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Application of Optimal Control Theory to Solve Three Stages Supply Chain Model

Hegazy Zaher[1], **Taher Taha Zaki**[2]

Institute of statistical Studies and Research
Operations Research Department
Cairo University, Egypt.

Abstract

This paper presents an optimal control Theory to manage three stages dynamic supply chain model with deteriorating items. The model is continuous time linear optimal control system to optimize both manufacture rate and dynamic price to maximize total supply chain revenue minus total cost. Assuming the demand is linear function of price. Supply chain cost consists of the vendor holding cost, the buyer holding cost and the vendor manufacture cost. Pontryagin Maximum Principle is applied to find the model optimal solution.

Keywords: Supply chain, linear optimal control, Pontryagin Maximum Principle.

1. Introduction

Many applications in operations research, management science, economics, and industrial engineering and finance involve the control of dynamic systems. i.e., systems those change with time. They named continuous time systems or discrete time systems. [13] described a model to adjust the inventory production system with deteriorating items in which the model is solved by two methods. The first method is solved using the control theory approach. The second method is solved using dynamic Programming technique. [11] Presented a production inventory model for both ameliorating and deteriorating items. The problem is an optimal control problem with inequality constraints, in which the model is derived and is solved using Pontryagin's maximum principle. [5] described a model to analyze and to achieve planned economic performance in a actual time vague and commotion. The execution environment is vital and modern issue in many supply chains. This study is considered as among the first papers addressing the operative perspective of the supply chain dynamic domain [2] studied a inventory production system with deteriorating items in which the inventory model is considered as linear optimal control problem and the model is solved analytically using optimal control theory. [4] Dealt with long-term demand-driven capacity planning in the reverse way of closed loop supply chains with remanufacturing. The model is solved using simulation-based system approach to find the total supply chain profit. [12] Presented a new scheme where the new policy iteration technique is used to solve the continuous time LQR problem without knowing about the system detail dynamics. The optimal control solution of linear time system is solved using algebraic Riccati equation.

[3] Presented optimal control of production inventory system considering deteriorating items. The main objective of this study is to determine minimum total production and inventory costs. [10] Focused on inventory system in supply chain using cooperative model to create predictive control to optimize overall supply chain objective. He developed a model predictive control algorithm which is suitable for stability system, and is applied in supply chain models. In this paper the optimal control model having general inequality constraints is considered and is solved using Pontryagin's maximum principle.

2. Assumptions and Notations

The model consists of a vendor who produces a homogeneous product keeping safety stock in his inventory and distributes it to a buyer who sells the product to end customer. The objective is to maximize total revenue minus supply chain total cost by optimize the dynamic price and manufacture rate. Assuming the price is linear decreasing function with the sales.

Inputs

T : Time horizon

$u_{\max}$: Maximum Vendor production.

h_1: Buyer holding cost.

h_2: Vendor holding cost.

θ : Deterioration rate of the product

c : The production cost coefficient

a, b : Coefficients used for the product at time t in linear relationship between price and sales

$s(t) = a - b\ p(t)$

$\hat{u}_{pr}$: Production goal level of the vendor

$\hat{x}_1$: Inventory goal level of the vendor (safety stock)

$\hat{x}_2$: Inventory goal level of the buyer (safety stock)

$P(t)$: Price of one unit of the product at time t (control variable)

$u_{pr}(t)$: Production rate at time t (control variable.)

x_1: Inventory level (number of units) of buyer at time t (state variable).

x_2: Inventory level (umber of units) of vendor at time t (state variable).

The problem seeks to maximize the revenue minus the inventory and productions costs. The objective function of the model can be written as:

Maximize

$$\int_0^T \left[p(t)s(t) - \frac{1}{2} h_1 \left(x_1(t) - \hat{x}_{1b}\right)^2 - \frac{1}{2} h_2 \left(x_2(t) - \hat{x}_2\right)^2 - \frac{c}{2} \left(u_{pr}(t) - \hat{u}_{pr}\right)^2 \right] dt \tag{1}$$

Subject to

$$\dot{x}_1 = -s(t) - \theta x_1(t) \tag{2}$$

$$\dot{x}_2 = u_{pr}(t) - s(t) - \theta x_2(t) \tag{3}$$

$$s(t) = a - b\, P(t) \qquad \forall \; t \in [0,T] \tag{4}$$

$$u_{pr}(t) \le u_{\max} \qquad \forall \; t \in [0,T] \tag{5}$$

$$p(t) \le \frac{a(t)}{b(t)} \tag{6}$$

$$u_{pr}(t),\; x_1(t),\; x_2(t),\; P(t) \ge 0 \tag{7}$$

With initial condition

$x_1(0) = x_{1o}$, $x_2(0) = x_{2o}$.

The first term in objective function represents income revenue, the second term represents holding cost of vendor, the thirds term represents holding cost of buyer and last term represents manufacture cost.

3. Mathematical Analysis

In this section, a maximum principle for problems involving mixed inequality constraints and the pure state variable inequality constraints is applied [1],[7],[9].

$$h(x,t) \ge 0 \tag{8}$$

Where the function $h : E^n x E^1 \to E^p$ is assumed to be continuously differentiable. Function h represents a set of p constraints $h_i(x,t) \ge 0, i = 1,2,3,...., p$.

The Lagrangian function can be stated as

$$L(x,u,\lambda,\mu,\eta) = H(x,u,\lambda,t) + \mu g(x,u,t) + \eta h^1(x,u,t) \tag{9}$$

Where Hamiltonian $H = F(x,u,t) + \lambda f(x,u,t)$ and λ is costate variable

μ satisfies the complementary slackness conditions

$\mu \geq 0$, $\mu g(x,u,t)=0$ and $\eta \in E^p$ Satisfies $\eta \geq 0, \eta h(x,t)=0$, $\dot{\eta} \leq 0$

Form the Lagrangian function as

$$L = p(t)s(t) - \frac{1}{2}h_1(x_1-\hat{x}_1)^2 - \frac{1}{2}h_2(x_2-\hat{x}_2)^2 - \frac{c}{2}(u_{pr}-\hat{u}_{pr})^2 - \lambda_1(s(t)+\theta_1 x_1)$$

$$+\lambda_2(u_{pr}-s(t)-\theta x_2) + \mu_1(t)x_1 + \mu_2(t)x_2 + \mu_3(t)(\frac{a}{b}-p(t)) + \mu_4(t)(u_{\max}-u_{pr})$$

$$+\eta_1(t)x_1 + \eta_2(t)x_2 \tag{10}$$

Where μ_1(t) , μ_2(t), μ_3(t) , μ_4(t), η_1(t), η_2(t)satisfy the complementary slackness conditions

$$\mu_1(\mathrm{t}) \geq 0 \ , \ \mu_1 u_{pr}(t) = 0 \tag{11}$$

$$\mu_2(\mathrm{t}) \geq 0 \ , \ \mu_2 p(t) = 0 \tag{12}$$

$$\mu_3(\mathrm{t}) \geq 0 \ , \ \frac{a(t)}{b(t)} - p(t) = 0 \tag{13}$$

$$\mu_4(t) \geq 0 \ , \ u_{\max} - u_{pr} = 0 \tag{14}$$

$$\eta_1 \geq 0 \ , \ \eta_1 x_1(t) = 0 \tag{15}$$

$$\eta_2 \geq 0 \ , \ \eta_2 x_2(t) = 0 \tag{16}$$

Differentiate lagrangian equation with respect to u_{pr}, p to obtain optimal production rate and optimal price respectively.

$$u_{pr}(t) = \hat{u}_{pr} + \frac{\lambda_2(t)}{c} \tag{17}$$

$$p(t) = \frac{a}{2b} + \frac{\lambda_1(t)}{2} + \frac{\lambda_2(t)}{2} \tag{18}$$

From equations (17), (18) and nonnegative constraints it is clear that

$\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$, $\mu_4(t)$, $\eta_1(t), \eta_2(t) = 0$

Substitute equations (17),(18) in equation (1),(2) to obtain state equations.

$$\dot{x}_1 = \frac{-a}{2} + \frac{\lambda_1(t)b}{2} + \frac{\lambda_2(t)b}{2} - \theta x_1(t) \tag{19}$$

$$\dot{x}_2 = \hat{u}_{pr} - \frac{a}{2} + \frac{\lambda_2(t)b}{2} + \frac{\lambda_2(t)}{c} + \frac{\lambda_1(t)b}{2} - \theta x_2 \tag{20}$$

To get costate equations $-\frac{\partial L}{\partial x_1}, -\frac{\partial L}{\partial x_2}$

$$\dot{\lambda}_1 = h_1(x_1(t)-\hat{x}_1) + \lambda_1\theta \tag{21}$$

$$\dot{\lambda}_2 = h_2(x_2(t)-\hat{x}_2) + \lambda_2\theta \tag{22}$$

The equations (19),(20),(21),(22) can be put in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -\theta & 0 & \frac{b}{2} & \frac{b}{2} \\ 0 & -\theta & \frac{b}{2} & \frac{b}{2}+\frac{1}{c} \\ h_1 & 0 & \theta & 0 \\ 0 & h_2 & 0 & \theta \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-a}{2} \\ \hat{u}_{pr}-\frac{a}{2} \\ -\hat{x}_1 h_1 \\ -\hat{x}_2 h_2 \end{bmatrix} \tag{23}$$

With initial and final conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

The equation (23) is in the form of

$\dot{x} = Ax + b$

Obtain the eigen values $\alpha_1, \alpha_2, \alpha_{3,} \alpha_4$ of matrix A then obtain the eigen vectors from

$$[A - \alpha_i I] y_i = 0 \tag{25}$$

The complete solution of equation (23) consists of homogenous solution x_h+ particular solution x_c using variation parameters technique.

$$x_h = \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} \\ y_{12} & y_{22} & y_{32} & y_{42} \\ y_{13} & y_{23} & y_{33} & y_{43} \\ y_{14} & y_{24} & y_{34} & y_{44} \end{bmatrix} \begin{bmatrix} c_1 e^{\alpha_1 t} \\ c_2 e^{\alpha_2 t} \\ c_3 e^{\alpha_3 t} \\ c_4 e^{\alpha_4 t} \end{bmatrix} \tag{26}$$

$$x_c = \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} \\ y_{12} & y_{22} & y_{32} & y_{42} \\ y_{13} & y_{23} & y_{33} & y_{43} \\ y_{14} & y_{24} & y_{34} & y_{44} \end{bmatrix} \begin{bmatrix} L_1 e^{\alpha_1 t} \\ L_2 e^{\alpha_2 t} \\ L_3 e^{\alpha_3 t} \\ L_4 e^{\alpha_4 t} \end{bmatrix} \tag{27}$$

Let z is the inverse of the matrix y

$$\begin{bmatrix} \dot{L}_1 e^{\alpha_1 t} \\ \dot{L}_2 e^{\alpha_2 t} \\ \dot{L}_3 e^{\alpha_3 t} \\ \dot{L}_4 e^{\alpha_4 t} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{12} & z_{22} & z_{32} & z_{42} \\ z_{13} & z_{23} & z_{33} & z_{43} \\ z_{14} & z_{24} & z_{34} & z_{44} \end{bmatrix} \begin{bmatrix} \frac{-a}{2} \\ \hat{u}_{pr}-\frac{a}{2} \\ -\hat{x}_1 h_1 \\ -x_2 h_2 \end{bmatrix} \tag{28}$$

From equation (28) by integration $\dot{L}_i$ to get L_i for all $i = 1,2,3,4$

$$L_i(t) = \int_o^t e^{-\alpha t_i} \left(z_{i1} \left[-\frac{a}{2} \right] + z_{i2} \left[\hat{u}_{pr} - \frac{a}{2} \right] - z_{i3} h_1 \hat{x}_1 - z_{i4} h_2 \hat{x}_{i4} \right) dt \tag{29}$$

The complete solution is given by:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{21} & y_{31} & y_{41} \\ y_{12} & y_{22} & y32 & y_{42} \\ y_{13} & y_{23} & y_{33} & y_{43} \\ y_{14} & y_{24} & y_{34} & y_{44} \end{bmatrix} \left[\begin{bmatrix} c_1 e^{\alpha_1 t} \\ c_2 e^{\alpha_2} t \\ c_3 e^{\alpha_3 t} \\ c_4 e^{\alpha_4 t} \end{bmatrix} + \begin{bmatrix} L_1 e^{\alpha_1 t} \\ L_2 e^{\alpha_2} t \\ L_3 e^{\alpha_3 t} \\ L_4 e^{\alpha_4 t} \end{bmatrix} \right] \tag{30}$$

4. Numerical Example

In this section numerical example is studied.

Assume $h_{1}=\$1$, h_2, $=\$2$, $c=\$3$ *per unit product* $\hat{u}_{pr}$=25, $\hat{x}_1$=40, $\hat{x}_2$=25, θ=0.02, T=5.

With initial condition $x_1(0)$=30, $x_2(0)$ =10,a=15 b=0.5.

Fig (1) is the graph of optimal production as function of time.

Fig (2) is the graph of optimal price as function of time.

Fig (3) is the graph of inventory level for both the vendor and the buyer.

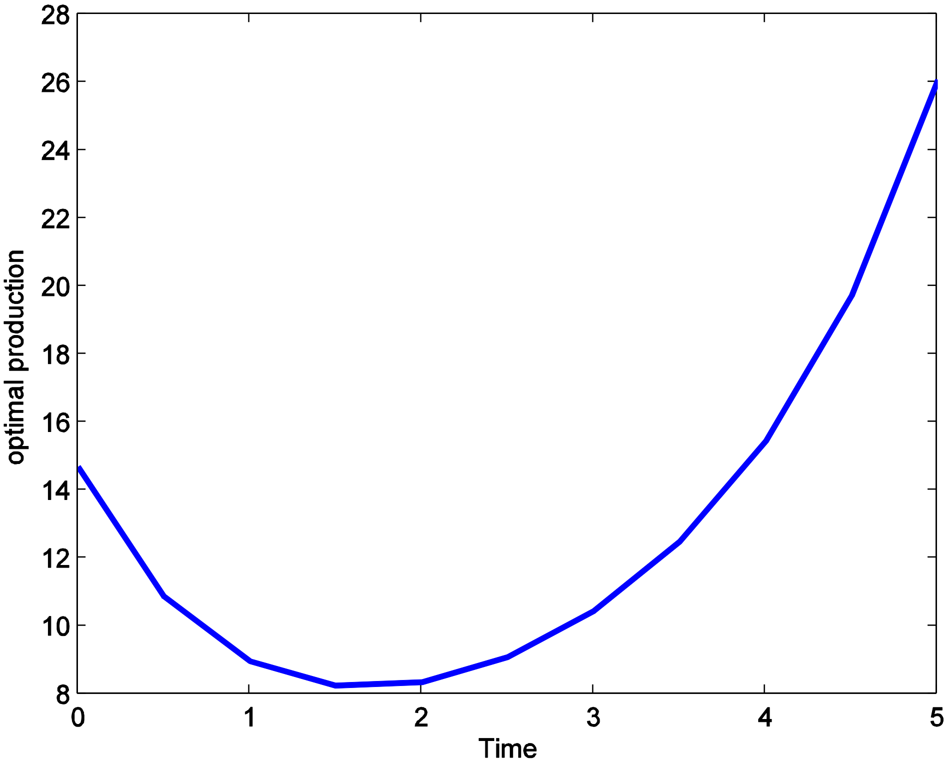

Fig (1) Optimal Production

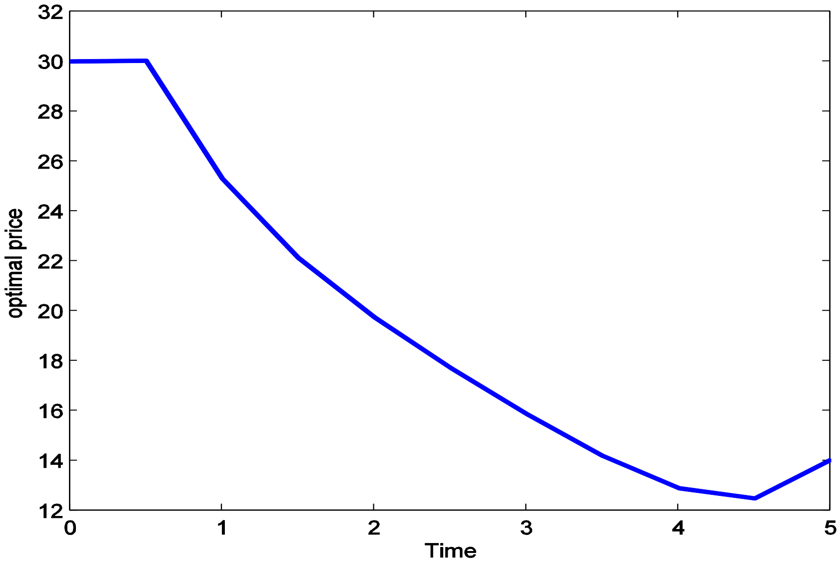

Fig (2) Optimal Price

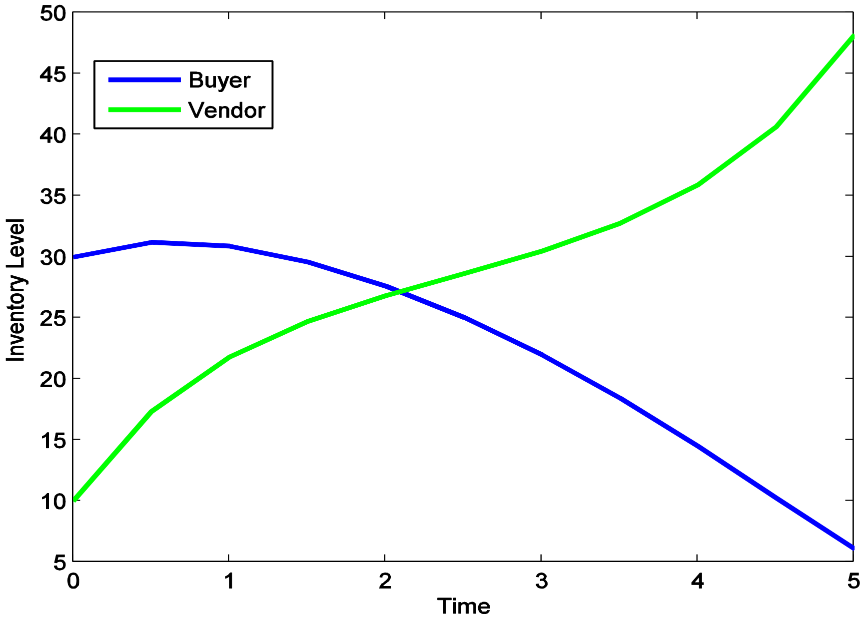

Fig (3) Inventory Level of the buyer and the vendor

5. Conclusion

This paper descried the solution of dynamic three stages supply chain (customers – distributors – manufacturers). The optimal solution is reached by adjusting dynamic price and production rate to maximize total revenue minus total holding cost. The model can be extended to include transportation cost, order cost and shortage cost.

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