 1)} (h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2} \left(1 - \frac{B_2}{B_1} + \frac{[2\lambda^2 - 4\lambda + 1 + \mu(3\lambda - 1)]B_1}{(2\lambda - 1)^2} \right).$$

Applying Lemma 4.1, we obtain

$$|a_3 - \mu a_2^2| = \frac{|B_1|}{2(3\lambda - 1)} |h_2 - \nu h_1^2| \\ \leq \frac{|B_1|}{3\lambda - 1} \max\{1, |1 - 2\nu|\}. \tag{4.2}$$

Substituting

$$B_1 = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}$$

and

$$B_2 = \frac{\beta - \alpha}{\pi} \sin \frac{2\pi(1 - \alpha)}{\beta - \alpha}$$

in (4.2), we can obtain the results as asserted.

Theorem 4.3 Let $0 \leq \alpha < 1 < \beta$ and let the function f given by (1.1) be in the class $\mathcal{G}(\alpha, \beta)$. Then for any complex number μ ,

$$|a_3 - \mu a_2^2| \leq \frac{2(\beta - \alpha)}{5\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \\ \times \max \left\{ 1, \left| \cos \frac{\pi(1 - \alpha)}{\beta - \alpha} - \frac{2(5\mu + 1)(\beta - \alpha)}{9\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha} \right| \right\}.$$

References

- [1] F. M. Al-Aboudi, n -Bazilevič functions, *Abstr. Appl. Anal.*, 2012 (2012), Article ID 383592, 1–10.
- [2] A. A. Amer and M. Darus, Distortion theorem for certain class of Bazilevič functions, *Internat. J. Math. Anal.*, 6 (2012), 591–597.
- [3] K. O. Babalola, On λ -pseudo starlike Functions, *Journal of Classical Analysis*, 3(2013), no.2, 137 - 147.
- [4] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York (1983).
- [5] A.W. Goodman, *Univalent functions*, Vol. I and II, Mariner, Tampa, Florida, 1983.
- [6] F. Keogh and E. Merkers, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, 20 (1969), 8 - 12.
- [7] Y. C. Kim and T. Sugawa, A note on Bazilevič functions, *Taiwanese J. Math.* 13 (2009), 1489–1495.
- [8] K. Kuroki and S. Owa, Notes on new class for certain analytic functions, *Advances in Mathematic: Scientific. Journal* 1 (2012), no. 2, 127 - 131.
- [9] S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*, Series of Monographs and textbooks in pure and applied mathematics, Vol. 225, Marcel Dekker Inc, New-York, Basel, 2000.
- [10] K. I. Noor and K. Ahmad, On higher order Bazilevič functions, *Internat. J. Modern Phys. B* 27 (4) (2013), Article ID 1250203, 1–14.
- [11] M.S. Robertson, On the theory of univalent functions, *Ann. Math.* 37 (1936), 374 – 408.
- [12] W. Rogosinski, On the coefficients of subordinate functions, *Proc. London Math. Soc.*, 48 (1943), 48 - 82.
- [13] Y. J. Sim and O.S. Kwon, Notes on analytic functions with a bounded positive real part, *Journal of Inequalities and Applications*, 370 (2013), 1 - 6.
- [14] Y. J. Sim and O.S. Kwon, On certain classes of convex functions, *Inter., J. Math., and Mathematical Sci.*, Volume 2013, Article ID 294378, 6 pages
- [15] R. Singh, On Bazilevič functions, *Proc. Amer. Math. Soc.* 28 (1973), 261–271.
- [16] B.A. Uralegaddi, M.D. Ganigi and S.M. Sarangi, Univalent functions with positive coefficients, *Tamkang J. Math.* 25 (1994), 225 - 230.