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Path Reduction Method For Solving Maximum Flow Problems

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ABSTRACT

In this paper, a new concept namely, max-flow matrix of a flow network is defined. Then, a new method namely, path reduction method is proposed for determining maximum flow in flow networks. The maximality of the flow value in the network by the proposed method is derived mathematically and the minimum cut set in the network is obtained using the proposed method. The maximum amount of flow and its related flow paths in the flow network are provided by the proposed method in which augmenting paths and preflows concepts are not used. Numerical example is presented for showing the efficiency and illustrating the solution procedure of the path reduction method. Further, the proposed method is extended to interval and fuzzy maximum flow problems.

Keywords: Maximum flow problem, Max-flow matrix, Path reduction method, Interval numbers, Triangular fuzzy numbers..

1. INTRODUCTION

Maximum flow problem in a flow network is one of the popular combinatorial optimization problems. It has been widely studied over the last 50 years due to its various applications in electrical power systems, communication networks, computer networks and logistic networks. The main objective of the maximum flow problem in a flow network is to find a flow that satisfies the capacity constraints and flow-balance constraints at all nodes other than source and sink, so that the amount of flow from the source to sink in the flow network is maximized. Fulkerson and Dantzig [15] proposed originally the maximum flow problem as a linear programming model and solved it by the simplex method.

In the literature, there are many efficient algorithms for solving maximum flow problems [2, 26] due to wise applicabity of these problems. In the efficient maximum flow algorithms, either augmenting paths [14, 12,] or preflows [22,25] are used. Augmenting-path algorithms push flow along a path from the source to the sink in the residual network and include Ford–Fulkerson's labeling algorithm [14] and Dinic's blocking flow algorithm [12]. Preflow-based algorithms push flow along edges in the residual network and include Karzanov's blocking flow algorithm [22] and Goldberg–Tarjan's push-relabeling algorithm [16,28].

Hochbaum [21] devloped the pseudoflow algorithm for the maximum flow problem, based on Lerchs and Grossman's alogithm for the maximum closure problem [24].Anderson and Hochbaum [7] proposed the highest label pseudoflow different that has the same strongly polynomial complexity, and performed an extensive computational study comparing the pseudoflow algorithm to push-relabel algorithm. Gu and Xu [19] presented the symbolic algorithms based on symbolic graph algorithms [13,20] for the maximum flow in general networks. Goldberg [18] described the two-level push-relabel algorithm and the generalized a practical algorithm for bipartite flow network problems. Goldberg et al.[17] investigated the incremental breadth-first search (IBFS) method for solving maximal-flow problems. Orlin [26] presented improved polynomial time algorithms for maximum flow network problems. Aditya Gaykar et al [1] presented parallel implementation of push-relabel in order to improve the efficiency of solving maximum flow problem on Beowulf cluster architecture.

Kim and Roush [23] originally developed the fuzzy flow theory and the conditions to obtain a optimal flow for fuzzy maximum flow problems. Chanas and Kolodziejczyk [8,9,10] developed three different algorithms for a network with crisp structure and fuzzy capacities. Diamond [11] proved a max- flow min-cut theorem for interval networks and proposed a Karp–Edmonds type algorithm for interval networks. Amit Kumar and Manjot Kaur [3,4,5,6] proposed various types of algorithms to find an optimal flow in fuzzy maximum flow problems using ranking function or fuzzy linear programming.

In this paper, a new concept namely, max-flow matrix of a network is introduced and then, we propose a new method namely, path reduction method for solving maximum flow problems. For showing the maximality of the flow value in the network by the proposed method, two theorems are derived mathematically and for finding minimum cut set in the network, one more theorem is presented. The maximum amount of flow in the network and its related paths in the network are determined by the path reduction method. In the proposed method, the augmenting paths, preflows and maxi-flow and min-cut theorem are not used. The efficiency and the solution procedure of the proposed method are shown by a numerical example. Further, the path reduction method is extended to interval and fuzzy maximum flow problems.

2. PRELIMINARIES

Let $N = (V, E)$ be a loop-free connected directed graph with node set V and arc set E and let $|V| = n$ and $|E| = m$. Then, the graph $N = (V, E)$ is called a flow network if the following conditions are satisfied :

(i) There exists a unique node $s \in V$ such that the in degree of s is zero. The node $s \in V$ is called the source.

(ii) There exists a unique node $t \in V$ such that the out degree of t is zero. The node $t \in V$ is called the sink.

(iii) There exists a function c from E to $Z^+ \cup \{0\}$, the set of all non-negative integers. For an $e = (u,v) \in E$, $c(e)$ or $c((u,v))$ is called the capacity of the edge e.

(iv) There exists a function f from E to $Z^+ \cup \{0\}$, the set of all non-negative integers and $f(e) \le c(e)$, for all $e \in E$. For an $e = (u,v) \in E$, $f(e)$ or $f((u,v))$ is the amount of flow from u to v or in $e \in E$.

(v) If there is no edge between *v* and *w*, then $c((u,v)) = 0$ and $f((u,v)) = 0$.

In this paper, the flow network $N = (V, E)$ is represented by $N = (V, E, c, f)$.

The edge $e \in E$ in $N = (V, E, c, f)$ is said to be *saturated* if $f(e) = c(e)$. Otherwise, the edge $e \in E$ is called unsaturated.

Let U be a subset of E in $N = (V, E, c, f)$. Then, U is said to be (s-t)-cut set if the removal of U from the network results in the separation of s and t. The sum of capacities of edges in U is called the capacity of U, denoted by $c(U)$.

Let U be an (s-t)-cut set with capacity d is said to be minimum (s-t)-cut set if there exists no other (s-t)-cut set S with capacity g such that $g < d$.

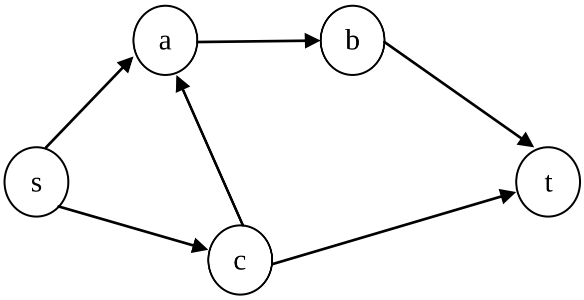
Let $P = ae_1a_1e_2a_2 \cdots a_{n-1}e_nb$ be a path from a node a to another node b in $N = (V, E, c, f)$. The flow in the path *P*, *flow*(*P*) is a non negative number k if $k \le f(e_i), i = 1,2,\ldots,n$. Therefore, the maximum flow in the path *P*, max *flow*(*P*) is the minimum of { $f(e_i), i = 1,2,\ldots,n$ }.

The maximum *flow* in an (s-t) path of the flow network $N = (V, E, c, f)$ is the minimum of the set of capacities of the edges in the path. The maximum *flow* in the flow network $N = (V, E, c, f)$ is the maximum amount of flow from source *s* to the sink *t* in $N = (V, E, c, f)$.

3. MAX-FLOW MATRIX

Let $N = (V, E, c, f)$ be a flow network with n nodes including source node s and the sink node t . The max-flow matrix of $N = (V, E, c, f)$ is an $(n-1) \times (n-1)$ matrix, (a_{ij}) where $a_{ij} = c((i, j))$, the capacity of the edge (i, j), $i = 2,3,4,\ldots,n-1,t$ and $j = s,2,3,\ldots,n-1$.

Example 3.1: Consider the following flow network



Edge	Capacity
(s,a)	5
(s,c)	8
(a,b)	7
(b,t)	10
(c,a)	6
(c,t)	9

Now, the max-flow matrix of the above network is given below:

$$\begin{array}{c} \\ s \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & t \end{array} \\ \begin{pmatrix} 5 & 0 & 8 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 6 & 0 & 0 & 9 \end{pmatrix} \end{array}$$

Remark 3. 1: If all entries in the i^{th} row of the max-flow matrix of a network are zero, then there is no inflow at i^{th} node, that is, in degree of i^{th} node is zero. Therefore, there is no outflow at i^{th} node

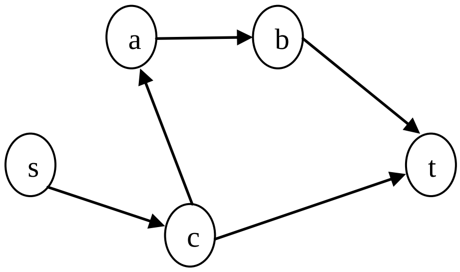
Remark 3.2: If all entries in the j^{th} column of the max-flow matrix of a network are zero, then there is no outflow at j^{th} node, that is, out degree of j^{th} node is zero. Therefore, there is no inflow at j^{th} node.

Let P be an (*s*-*t*)- path in the flow network $N = (V, E, c, f)$ with maximum flow p. The difference of N from P denoted by $N \setminus P$ is defined as a flow network $N \setminus P = (V, E_1, c_1, f)$ where $E_1 = E \setminus \{(a,b) \in P : c(a,b) = p\}$ and $c_1(a,b) = c(a,b)$, if $(a,b) \notin E_1$ and $c_1(a,b) = c(a,b) - p$, if $(a,b) \in E_1$.

Example 3.2 Consider the flow network of the Example 3.1..

Let P = *s*-*a*-*b*-*t* be an (*s*-*t*) path in the flow network. Then, the maximum amount flow in the path P is 5.

Now, the flow network $N_1 = N \setminus P$ is given below:



Edge	Capacity
(s,c)	8
(a,b)	2
(b,t)	5
(c,a)	6
(c,t)	9

Now, the max-flow matrix of $N \setminus P$ from the max-flow matrix of N is given below:

$$\begin{array}{c} \\ s \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & t \end{array} \\ \begin{pmatrix} 0 & 0 & 8 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 6 & 0 & 0 & 9 \end{pmatrix} \end{array}$$

4. THE PATH REDUCTION METHOD

Now, we prove the following theorems which are used in the proposed method for finding the maximum flow in the network.

Theorem 4.1: Let (s,a) be an edge in $N = (V, E, c, f)$ having the maximum outflow at the source node s. If $P_i, i = 1,2,...,k$ are (s,t)-paths having (s,a) as a starting edge with maximum flow $p_i, i = 1,2,...,k$ such that P_k is the last (s,t) path having (s,a) as a starting edge, then the maximum flow in N is $\sum_{i=1}^{k} p_i +$ the maximum flow in N_1 where $N_1 = N \setminus P_1 \setminus P_2 \cdots \setminus P_k$.

Proof: Now, from the construction of $N \setminus P$, we observe that the flow problem $N \setminus P = (V, E_1, c_1, f)$ is obtained from $N = (V, E, c, f)$ by subtracting the maximum flow amount p of the path P from the capacities of all the edges in the path P.

Since the edge (s,a) in $N = (V, E, c, f)$ has the maximum outflow at the source node *s,* we have the maximum flow in $N \setminus P_1$ = the maximum flow in $N - p_1$.

This implies that, the maximum flow in N = $p_1 +$ the maximum flow in N / P_1. (1)

Now, using the relation (1), we obtain the following results.

$$\left.\begin{array}{l}\text{The maximum flow in } N/P_1 = p_2 + \text{ the maximum flow in } (N/P_1)\setminus P_2 \\ \text{The maximum flow in } (N/P_1)/P_2 = p_3 + \text{the maximum flow in } ((N/P_1)\setminus P_2)\setminus P_3 \\ \vdots \\ \text{The maximum flow in } ((((N\setminus P_1)\setminus P_2)\cdots)\setminus P_{k-1}) = p_k + \text{the maximum flow in } ((((N\setminus P_1)\setminus P_2)\cdots)\setminus P_k)\end{array}\right\} \quad (2)$$

Now, from (1) and (2), we can conclude that the maximum flow in N is $\sum_{i=1}^{k} p_i +$ the maximum flow in N_1 where $N_1 = N \setminus P_1 \setminus P_2 \cdots \setminus P_k$.

Hence the theorem.

Theorem 4.2: Let $\{a_1, a_2, ..., a_r\} \subset V$ be in the flow network $N = (V, E, c, f)$ such that $a_j, j = 1,2,...,r$ are adjacent to the source node s and $c((s,a_1)) \geq c((s,a_2)) \geq ... \geq c((s,a_r))$. If $P_i^1, i = 1,2,...,k_1$ are (s,t)-paths in flow network N having (s,a_1) as a starting edge with maximum flow $p_i^1, i = 1,2,...,k_1$ such that P_{k_1} is the last (s,t) path and if $P_i^j, i = 1,2,...,k_j$ are (s,t)-paths in $N_{j-1} = ((((N \setminus P_1^{j-1}) \setminus P_2^{j-1})\cdots) \setminus P_{k_1}^{j-1})$ having (s,a_j) as a starting edge with maximum flow $p_i^j, i = 1,2,...,k_j$ such that P_{k_j} is the last (s,t) path , $j = 2,3...,r$, then the maximum flow in N is $\sum_{i=1}^{k_1} p_i^1 + \sum_{i=1}^{k_2} p_i^2 + \cdots + \sum_{i=1}^{k_r} p_i^r$.

Proof : Now, since $P_i^1, i = 1,2,...,k_1$ are (s,t)-paths having (s,a_1) as a starting edge with maximum flow $p_i^1, i = 1,2,...,k_1$ such that P_{k_1} is the last (s,t) path in N and by the Theorem 4.1., we have the maximum flow in N is $\sum_{i=1}^{k_1} p_i^1$ + the maximum flow in N_1 where $N_1 = ((((N \setminus P_1^1) \setminus P_2^1)\cdots) \setminus P_{k_1}^1)$.

Now, since $P_i^2, i = 1,2,...,k_2$ are (s,t)-paths having (s,a_2) as a starting edge with maximum flow $p_i^2, i = 1,2,...,k_2$ such that P_{k_2} is the last (s,t) path in N_1 and by the Theorem 4.1., we have the maximum flow in N_1 is $\sum_{i=1}^{k_2} p_i^2$ + the maximum flow in N_2

where $N_2 = ((((N_1 \setminus P_1^2) \setminus P_2^2)\cdots) \setminus P_{k_2}^2)$.

$\ddots \qquad \ddots \qquad \ddots$

Now, since $P_i^r, i=1,2,...,k_r$ are (s,t)-paths having (s,a_r) as a starting edge with maximum flow $p_i^r, i=1,2,...,k_r$ such that P_{k_r} is the last (s,t) path in N_{r-1} and by the Theorem 4.1., we have the maximum flow in N_{r-1} is $\sum_{i=1}^{k_r} p_i^r$ + the maximum flow in N_r where $N_r = ((((N_{r-1} \setminus P_1^r) \setminus P_2^r) \cdots) \setminus P_{k_r}^r)$.

Since N_r has no (s-t)-path, the maximum flow in $N_r = 0$.

Therefore, the maximum flow in N_{r-1} is $\sum_{i=1}^{k_r} p_i^r$.

Now, the maximum flow in $N = \sum_{i=1}^{k_1} p_i^1$ + the maximum flow in N_1

$$= \sum_{i=1}^{k_1} p_i^1 + \sum_{i=1}^{k_2} p_i^2 \text{ + the maximum flow in } N_2$$

$\ddots$

$$= \sum_{i=1}^{k_1} p_i^1 + \sum_{i=1}^{k_2} p_i^2 + \sum_{i=1}^{k_3} p_i^3 + \cdots + \sum_{i=1}^{k_r} p_i^r .$$

Hence the theorem is proved.

Now, we propose a new method namely, path reduction method for finding the maximum flow in the network and also, identifying the used (s-t)-flow paths in the network.

The proposed method proceeds as follows.

Step 1: Construct the max-flow matrix for the given flow network.
Step 2: Find the maximum value in the sth row of the max-flow matrix obtained in the Step 1.. Say x_1 which equals to $c((s,i))$. If more than one occur, select any one.
Step 3: Find the maximum value in the ith row of the max-flow matrix . Say x_2 which equals to $c((i,j))$. If more than one occur, select any one.
Step4: : Find the maximum value in the jth row of the max-flow matrix . Say x_3 which equals to $c((j,j_1))$. If more than one occur, select any one.
Step 5: Continue the process of the Step 3. and the Step 4. until no such row occurs. Let $\{x_1, x_2, \cdots, x_k\}$ be a sequence of maximum values of the rows obtained such that $x_1 = c(s,i)$ and $x_k = c(p,q)$.
Step 6: Construct a path P_1 from s^{th} node to q^{th} node using the sequence $\{x_1, x_2, \cdots, x_k\}$ such that $x_i's$ are the capacities of the edges of the path P_1, $x_1 = c(s,i)$ and $x_k = c(p,q)$
Step 7: If $q \neq t$, sink, the path P_1 is not (s-t)-path and remove qth row and qth column

in the max-flow matrix. If all row values or all column values related to a node are zero, remove the row and column of the node in the max-flow matrix. Then, go to the Step 2.

If $q = t$,sink, the path P_1 is an (s-t)-path. Then, go to the Step 8..

Step 8: Find the minimum value of $\{x_1, x_2, \cdots, x_k\}$. Say x_v^1 which equals to the capacity of an edge or edges in the path P_1 and also, it is the maximum flow in the path P_1.

Step 9 : Construct the modified max-flow matrix of the reduced flow network $N \setminus P_1$.

Step 10: Remove the row and column of the node in the max-flow matrix of $N \setminus P_1$ whose all row or column values are zero.

Step 11: (a) If $x_1 - x_v^1 = 0$, that is, the edge (s,i) is saturated, go to the Step 2..

(b) If $x_1 - x_v^1 \neq 0$ and ith row exists, go to the Step 3..

(c) If $x_1 - x_v^1 \neq 0$ and ith row does not exist, go to the Step 2..

Step 12: Stop the computation if s^{th} row or t^{th} column of the max-flow matrix of the reduced flow network has no non-zero value. If not, repeat the Step 2. to the Step 11..

Step 13. The maximum flow in the network is $\sum_i x_v^i$ where x_v^i is the maximum flow in the path P_i which is obtained in the Step 7. (by the Theorem 4. 2.).

Algorithm Information

Let m be the number of edges in the network and k be the maximum number of (s-t) paths in the network. Over all maximum number of comparisons made in the algorithm is $(m-1)+(m-2)+.....+(m-k-1)$, that is, $mk - \frac{(k-1)k}{2}$. The path reduction algorithm has the complexity of $O\left(mk - \frac{(k-1)k}{2} + (k-1)\right)$, that is, $O((m-k)k)$.

Now, the following theorem helps us to compute the minimum (s,t)-cut set of the flow network.

Theorem 4.3: Let $M_i, i = 1,2,...,k$ be (s,t)-paths in N obtained by the path reduction method with maximum flow $q_i, i = 1,2,...,k$ such that the maximum in N = $\sum_{i=1}^{k} q_i$. Let U be a collection edges in $M_i, i = 1,2,...,k$ such that select one edge e in each of $M_i's$ and the edge e should satisfy any one of the following properties:

P1: e is a common edge of q number of $M_i's$ (s,t)-paths such that $p_1 + p_2 + ... + p_q = c(e)$ where p_i is the maximum flow of the M_i th (s,t)-path.

P2: e is a saturated edge of an M_i th (s,t)-path.

Then, U is a minimum (s-t)-cut set of the flow network N.

Proof: Now, from the construction of U, we can conclude that the removal of U in N separates the nodes s and t. Therefore, U is an (s-t)-cut set of the flow network N.

Now, c(U) = $\sum_{e \in U} c(e) = \sum_{\substack{e \in U \\ e \text{ is a commom edge}}} c(e) + \sum_{\substack{e \in U \\ e \text{ is a saturted edge}}} c(e)$

= sum of maximum flows in $M_i's$ having common edges

+ sum of maximum flows in $M_i's$ having saturated edges

= sum of flows in Mi's paths of the flow network N.

= $\sum_{i=1}^{k} q_i$, the maximum flow in N.

Suppose that U is not a minimum cut set of the flow network N.

Then, there exists a cut set , W in N such that $c(W) < c(U) =$ Maximum flow in N.

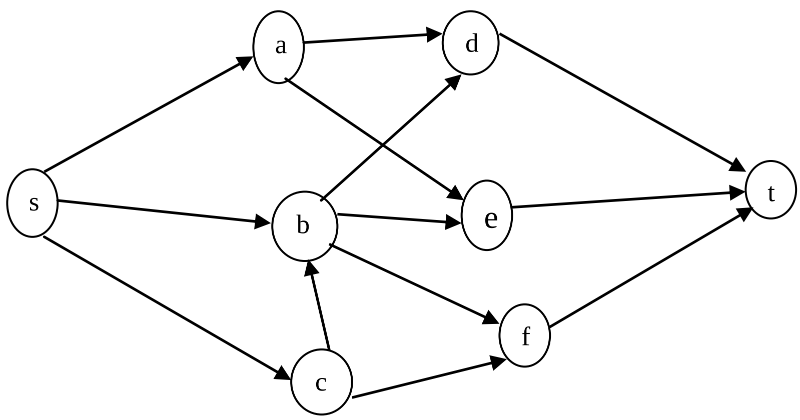
This is not possible. Therefore, the set U is a minimum cut set of the flow network N.

Hence the theorem is proved.

Remark 4.1: The Theorem 4.2.and 4.3. jointly is an alternative theorem for the max-flow min-cut theorem [2, 14].

Now, we illustrate the solution procedure of the path reduction method with help of the following numerical example.

Example 4.1: Consider the following maximum flow problem:



Edge	Capacity		Edge	Capacity		Edge	Capacity
(s,a)	2		(b,d)	10		(d,t)	4
(s,b)	10		(b,e)	3		(e,t)	9
(s,c)	8		(b,f)	2		(f,t)	10
(a,d)	6		(c,b)	5			
(a,e)	6		(c,f)	3			

Now, the max-flow matrix of the given problem is given as follow

$$\begin{array}{c} \\ s \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & t \end{array} \\ \begin{pmatrix} 2 & 10 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 10 & 3 & 2 & 0 \\ 0 & 5 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix} \end{array}$$

Now, the maximum amount to be send at starting node = sum of the first row =20.

The maximum amount to be received at the sink node = sum of the last column =23.

Iteration -1

Now, using the Step 2. to the Step 11. of the proposed method , we obtain the (s-t) paths, $s-b-d-t$, $s-b-e-t$ and $s-b-f-t$ with maximum flows 4, 3 and 2 respectively.

Now, applying the Step 10. , we obtain the following reduced max-flow matrix .

$$\begin{array}{c} \\ s \\ a \\ c \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccccc} a & c & e & f & t \end{array} \\ \begin{pmatrix} 2 & 8 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \end{array}$$

Iteration-2:

Now, using the Step 2. to the Step 11. of the proposed method , we obtain the (s-t) path, $s-c-f-t$ with maximum flow 3.

Now, applying the Step 10. , we obtain the following reduced max-flow matrix .

$$\begin{array}{c} \\ s \\ a \\ e \end{array} \begin{array}{c} \begin{array}{ccc} a & e & t \end{array} \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} \end{array}$$

Iteration -3:

Now, using the Step 2. to the Step 11. of the proposed method , we obtain the (s-t) path, $s-a-e-t$ with maximum flow 2 .

Now, applying the Step 10. , we obtain the following reduced max-flow matrix .

$$\begin{array}{c} \\ s \\ a \\ e \end{array} \begin{array}{c} \begin{array}{ccc} a & e & t \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix} \end{array}$$

Now, since the flow values of the sth row of the above max-flow matrix are zero and by the Step 12. of the proposed method, we stop the computation.

Now, the (s-t) paths in the flow network with their maximum flow value are given below

(s-t) paths	Maximum flow value
$s-b-d-t$	4
$s-b-e-t$	3
$s-b-f-t$	2
$s-c-f-t$	3
$s-a-e-t$	2
Maximum total flow	14

Therefore, the maximum flow value in the given network is 14 by the Step 13. of the path reduction method and by the Theorem 4.3., the minimum (s,t)-cut set is $\{(d, t), (b, e), (b, f), (c, f), (s, a)\}$.

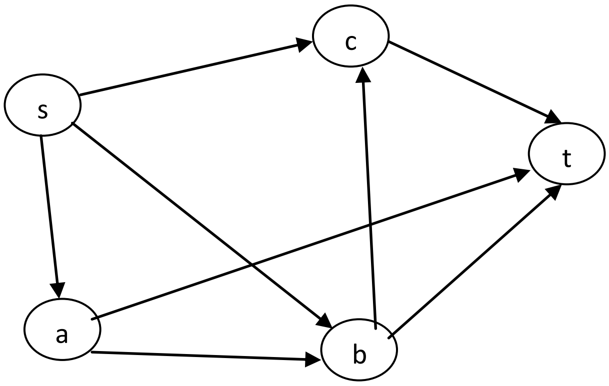
Remark 4.2: If we increase inflow capacity at the node '*a*' by 4, the maximum flow in the network is increased by 4, that is, the maximum flow in the network is 18.

5. INTERVAL FLOW NETWORK PROBLEMS

Consider a maximum flow network problem with interval capacities and interval flow values called, an interval maximum flow network problem. A crisp maximum flow network problem from given interval flow network problem is constructed as the interval network with the upper value of interval capacity of each of edge, called the upper level maximum flow network problem. Then, using the path reduction method, the maximum flow in the upper level maximum flow network problem and its related flow paths and also, thw minimum (s,t)-cut set are obtained. Based on the results of the upper level maximum flow network problem, the maximum flow and its related paths and also, minimum (s,t)-cut set for the given interval flow network problem are obtained.

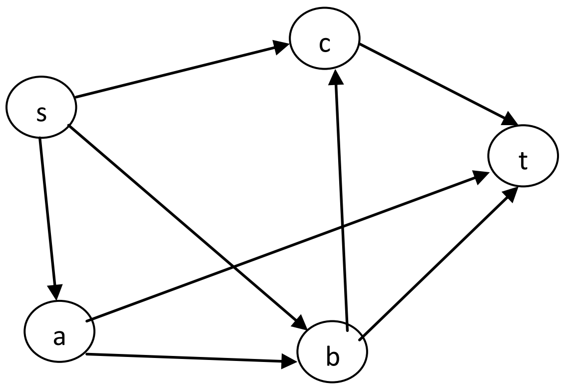
The solution procedure for solving the interval flow network problem is illustrated using the following example.

Example 5.1: Consider the following maximum flow network with interval numbers:



Edge	Capacity		Edge	Capacity
(s,a)	[10,20]		(a,t)	[25,30]
(s,b)	[15,30]		(b,c)	[5,10]
(s,c)	[5,10]		(b,t)	[10,20]
(a,b)	[30,40]		(c,t)	[10,20]

Now, the upper level maximum flow network problem is given below:



Edge	Capacity		Edge	Capacity
(s,a)	20		(a,t)	30
(s,b)	30		(b,c)	10
(s,c)	10		(b,t)	20
(a,b)	40		(c,t)	20

Now, by the path reduction method, the maximum flow and its related (s-t) paths for the upper level maximum flow network problem are given below:

(s-t)-paths	Maximum flow value
$s-b-t$	20
$s-b-c-t$	10
s-a-t	20
$s-c-t$	10
Maximum total flow	60

Also the minimum (s,t)-cut set = { (*s*,*a*), (*b*,*c*) ,(*s*,*c*), (*b*,*t*)} or { (s,a),(s,b),(s,c)}.

Now, using the (s-t)-paths for upper level maximum flow network problem, we obtain the lower and upper maximum flows for each of the paths using the interval maximum flow network problem.

(s-t) paths	Upper Maximum flow value	Lower Maximum flow value
$s-b-t$	20	10
$s-b-c-t$	10	5
s-*a*-*t*	20	10
$s-c-t$	10	5
Maximum total flow	60	30

Therefore, the maximum flow in the given interval maximum flow network problem with its related paths are given below:

(s-t) paths	Maximum flow value
$s-b-t$	[10,20]
$s-a-t$	[10,20]
$s-c-t$	[5,10]
$s-b-c-t$	[5,10]
Maximum total flow	[30,60]

Therefore, the maximum flow value in the given interval maximum flow network problem is [30,60] and the minimum cut set = {(*b*,*t*) ,(*s*,*a*), (*b*,*c*) (*s*,*c*)} or { (s,a), (s,b), (s,c)}.

Remark 5.1: In the computation of maximum flow for interval maximum flow problems, if you consider two completely overlapping intervals , say A and B and the upper value of A is selected, select the lower value of A instead to select the lower value of B.

6. FUZZY FLOW NETWORK PROBLEMS

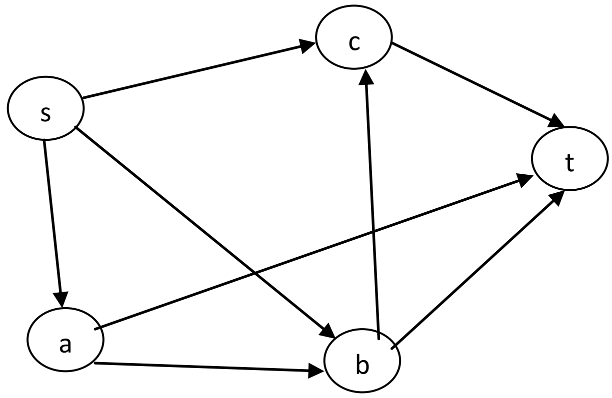
A triangular fuzzy number (a,b,c) can be represented as an interval number form as follows.

$$(a,b,c) = [a+(b-c)\lambda, c-(c-b)\lambda],\ 0 \le \lambda \le 1. \quad (1)$$

When the given maximum flow network problem is a fuzzy maximum flow network problem, each edge capacity is a triangular fuzzy number. Using the relation (1), the fuzzy maximum flow network problem is converted into an interval maximum flow network problem. Then, the maximum flow and its related flow paths and also, the minimum (s,t)-cut set are obtained from the results of the interval maximum flow network problem which is solved as per the Section 4..

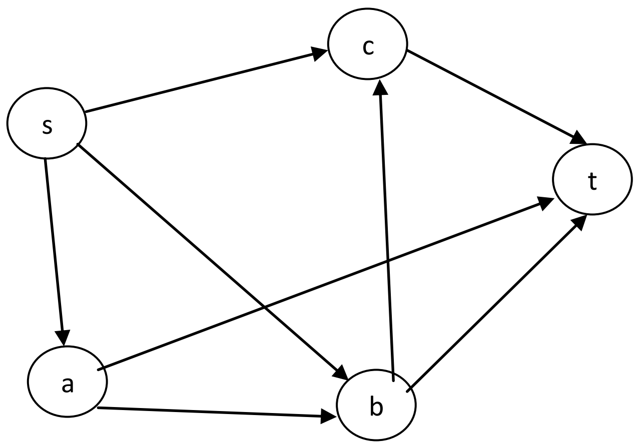
The solution procedure for obtaining the maximum flow in a fuzzy maximum flow network problem is demonstrated with help of the following numerical example.

Example 6.1: Consider the following fuzzy maximum flow network problem in which edge capacities are fuzzy triangular numbers:



Edge	Capacity
(s,a)	(10,20,30)
(s,b)	(15,30,45)
(s,c)	(5,10,15)
(a,b)	(30,40,50)
(a,t)	(25,30,35)
(b,c)	(5,10,15)
(b,t)	(10,20,30)
(c,t)	(10,20,30)

Now, the interval maximum flow network problem corresponding to the given problem is given below:



Edge	Capacity
(s,a)	$[10+10\lambda, 30-10\lambda]$
(s,b)	$[15+15\lambda, 45-15\lambda]$
(s,c)	$[5+5\lambda, 15-5\lambda]$
(a,b)	$[30+10\lambda, 50-10\lambda]$
(a,t)	$[25+5\lambda, 35-5\lambda]$
(b,c)	$[5+5\lambda, 15-5\lambda]$
(b,t)	$[10+10\lambda, 30-10\lambda]$
(c,t)	$[10+10\lambda, 30-10\lambda]$

Now, the maximum flow and its related paths and also, the minimum (s,t)-cut set for the above interval maximum flow network problem are given below:

(s-t) paths	Maximum flow value
$s-b-t$	$[10+10\lambda, 30-10\lambda]$
$s-a-t$	$[10+10\lambda, 30-10\lambda]$
$s-c-t$	$[5+5\lambda, 15-5\lambda]$
$s-b-c-t$	$[5+5\lambda, 15-5\lambda]$
Maximum total flow	$[30+30\lambda, 90-30\lambda]$

Therefore, the maximum flow value in the interval flow network is $[30+30\lambda, 90-30\lambda]$ and the minimum cut set = {(*b*,*t*) ,(*s*,*a*), (*b*,*c*) (*s*,*c*)}.

This implies that the maximum flow and its related paths and also, the minimum (s,t)-cut set for the given fuzzy maximum flow network problem are given below:

(s-t) paths	Maximum flow value
$s-b-t$	(10,20,30)
$s-a-t$	(10,20,30)
$s-c-t$	(5,10,15)
$s-b-c-t$	(5,10,15)
Maximum total flow	(30,60,90)

Therefore, the maximum flow value in the given fuzzy maximum flow network problem the minimum cut set = {(*b*,*t*) ,(*s*,*a*), (*b*,*c*) (*s*,*c*)}.

7. CONCLUSION

In this paper, we consider a maximum flow problem which is a classical combinatorial optimization. Maximum flow problem has many applications abound in different fields. A new concept namely, a max-flow matrix of a flow network and a new method namely, path reduction method for solving maximum flow problems are presented. Two theorems are derived mathematically for showing the maximality of the flow value in the network by the proposed method. For finding minimum cut set, one more theorem is given. The proposed method provides the maximum flow in the flow network and also, its related flow paths. The proposed approach differs from all the existing methods because of not using augmenting paths and preflows. Numerical example is given for illustrating the solution procedure of the proposed method. Further, the path reduction method for solving maximum flow problems is extended to interval and fuzzy maximum flow problems.

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