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New subclass of bi-univalent functions of complex order involving $q-$ hypergeometric functions with fixed point.

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In the present paper, a new subclass of bi-starlike function of complex order associated with $q-$ hypergeometric functions are introduced and coefficient estimates for functions in this new class are obtained. Several new(or known) consequences of the results are also pointed out.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disc $\Delta = \{z: |z| < 1\}$.Further,let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in Δ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)(0 \le \alpha < 1)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)(0 \le \alpha < 1)$ of convex functions of order α in Δ.

The Convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.2}$$

where f is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$. In terms of the Hadamard product (or Convolution).

An analytic function f is subordinate to an analytic function h, written $f(z) \prec h(z)$, provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = h(w(z))$.

Let $\xi(|\xi| = d)$ be a fixed point in the unit disc Δ, and denote by $\mathcal{H}(\Delta)$ the class of functions which are regular and

$\mathcal{A}(\xi) = \{f \in H(\Delta): f(\xi) = f'(\xi) - 1 = 0\}$

consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \tag{1.3}$$

which are analytic in Δ. Also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi): f \text{ is univalent in } \Delta\}$, the subclass of $\mathcal{A}(\xi)$. Note that $\mathcal{S}_0 = \mathcal{S}$ be the subclasses of $\mathcal{A} = \mathcal{A}(0)$ consisting of univalent functions in Δ. By $\mathcal{S}_\xi^*(\beta)$ and $\mathcal{K}_\xi(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\Re\left\{\frac{(z-\xi)f'(z)}{f(z)}\right\} > \beta, \quad \Re\left\{1 + \frac{(z-\xi)f''(z)}{f'(z)}\right\} > \beta \qquad and \qquad (z - \xi) \in \Delta$$

for $0 \leq \beta < 1$ introduced and studied by Kanas and Ronning [8].The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi^*(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [6] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$ respectively, are the well-known standard classes of convex and starlike functions.

Motivated by Ma and Minda [11], we consider an analytic function $\phi_\xi(z)$ with positive real part in the unit disk $\Delta, \phi(\xi) = 1, \phi'(\xi) > 0$, and ϕ_ξ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{(z-\xi)\ f\prime(z)}{f(z)} \prec \phi_\xi(z)$.

Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $\left(1 + \frac{(z-\xi)^2\ f\prime(z)}{f\prime(z)}\right) \prec \phi_\xi(z)$, where ξ is an arbitrary fixed point in Δ.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}_\Sigma^*(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi_\xi(z) = 1 + B_1(z - \xi) + B_2(z - \xi)^2 + B_3(z - \xi)^3 + \cdots, \quad (B_1 > 0). \tag{1.4}$$

More recently, Purohit and Raina [16] introduced a generalized q −Taylor's formula in fractional q −calculus and derived certain q −generating functions for q − hypergeometric functions.

For complex parameters $a_1, \dots, a_l$ and $b_1, \dots, b_m$ $(b_j \neq 0, -1, \dots; j = 1,2, \dots, m)$ the *q- hypergeometric function* ${}_l\Psi_m(z)$ is defined by

$$ {}_l\Psi_m(a_1,\dots a_l;b_1,\dots,b_m;q,z):=\sum_{n=0}^{\infty}\frac{(a_1,q)_n\dots(a_l,q)_n}{(b_1,q)_n\dots(b_m,q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n $$

with $\binom{n}{2}=\frac{n(n-1)}{2}$ where $q\neq 0$ when $l>m+1(l,m\in\mathbb{N}_0=\mathbb{N}\cup\{0\};z\in\Delta$.

The $q-$ shifted factorial is defined for $a,q\in\mathbb{C}$ as a product of n factors by

$$ (a;q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & n\in\mathbb{N} \end{cases} $$

and in terms of basic analogue of the gamma function

$$ (q^a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, n>0. \tag{1.5} $$

It is interest to note that $\lim_{q\to 1^-}\frac{(q^a;q)_n}{(1-q)^n}=(a)_n=a(a+1)\dots(a+n-1)$ the familiar Pochhammer symbol.

Now for $z\in\Delta, 0<|q|<1$, and $l=m+1$, we get the basic hypergeometric function given below (see [2,3])

$$ {}_l\psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z)=\sum_{n=0}^{\infty}\frac{(a_1,q)_n\dots(a_l,q)_n}{(q,q)_n(b_1,q)_n\dots(b_m,q)_n}z^n $$

which converges absolutely in the open unit disk Δ. Let

$$ \mathcal{I}(a_l,b_m;q;z)=z\,{}_l\psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z)=\sum_{n=0}^{\infty}\Upsilon_n^{l,m}[a_1,q]z^{n+1} \tag{1.6} $$

where for convenience,

$$ \Upsilon_n^{l,m}[a_1,q]=\frac{(a_1;q)_n\dots(a_l;q)_n}{(q;q)_n(b_1;q)_n\dots(b_m,q)_n}. \tag{1.7} $$

The operator $\mathcal{I}(a_l,b_m;q)f(z)$ was studied recently by Mohammed and Darus [13].For $a_i=q^{\alpha_i}, b_j=q^{\beta_j},\alpha_i,\beta_j\in\mathbb{C}$, and $\beta_j\neq 0,-1,-2,\dots,(i=1,\dots,l,j=1,\dots,m)$ and $q\to 1$, (for $l=m+1$)we obtain the well-known Dziok-Srivastava linear operator [3,2] and special cases cited therein.

For $(z-\xi)\in\Delta$, $|q|<1$, and $l=m+1$ by convolution (Hadamard product) we define

$$ \mathcal{I}(a_l,b_m;q)f(z)=(z-\xi)\quad {}_l\psi_m(a_1;\dots a_l;b_1,\dots,b_m;q,z-\xi). $$
$$ =\sum_{n=0}^{\infty}\Upsilon_n^{l,m}[a_1,q](z-\xi)^{n+1} \tag{1.8} $$

Shortly, we let

$$ \mathcal{I}_m^l f(z)=(z-\xi)+\sum_{n=2}^{\infty}\Upsilon_n^{l,m}[a_1,q]a_n(z-\xi)^n \tag{1.9} $$

where

$$ \varphi_n=\Upsilon_m^l(n)=\Upsilon_n^{l,m}[a_1,q]=\frac{(a_1;q)_{n-1}\dots(a_l;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\dots(b_m,q)_{n-1}} \tag{1.10} $$

unless otherwise stated.

It is well known that every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by

$f^{-1}(f(z))=z$, $(z\in\Delta)$ and $f(f^{-1}(w))=w$, $(|w|<r_0(f);r_0(f)\geq 1/4)$

where

$$ f^{-1}(w)=g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.11} $$

A function $f(z)\in\mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ.Let Σ denote the class of bi-univalent functions in Δ given by (1.1). Earlier, Brannan and Taha [1] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}_{\Sigma}^{*}(\alpha)$ and bi-convex function of order α denoted by $\mathcal{K}_{\Sigma}(\alpha)$ corresponding to the function classes $\mathcal{S}^{*}(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. For each of the function classes $\mathcal{S}_{\Sigma}^{*}(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [1,19]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|,(n \in \mathbb{N}\backslash\{1,2\};\quad \mathbb{N}:=\{1,2,3,\cdots\})$is still an open problem (see[1, 19, 9, 12]). Recently many researchers (see [5, 7, 10, 15, 17, 18, 20, 21]) have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates $|a_2|$ and $|a_3|$.

Let a function $f \in \mathcal{A}(\xi)$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z-\xi$, $(z \in \Delta)$ and $f(f^{-1}(w)) = (w-\xi)$ $(|(w-\xi)| < r_0(f); r_0(f) \geq 1/4)$where

$$f^{-1}(w) = g(w) = (w-\xi) - a_2(w-\xi)^2 + (2a_2^2 - a_3)(w-\xi)^3 - (5a_2^3 - 5a_2a_3 + a_4)(w-\xi)^4 + \cdots. \quad (1.12)$$

A function $f(z) \in \mathcal{A}(\xi)$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ. Let Σ_{ξ} denote the class of $\xi-$ bi-univalent functions in Δ given by (1.3). Using the techniques of Deniz [4], in the present paper we define a new subclass of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, q-hypergeometric functions operator $\mathcal{J}_m^l(=\mathcal{J}(a_l, b_m; q; z))$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions $f \in \Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$. Several related classes are also considered, and connection to earlier known results are stated.

Definition 1.1 *For* $0 \leq \lambda \leq 1; 0 \leq \mu \leq 1$, *a function* $f(z) \in \Sigma$ *given by (1.1) is said to be in the class* $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$ *if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l f(z))'\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu-1} - 1\right] \prec \phi_{\xi}(z) \quad (1.13)$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l f(w))'\left(\frac{\mathcal{J}_m^l f(w)}{w-\xi}\right)^{\mu-1} - 1\right] \prec \phi_{\xi}(w) \quad (1.14)$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z, w \in \Delta$, $\xi(|\xi| = d)$ and the function g is given by (1.12).
For, suitable choices of λ and μ we can state various subclasses of Σ as illustrated below:

Example 1.1 *For* $\lambda = 1; 0 \leq \mu < 1$,*a function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $\Sigma_{\phi,q}^{l,m}(\mu, 1, \gamma,) \equiv \mathcal{B}_{\Sigma,q}^{l,m}(\mu,\gamma,\phi)$, *if the following conditions are satisfied :*

$$1 + \frac{1}{\gamma}\left[(\mathcal{J}_m^l f(z))'\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu-1} - 1\right] \prec \phi_{\xi}(z)$$

and

$$1 + \frac{1}{\gamma}\left[(\mathcal{J}_m^l g(w))'\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu-1} - 1\right] \prec \phi_{\xi}(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z, w \in \Delta, \xi(|\xi| = d)$ and the function g is given by (1.12).
Example 1.2 *For* $\lambda = 1; \mu = 0$, *a function* $f(z) \in \Sigma$ *given by (1.1)is said to be in the*

class $\Sigma_{\phi,q}^{l,m}(0,1,\gamma,) \equiv \mathcal{S}_{\Sigma,q}^{l,m}(\gamma,\phi)$ *if it satisfies the following conditions:*

$$1+\frac{1}{\gamma}\left[\left(\frac{(z-\xi)(\mathcal{J}_m^l f(z))'}{\mathcal{J}_m^l f(z)}\right)-1\right] \prec \phi_\xi(z)$$

and

$$1+\frac{1}{\gamma}\left[\left(\frac{(w-\xi)(\mathcal{J}_m^l g(w))'}{\mathcal{J}_m^l g(w)}\right)-1\right] \prec \phi_\xi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z,w \in \Delta$ $\xi(|\xi|=d)$and the function g is given by (1.12).

Example 1.3 *By taking* $\mu = 1$, *a function* $f(z) \in \Sigma$ *given by (1.1)is said to be in the class* $\Sigma_{\phi,q}^{l,m}(1,\lambda,\gamma) \equiv \mathcal{F}_{\Sigma,q}^{l,m}(\lambda,\gamma,\phi)$ *if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)+\lambda(\mathcal{J}_m^l f(z))'-1\right] \prec \phi_\xi(z)$$

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)+\lambda(\mathcal{J}_m^l g(w))'-1\right] \prec \phi_\xi(w)$$

where $0 \le \lambda \le 1, \gamma \in \mathbb{C}\backslash\{0\}$ $z,w \in \Delta$, $\xi(|\xi|=d)$and the function g is given by (1.12).

Example 1.4 *Taking* $\mu = \lambda = 1$, *a function* $f(z) \in \Sigma$ *given by (1.1)is said to be in the class* $\Sigma_{\phi,q}^{l,m}(1,1,\gamma) \equiv \mathcal{H}_{\Sigma,q}^{l,m}(\gamma,\phi)$ *if it satisfies the following conditions:*

$$1+\frac{1}{\gamma}[(\mathcal{J}_m^l f(z))'-1] \prec \phi_\xi(z)$$

$$1+\frac{1}{\gamma}[(\mathcal{J}_m^l g(w))'-1] \prec \phi_\xi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $z,w \in \Delta$, $\xi(|\xi|=d)$ and the function g is given by (1.12).

It is of interest to note that by taking $l = 2$,$m = 1$ and allowing $q \to 1^-$ we get $\Sigma_{\phi,q}^{l,m} \equiv \Sigma(\phi)$.

Example 1.5 *For* $\mu+1=\lambda=1$, *a function* $f(z) \in \Sigma$ *given by (1.1)is said to be in the class* $\mathcal{S}_\Sigma^\xi(\gamma,\phi)$ *if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}\left(\frac{(z-\xi)f'(z)}{f(z)}-1\right) \prec \phi_\xi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{(w-\xi)g'(w)}{g(w)}-1\right) \prec \phi_\xi(w),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z,w \in \Delta$, $\xi(|\xi|=d)$and the function g is given by (1.12).

Example 1.6 *Taking* $\mu = \lambda = 1$, *a function* $f(z) \in \Sigma$ *given by (1.1)is said to be in the class* $\mathcal{H}_\Sigma^\xi(\gamma,\phi)$ *if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}(f'(z)-1) \prec \phi_\xi(z)$$

and

$$1+\frac{1}{\gamma}(g'(w)-1) \prec \phi_\xi(w),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z,w \in \Delta$, $\xi(|\xi|=d)$and the function g is given by (1.12).

2. Coefficient estimates for the function class $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$

In order to prove our main result, we recall the following lemma.

Lemma 2.1 *[14] If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in Δ for which $\Re(h(z)) > 0$ and*

$h_\xi(z) = 1 + c_1(z-\xi) + c_2(z-\xi)^2 + \cdots$ for $(z-\xi) \in \Delta$.

Define the functions $p_\xi(z)$ and $q_\xi(z)$ by

$$p_\xi(z) := \frac{1+u_\xi(z)}{1-u_\xi(z)} = 1 + p_1(z-\xi) + p_2(z-\xi)^2 + \cdots$$

and

$$q_\xi(z) := \frac{1+v_\xi(z)}{1-v_\xi(z)} = 1 + q_1(w-\xi) + q_2(w-\xi)^2 + \cdots.$$

It follows that,

$$u_\xi(z) := \frac{p_\xi(z)-1}{p_\xi(z)+1} = \frac{1}{2}\left[p_1(z-\xi) + \left(p_2 - \frac{p_1^2}{2}\right)(z-\xi)^2 + \cdots\right]$$

and

$$v_\xi(z) := \frac{q_\xi(z)-1}{q_\xi(z)+1} = \frac{1}{2}\left[q_1(w-\xi) + \left(q_2 - \frac{q_1^2}{2}\right)(w-\xi)^2 + \cdots\right].$$

Then $p_\xi(z)$ and $q_\xi(z)$ are analytic in Δ with $p_\xi(0) = 1 = q_\xi(0)$.

Since $u_\xi, v_\xi : \Delta \to \Delta$, the functions $p_\xi(z)$ and $q_\xi(z)$ have a positive real part in Δ, and $|p_{\xi,i}| \le 2$ and $|q_{\xi,i}| \le 2$ for each i.

Theorem 2.1 *Let $f(z)$ is given by (1.1) be in the class $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$. Then*

$$|a_2| \le \frac{|\gamma| B_1 \sqrt{2B_1}}{\sqrt{|\gamma| |[2(2\lambda+\mu)\varphi_3 + (\mu-1)(2\lambda+\mu)\varphi_2^2] B_1^2 + 2(\lambda+\mu)^2 (B_1 - B_2)\varphi_2^2|}} \tag{2.1}$$

and

$$|a_3| \le \frac{|\gamma|^2 B_1^2}{(\lambda+\mu)^2 \varphi_2^2} + \frac{|\gamma| B_1}{(2\lambda+\mu)\varphi_3} \tag{2.2}$$

where $\gamma \in \mathbb{C}\backslash\{0\}; 0 \le \lambda \le 1, 0 \le \mu \le 1$ and φ_n is given by (1.10).

Proof. It follows from (1.13) and (1.14) that

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l f(z))'\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu-1} - 1\right] = \phi_\xi(u_\xi(z)) \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l g(w))'\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu-1} - 1\right] = \phi_\xi(v_\xi(w)), \tag{2.4}$$

where $p_\xi(z)$ and $q_\xi(w)$ in $\mathcal{P}$ and have the following forms:

$$\phi_\xi(u_\xi(z)) = \phi_\xi\left(\frac{1}{2}\left[p_1(z-\xi) + \left(p_2 - \frac{p_1^2}{2}\right)(z-\xi)^2 + \cdots\right]\right) \tag{2.5}$$

and

$$\phi_\xi(v_\xi(w)) = \phi_\xi\left(\frac{1}{2}\left[q_1(w-\xi) + \left(q_2 - \frac{q_1^2}{2}\right)(w-\xi)^2 + \cdots\right]\right). \tag{2.6}$$

Now, equating the coefficients in (2.3) and (2.4), we get

$\frac{1}{\gamma}(\lambda+\mu)\varphi_2 a_2 = \frac{1}{2}B_1 p_{1,\xi}$ (2.7)

$\frac{1}{\gamma}\left[2\lambda+\mu)\varphi_3 a_3 + \frac{(\mu-1)(2\lambda+\mu)\varphi_2^2 a_2^2}{2}\right] = \frac{1}{2}B_1(p_{2,\xi} - \frac{p_{1,\xi}^2}{2}) + \frac{1}{4}B_2 p_{1,\xi}^2$ (2.8)

$-\frac{1}{\gamma}(\lambda+\mu)\varphi_2 a_2 = \frac{1}{2}B_1 q_{1,\xi}$ (2.9)

and

$\frac{1}{\gamma}\left[\left(\frac{4(2\lambda+\mu)\varphi_3+(2\lambda+\mu)(\mu-1)}{2}\varphi_2^2\right)a_2^2 - (2\lambda+\mu)\varphi_3 a_3\right] = \frac{1}{2}B_1(q_{2,\xi} - \frac{q_{1,\xi}^2}{2}) + \frac{1}{4}B_2 q_{1,\xi}^2.$
(2.10)

From (2.7) and (2.9), we get

$p_{1,\xi} = -q_{1,\xi}$ (2.11)

and

$8(\lambda+\mu)^2\varphi_2^2 a_2^2 = \gamma^2 B_1^2(p_{1,\xi}^2 + q_{1,\xi}^2).$ (2.12)

Now from (2.8), (2.10) and (2.12), we obtain

$(2\{\gamma[2(2\lambda+\mu)\varphi_3 + (\mu-1)(2\lambda+\mu)\varphi_2^2]B_1^2 + 2(\lambda+\mu)^2(B_1-B_2)\varphi_2^2\})a_2^2 = \gamma^2 B_1^3(p_{2,\xi} + q_{2,\xi}).$ (2.13)

Applying Lemma (2.1) for the coefficients $p_{2,\xi}$ and $q_{2,\xi}$, we have the desired inequality given in (2.1). Next, in order to find the bound on $|a_3|$, by subtracting (2.8) from (2.10) and using (2.11), we get

$\frac{2}{\gamma}(2\lambda+\mu)\varphi_3(a_3 - a_2^2) = \frac{B_1}{2}(p_{2,\xi} - q_{2,\xi}).$

Upon substituting the value of a_2^2 from (2.12), we get

$a_3 = \frac{\gamma^2 B_1^2(p_{1,\xi}^2+q_{1,\xi}^2)}{8(\lambda+\mu)^2\varphi_2^2} + \frac{\gamma B_1(p_{2,\xi}-q_{2,\xi})}{4(2\lambda+\mu)\varphi_3}.$

Applying Lemma (2.1) once again for the coefficients $p_{1,\xi}, p_{2,\xi}, q_{1,\xi}$ and $q_{2,\xi}$, we get

$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(\lambda+\mu)^2\varphi_2^2} + \frac{|\gamma| B_1}{(2\lambda+\mu)\varphi_3}.$

Corollary 2.1 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma,q}^{l,m}(\mu,\gamma,\phi)$, then*

$|a_2| \leq \frac{|\gamma| B_1\sqrt{2B_1}}{\sqrt{|\gamma(2+\mu)(2\varphi_3+(\mu-1)\Gamma_2^2)B_1^2+2(1+\mu)^2(B_1-B_2)\varphi_2^2|}}$

and

$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1+\mu)^2\varphi_2^2} + \frac{|\gamma| B_1}{(2+\mu)\varphi_3}$

where φ_n is given by (1.10).

Corollary 2.2 *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma,q}^{l,m}(\gamma,\phi)$, then*

$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|\gamma(2\varphi_3-2\varphi_2^2)B_1^2+(B_1-B_2)\varphi_2^2|}}$

and

$|a_3| \leq \frac{|\gamma|^2 B_1^2}{\varphi_2^2} + \frac{|\gamma| B_1}{2\varphi_3}$

where φ_n is given by (1.10).

Corollary 2.3 *Let $f(z)$ given by (1.1) be in the class $\mathcal{F}_{\Sigma}^{a,b;c}(\lambda,\gamma,\phi)$, then*

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|[|\gamma(2\lambda+1)\varphi_3)]B_1^2+(\lambda+1)^2(B_1-B_2)\varphi_2^2|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2B_1^2}{(\lambda+1)^2\varphi_2^2} + \frac{|\gamma|B_1}{(2\lambda)\varphi_3}$$

where φ_n is given by (1.10).

Corollary 2.4 *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\Sigma,q}^{l,m}(\gamma,\phi)$, then*

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma 3\varphi_3 B_1^2+4(B_1-B_2)\varphi_2^2|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2B_1^2}{4\varphi_2^2} + \frac{|\gamma|B_1}{3\varphi_3}.$$

where φ_n is given by (1.10).

Remark 2.1 *By taking $\gamma = (1-\alpha)cos\beta e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$; $0 \leq \alpha < 1$ we define a the class $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma) \equiv \Sigma_{\phi,q}^{l,m}(\alpha,\beta,\lambda,\mu)$ called the generalized class of β −bi-spirallike functions of order $\alpha(0 \leq \alpha < 1)$ if it satisfy the following conditions:*

$$e^{i\beta}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l f(z))'\left(\frac{\mathcal{J}_m^l f(z)}{z-\xi}\right)^{\mu-1}\right] \prec \phi_{\xi}(z)cos\beta + isin\beta$$

and

$$e^{i\beta}\left[(1-\lambda)\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu} + \lambda(\mathcal{J}_m^l g(w))'\left(\frac{\mathcal{J}_m^l g(w)}{w-\xi}\right)^{\mu-1}\right] \prec \phi_{\xi}(w)cos\beta + isin\beta.$$

where $0 \leq \lambda \leq 1; 0 \leq \mu \leq 1$, and $z, w \in \Delta$.

By choosing $\phi_{\xi}(z) = \left(\frac{1+(z-\xi)}{1-(z-\xi)}\right)$,for function $f \in \Sigma_{\phi,q}^{l,m}(\alpha,\beta,\lambda,\mu)$ given by (1.1) we can obtain the estimates $|a_2|$ and $|a_3|$ on lines similar to that of Theorem 2.1.

3. Corollaries and Concluding remarks

For the class of strongly starlike functions of order $\alpha(0 < \alpha \leq 1)$, let the function ϕ be given by

$$\phi_{\xi}(z) = \left(\frac{1+(z-\xi)}{1-(z-\xi)}\right)^{\alpha} = 1 + 2\alpha(z-\xi) + 2\alpha^2(z-\xi)^2 + \cdots \tag{3.1}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. From the corollaries 2.1 to 2.4 we state the following corollary.

Corollary 3.1 *Let $f(z)$ is given by (1.1) be in the class $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$. Then*

$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{|\gamma\alpha(2\lambda+\mu)\varphi_3+(\mu-1)(2\lambda+\mu)\varphi_2^2]+(\lambda+\mu)^2(1-\alpha)\varphi_2^2|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2\alpha^2}{(\lambda+\mu)^2\varphi_2^2} + \frac{2|\gamma|\alpha}{(2\lambda+\mu)\varphi_3}$$
where $\gamma \in \mathbb{C}\backslash\{0\}; 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1$ and φ_n is given by (1.10).

Corollary 3.2 *Let $f(z)$ given by (1.1), by choosing $\phi_\xi(z)$ of the form (3.1), and by Theorem 2.1, we state the following:*
(1) For functions $f \in \mathcal{B}_{\Sigma,q}^{l,m}(\mu,\gamma,\alpha)$, we have
$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{[(1-\alpha)(\mu+1)^2+2|\gamma|(2+\mu)(\mu-1)]\varphi_2^2+4|\gamma|2+\mu)}}$$
and
$$|a_3| \leq \frac{4|\gamma|^2\alpha^2}{(1+\mu)^2\varphi_2^2} + \frac{2|\gamma|\alpha}{(2+\mu)\varphi_3}.$$
(2) For functions $f \in \mathcal{S}_{\Sigma,q}^{l,m}(\gamma,\alpha)$, we have
$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{[(1-\alpha)-2|\gamma|\alpha]\varphi_2^2+4|\gamma|\alpha\varphi_3}}$$
and
$$|a_3| \leq \frac{4|\gamma|^2\alpha^2}{\varphi_2^2} + \frac{|\gamma|\alpha}{\varphi_3}$$
(3) For functions $f \in \mathcal{F}_{\Sigma,q}^{l,m}(\lambda,\gamma,\alpha)$, we have
$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{(1-\alpha)(\lambda+1)^2\varphi_2^2+2|\gamma|\alpha(2\lambda+1)\varphi_3}}$$
and
$$|a_3| \leq \frac{4|\gamma|^2\alpha^2}{(\lambda+1)^2\varphi_2^2} + \frac{2|\gamma|\alpha}{(2\lambda+1)\varphi_3}$$
(4) For functions $f \in \mathcal{H}_{\Sigma,q}^{l,m}(\gamma,\alpha)$ then, we have
$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{4(1-\alpha)\varphi_2^2+6|\gamma|\alpha\varphi_3}}$$
and
$$|a_3| \leq \frac{|\gamma|^2\alpha^2}{\varphi_2^2} + \frac{2|\gamma|\alpha}{3\varphi_3}$$
where φ_n is given by (1.10).

On the other hand if we take
$$\phi_\xi(z) = \frac{1+(1-2\beta)(z-\xi)}{1-(z-\xi)} = 1 + 2(1-\beta)(z-\xi) + 2(1-\beta)(z-\xi)^2 + \cdots \quad (0 \leq \beta < 1), \tag{3.2}$$
then $B_1 = B_2 = 2(1-\beta)$. From the corollaries2.1 to 2.4we state the following corollary.

Corollary 3.3 *Let $f(z)$ is given by (1.1) be in the class $\Sigma_{\phi,q}^{l,m}(\mu,\lambda,\gamma)$. Then*
$$|a_2| \leq \frac{2|\gamma|\sqrt{(1-\beta)}}{\sqrt{|\gamma[2(2\lambda+\mu)\varphi_3+(\mu-1)(2\lambda+\mu)\varphi_2^2]|}}$$
and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(\lambda+\mu)^2 \varphi_2^2} + \frac{|\gamma| B_1}{(2\lambda+\mu)\varphi_3}$$

where $\gamma \in \mathbb{C}\backslash\{0\}; 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1$ and φ_n is given by (1.10).

Corollary 3.4 *By choosing $\phi_\xi(z)$ of the form (3.2), and by Theorem2.1,we state the following results:*

(1) For functions $f \in \mathcal{B}_{\Sigma,q}^{l,m}(\mu, \gamma, \beta)$, we have

$$|a_2| \leq \frac{2|\gamma|\sqrt{(1-\beta)}}{\sqrt{|\gamma(2+\mu)[2\varphi_3 + (\mu-1)\varphi_2^2]|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2(1-\beta)^2}{(1+\mu)^2\varphi_2^2} + \frac{2|\gamma|(1-\beta)}{(2+\mu)\varphi_3}.$$

(2) For functions $f \in \mathcal{S}_{\Sigma,q}^{l,m}(\gamma, \beta)$, we have

$$|a_2| \leq \sqrt{\frac{2|\gamma|(1-\beta)}{|2\varphi_3 - \varphi_2^2|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2(1-\beta)^2}{\varphi_2^2} + \frac{|\gamma|(1-\beta)}{\varphi_3}$$

(3) For functions $f \in \mathcal{F}_{\Sigma,q}^{l,m}(\lambda, \gamma, \beta)$, we have

$$|a_2| \leq \sqrt{\frac{2|\gamma|(1-\beta)}{(2\lambda+1)\varphi_3}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2(1-\beta)^2}{(\lambda+1)^2\varphi_2^2} + \frac{2|\gamma|(1-\beta)}{(2\lambda+1)\varphi_3}$$

(4) For functions $f \in \mathcal{H}_{\Sigma,q}^{l,m}(\gamma, \alpha)$ then, we have

$$|a_2| \leq \sqrt{\frac{2|\gamma|(1-\beta)}{3\varphi_3}}$$

and

$$|a_3| \leq \frac{|\gamma|^2(1-\beta)^2}{\varphi_2^2} + \frac{2|\gamma|(1-\beta)}{3\varphi_3}$$

where φ_n is given by (1.10).

Concluding Remarks:

We can define various other subclass of $\Sigma_{\phi,q}^{l,m}(\mu, \lambda, \gamma)$ involving these operators and specializing the parameters λ, μ, l, m one can state the various other interesting subclasses of $\Sigma_{\phi,q}^{l,m}(\mu, \lambda, \gamma)$ as illustrated in the Example1.1 to Example1.4. Further various interesting results (as in Theorem 2.1)and the corresponding corollaries as mentioned above can be derived easily and so we omit the details.Also,from the corollaries 3.2, and 3.4, by taking $\gamma = 1$, we can obtain the results studied earlier in the literature (see [1, 5, 7, 9, 10, 12, 15, 17, 18, 19, 20, 21]).

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