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The Ho-Kalman Model As A Transformer Of Linear Recurring Sequences

Zenonas Navickas[a], Mantas Landauskas[1b], Rasa Smidtaite[c], Minvydas K. Ragulskis[d]

[a] Department of Applied Mathematics, Kaunas University of Technology, Studentu 50-325, Kaunas, LT-51368, Lithuania, +37068223789, zenonas.navickas@ktu.lt
[b] Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-146, Kaunas, LT-51368, Lithuania, +37067713231, mantas.landauskas@ktu.lt
[c] Department of Applied Mathematics, Kaunas University of Technology, Studentu 50-325, Kaunas, LT-51368, Lithuania, +37067113831, rasa.smidtaite@ktu.lt
[d] Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-147, Kaunas, LT-51368, Lithuania, +37069822456, minvydas.ragulskis@ktu.lt

Abstract

This paper demonstrates that the Ho-Kalman model is a transformer of linear recurring sequences. It is shown that the number of roots of the characteristic equation for the output system comprises the roots of the input system and eigenvalues of the system matrix. Different scenarios among the roots of the input signal and the eigenvalues of the system matrix yield a rich variety in the output signals. These theoretical results provide the insight in the complexity of processes governed by the Ho-Kalman model. A number of computational experiments are used to illustrate the theoretical results.

Keywords: Ho-Kalman model; the order of a sequence; algebraic equation.

Classification: 93B17; 94A12; 93C65.

1 Introduction

The identification of the underlying state-space model of the system from a sequence of output data is a challenging system-theoretical problem. A first solution to this theoretical problem that later became known as the reduced-order state-space

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[1] Corresponding author. Email: mantas.landauskas@ktu.lt

realization problem for a time-invariant linear dynamical system was provided in 1965 in [1]. A particular factorization of the Hankel matrix of the discrete-time process into the product of the observability matrix and the controllability matrix is known as the Ho-Kalman realization method [1]. It took years of research to go from the theoretical results described in [1] to a numerically reliable realization algorithm [2].

The combination of deterministic realization theory based on the factorization of the Hankel matrix, with the theory of Markovian and innovations representations, gave rise to the stochastic theory of minimal realizations. The stochastic realization problem was studied intensively during the early 1970s in connection with innovations theory and spectral factorization theory [3, 4]. These methods are substantially based on special factorizations of the Hankel matrix, which maps the past outputs of the discrete-time process into the future outputs.

The Ho-Kalman algorithm has become very popular, not only as an exact realization algorithm, but also by providing methods for approximate realization [5], and by giving rise to interesting developments in the analysis of balancing properties of state space models [6]. The respective realization algorithms have been a principal landmark for the development of the so-called subspace identification methods [7]. Starting from the classical Ho-Kalman algorithm that solves the problem using Markov parameter expansions, a generalization is obtained by analyzing the matrix representations of the Hankel operators in generalized orthonormal bases [8]. Notwithstanding the growing popularity of the above methods, some aspects of their applicability are not yet fully understood, especially in regards to the applicability of these methods to systems driven by excitations different from the one hypothesized [9].

1.1 The Main Objective Of This Paper

The main objective of this paper is to demonstrate that the Ho-Kalman model is a transformer of linear recurrent sequences. The linear iterative Ho-Kalman model reads:

$$\begin{aligned} x(0) &= 0,\ y(0) = 0; \\ x(n+1) &= Gu(n) + \Phi x(n); \\ y(n+1) &= Hx(n+1); \\ n &= 0,1,2,\ldots \end{aligned} \tag{1}$$

where; $G \in \mathbb{C}^{m\times q}$; $\Phi \in \mathbb{C}^{m\times m}$; $H \in \mathbb{C}^{p\times m}$. Vectors $u(n) \in \mathbb{C}^{q}$, $x(n) \in \mathbb{C}^{m}$ and $y(n) \in \mathbb{C}^{p}$ describe the input, the state and the output of the system (it is usually assumed that $p \le q$). Model (1) yields recurrent equalities:

$$\begin{aligned} x(0) &= 0; \\ x(1) &= Gu(0); \\ x(2) &= Gu(1) + \Phi Gu(0); \\ &\ldots \\ x(n) &= \sum_{j=0}^{n-1} \Phi^{j} Gu(n-1-j); \\ n &= 1,2,3,\ldots. \end{aligned} \tag{2}$$

Thus (1) can be rearranged into

$$y(0) = 0, y(n+1) = \sum_{j=0}^{n} H\Phi^{j} G u(n-j), n = 0,1,2,\ldots. \tag{3}$$

Then $U := (u(n))_{n=0}^{\infty}$ and $Y := (y(n))_{n=0}^{\infty}$ denote input and output sequences of vectors. We will show that if U is a linear recurring sequence then Y is also a linear recurring sequence. Moreover, we will prove the relationship between the algebraic complexity of the input and the output sequences.

2 Preliminaries

2.1 Elementary Operations With Sequences

The following elementary operations hold for two sequences $(p_j)_{j=0}^{\infty}$ and $(q_j)_{j=0}^{\infty}$; $p_j, q_j \in \mathbb{C}$:

1. linear mixture:

$$a(p_j)_{j=0}^{\infty} + b(q_j)_{j=0}^{\infty} := (ap_j + bq_j)_{j=0}^{\infty}, a, b \in \mathbb{C}; \tag{4}$$

2. shift operators:

$$P(p_0, p_1, p_2, \ldots) = (0, p_0, p_1, p_2, \ldots);$$

3. convolution:

$$(p_j)_{j=0}^{\infty} * (q_j)_{j=0}^{\infty} := \left(\sum_{r=0}^{j} p_r \cdot q_{j-r} \right)_{j=0}^{\infty}; \tag{5}$$

the operation of convolution is commutative, associative and distributive. Also, the operation of convolution has a priority over the operation of the summation.

4. differentiation:

$$\frac{\partial (p_j)_{j=0}^{\infty}}{\partial \lambda} := \left(\frac{\partial p_j}{\partial \lambda} \right)_{j=0}^{\infty}, \tag{6}$$

if only elements $p_j = p_j(\lambda)$ are functions of λ and derivatives $\frac{\partial p_j(\lambda)}{\partial \lambda}; j = 0,1,\ldots$ do exist.

The following equalities do hold:

1. shift operators:

$$P^{k+l}\left((p_j)_{j=0}^{+\infty} * (q_j)_{j=0}^{+\infty}\right) = \left(P^k (p_j)_{j=0}^{+\infty}\right) * \left(P^l (q_j)_{j=0}^{+\infty}\right); k, l = 0,1,2,\ldots \tag{7}$$

2. differentiation:

$$\frac{\partial}{\partial\lambda}\left(P(p_j)_{j=0}^{+\infty}\right) = P\left(\frac{\partial p_j}{\partial\lambda}\right)_{j=0}^{+\infty} \tag{8}$$

Proof. Subject to

$$\begin{aligned} & (0, p_0, p_1, \ldots) * (q_0, q_1, q_2, \ldots) \\ = & (p_0, p_1, p_2, \ldots) * (0, q_0, q_1, \ldots) \\ = & (0, p_0q_0, p_0q_1 + p_1q_0, \ldots) \end{aligned}$$

the equality Eq. (14) holds.
Analogous proof is valid for Eq. (14).

2.2 The Order Of A Sequence

Let us consider a sequence:

$$(p_j)_{j=0}^{\infty}; p_j \in \mathbb{C}. \tag{9}$$

The Hankel transform of the scalar sequence produces the sequence of determinants of Hankel catalecticant matrixes $d_n; n = 0,1,\ldots$, where:

$$d_{n+1} := \det\begin{bmatrix} p_0 & p_1 & \cdots & p_n \\ p_1 & p_2 & \cdots & p_{n+1} \\ & & \cdots & \\ p_n & p_{n+1} & \cdots & p_{2n} \end{bmatrix}; n = 0,1,2,\ldots. \tag{10}$$

Definition 2.1 It is admitted that $Hr(0,0,\ldots) = 0$, i.e. $d_1 = d_2 = \ldots = 0$.

The order of the sequence $(p_j)_{j=0}^{\infty}$ is $m \in \mathbb{Z}_0$:

$$Hr(p_j)_{j=0}^{\infty} = m \tag{11}$$

if the sequence of determinants of Hankel matrixes has the following structure:

$$(d_1, d_2, \ldots, d_m, 0, 0, \ldots) \tag{12}$$

where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = \ldots = 0$, $m = 1,2,3,\ldots$.

If $d_m \neq 0$, $\forall m$, it is assumed that $Hr(0,0,\ldots) = +\infty$.

Note that $Hr(p_j)_{j=0}^{\infty} = 0$ yields $p_j \equiv 0$.

Definition 2.2 Let $Hr(p_j)_{j=0}^{\infty} = m$. Then the characteristic Hankel equation for the sequence $(p_j)_{j=0}^{\infty}$ is defined as [2]:

$$\det\begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ \ldots & \ldots & \ldots & \ldots \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \rho & \ldots & \rho^m \end{bmatrix} = 0. \tag{13}$$

The expansion of the determinant in Eq. (13) yields an m-th order algebraic equation for the determination of roots of the characteristic equation:

$$A_m\rho^m + A_{m-1}\rho^{m-1} + \ldots + A_1\rho + A_0 = 0; \tag{14}$$

where $A_m \neq 0$ because $d_m \neq 0$.

Theorem 2.3 Let $Hr(p_j)_{j=0}^{\infty} = m$ and the recurrence indexes of roots $\rho_1, \rho_2, \ldots, \rho_l$ of the characteristic equation (Eq. (14)) are $m_1, m_2, \ldots, m_l$ accordingly; $\sum_{r=1}^{l} m_r = m$. Then the following equality holds true:

$$p_j = \sum_{r=1}^{l}\sum_{k=0}^{m_r-1} \mu_{rk}\binom{j}{k}\rho_r^{j-k}; \tag{15}$$

where $\mu_{rk}, \rho_r \in \mathrm{C}$; $\mu_{r,m_r-1} \neq 0$.

Rigorous proof of this theorem is given in [2].

Note that $\mu_{rk}\binom{j}{k}\rho_r^{j-k} = 0$ if only $\binom{j}{k} = 0$ (what is true for all $0 \le j < k$). Moreover, $0^0 = 1$; $0^1 = 0^2 = \ldots = 0$.

Coefficients μ_{rk} can be found by solving the linear algebraic system of equations ($\rho_1, \rho_2, \ldots, \rho_l$ are determined beforehand):

$$\sum_{r=1}^{l}\sum_{k=0}^{m_r-1}\binom{j}{k}\rho_r^{j-k}\mu_{rk} = p_j;\ j = 0,1,\ldots, m-1. \tag{16}$$

This linear system of algebraic equations has one and the only one solution [2].

2.3 Idempotents and nilpotents of square matrices of order 2

Let us consider a square matrix of order 2:

$$X := \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, x_{11}, \ldots, x_{22} \in \mathbb{C}. \tag{17}$$

The eigenvalues of X are denoted as $\lambda_1, \lambda_2 \in \mathbb{C}$:

2.3.1 The case when eigenvalues are different

Since $\lambda_1 \neq \lambda_2$, it is possible to construct two idempotents D_k:

$$D_k := \frac{1}{\lambda_k - \lambda_l}\left(X - \lambda_l I\right); \quad k,l = 1,2; \quad k \neq l; \tag{18}$$

where I is the identity matrix. Matrices D_k do satisfy the following equalities:

1. $\det D_k = 0$;
2. $D_1 + D_2 = I$ (D_1 and D_2 are conjugate idempotents);
3. $D_k \cdot D_l = \delta_{kl} D_k$; $k,l = 1,2$; $\delta_{kl} := \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases}$
4. The matrix X can be expressed as $X = \lambda_1 D_1 + \lambda_2 D_2$.

2.3.2 The case when eigenvalues are equal

Let us denote $\lambda_1 = \lambda_2 = \lambda_0$. Then the nilpotent N reads:

$$N := X - \lambda_0 I \tag{19}$$

The nilpotent N does satisfy the following relationships:

1. $N^2 = \Theta$; $\Theta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
2. $\det N = 0$.
3. The matrix X can be expressed as: $X = \lambda_0 I + N$.

Note that a scalar matrix $X = \lambda_0 I$ can be expressed in the form $\lambda_0 I = \lambda_0 D_1 + \lambda_0 D_2$ where $\left(D_1; D_2\right)$ is any pair of conjugate idempotents – or (alternatively) in the form $\lambda_0 I = \lambda_0 I + N$ where the $N = \Theta$. Note that nilpotents N and cN are similar nilpotents ($c \in \mathbb{C}$).

2.3.3 Arithmetic operations with matrices

Let $\left(D_1; D_2\right)$ is a pair for idempotents of two different matrices X_1 and Y_1. Then D_1 and D_2 are also idempotents of matrices $X_1 \cdot Y_1$ and $X_1 + Y_1$.

Let $X_1 = \lambda_1 I + N$ and $X_2 = \lambda_2 I + \alpha N$ be nilpotent matrices having similar nilpotents. Then the matrices $X_2 \cdot Y_2$ and $X_2 + Y_2$ are nilpotent matrices with similar nilpotents.

Moreover, powers of X_1^n and X_2^n read:

$$X_1^n = \lambda_1^n D_1 + \lambda_2^n D_2; X_2^n = \lambda_0^n I + n\lambda_0^{n-1} N; n = 0,1,2, \ldots \tag{20}$$

Detailed proofs of these properties of square matrices of order 2 are given in [1].

2.4 Unilateral *Z*-transform and its properties

Let us consider a scalar sequence $(p_j)_{j=0}^{\infty}$. Unilateral *Z*-transform of this sequence is defined as:

$$Z(p_j)_{j=0}^{\infty} := \sum_{j=0}^{+\infty} p_j \frac{1}{z^j}; z \in \mathrm{C}. \tag{21}$$

The region of convergence of Eq. (21) is the causal region $|z| > \varepsilon$, where ε is a finite positive number. The following properties do hold true:

1. linear mixture:

$$Z\left(a(p_j)_{j=0}^{\infty} + b(q_j)_{j=0}^{\infty}\right) = aZ(p_j)_{j=0}^{\infty} + bZ(p_j)_{j=0}^{\infty}; a, b \in \mathrm{C} \tag{22}$$

2. convolution:

$$Z\left((p_j)_{j=0}^{\infty} * (q_j)_{j=0}^{\infty}\right) = Z(p_j)_{j=0}^{\infty} \cdot Z(q_j)_{j=0}^{\infty}; \tag{23}$$

3. *Z*-transform of the shifted sequence:

$$Z\left(P^k (p_j)_{j=0}^{\infty}\right) = \frac{1}{z^k} Z(p_j)_{j=0}^{\infty}; k = 0,1,2, \ldots. \tag{24}$$

Proof.

$$Z\left(P(p_j)_{j=0}^{\infty}\right) = Z\left(0, p_0, p_1, \ldots\right) = \frac{1}{z}\left(p_0 \frac{1}{z^0}, p_1 \frac{1}{z}, \ldots\right) = \frac{1}{z} Z(p_j)_{j=0}^{\infty}.$$

4. differentiation:

$$\frac{\partial Z(p_j)_{j=0}^{\infty}}{\partial \lambda} = Z\left(\frac{\partial p_j}{\partial \lambda}\right)_{j=0}^{\infty}$$

if only elements $p_j = p_j(\lambda)$ are functions of λ and derivatives $\frac{\partial \left(p_j\right)}{\partial \lambda}; j = 0,1, \ldots$ do exist.

5. differentiation of convolution:

$$\frac{\partial Z\left((p_j)_{j=0}^{\infty} * (q_j)_{j=0}^{\infty}\right)}{\partial \lambda} = \frac{\partial \left(Z(p_j)_{j=0}^{\infty}\right)}{\partial \lambda} \cdot Z(q_j)_{j=0}^{\infty} + Z(p_j)_{j=0}^{\infty} \cdot \frac{\partial Z(q_j)_{j=0}^{\infty}}{\partial \lambda}; \tag{25}$$

if derivatives $\frac{\partial \left(p_j\right)}{\partial \lambda}; \frac{\partial \left(q_j\right)}{\partial \lambda}; j = 0,1, \ldots$ do exist.

Example 1. $Z(\rho^j)_{j=0}^{\infty} = \sum_{j=0}^{+\infty}\left(\frac{\rho}{z}\right)^j = \frac{z}{z-\rho}$; the region of convergence is $|z| > |\rho|$.

3. The Z-Transform And Linear Recurrent Sequences

Lemma 3.1 Eq. (21)-(23) yield following relationships:

$$\frac{\partial^k Z(\rho^j)_{j=0}^{\infty}}{\partial \rho^k} = Z\left(k!\binom{j}{k}\rho^{j-k}\right)_{j=0}^{\infty} = \frac{k!z}{(z-\rho)^{k+1}}; k = 0,1,2,\ldots.$$

Thus,

$$Z\left(\sum_{k=1}^{n}\sum_{r=0}^{n_k-1}\mu_{kr}\binom{j}{k}\rho_k^{j-r}\right)_{j=0}^{\infty} = \sum_{k=1}^{n}\sum_{r=0}^{n_k-1}\mu_{kr}\frac{z}{(z-\rho_k)^{r+1}} \tag{26}$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are all different; *n*, $n_k \in \mathrm{N}$; $\mu_{kr} \in \mathrm{C}$; $z > \max_k |\rho_k|$. The proof follows from previous definitions.

Lemma 3.2 Unilateral Z-transform of the convolution yields:

$$Z\left((\rho_1^j)_{j=0}^{\infty} * (\rho_2^j)_{j=0}^{\infty}\right) = \frac{\rho_1}{\rho_1-\rho_2}\cdot\frac{z}{z-\rho_1} + \frac{\rho_2}{\rho_2-\rho_1}\cdot\frac{z}{z-\rho_2} \tag{27}$$

for $\rho_1 \neq \rho_2$, and

$$Z\left((\rho^j)_{j=0}^{\infty} * (\rho^j)_{j=0}^{\infty}\right) = \frac{z}{z-\rho} + \rho\frac{z}{(z-\rho)^2} \tag{28}$$

for $\rho_1 = \rho_2 = \rho$.

Proof. The proof of Eq. (27) follows from the equality:

$$\frac{z}{z-\rho_1}\cdot\frac{z}{z-\rho_2} = z\left(\frac{z}{(z-\rho_1)(z-\rho_2)}\right) = z\left(\frac{\rho_1}{\rho_1-\rho_2}\cdot\frac{1}{z-\rho_1} + \frac{\rho_2}{\rho_2-\rho_1}\cdot\frac{1}{z-\rho_2}\right)$$

Analogously, the proof of Eq. (28) follows from the equality:

$$\left(\frac{z}{z-\rho}\right)^2 = z\cdot\left(\frac{z}{(z-\rho)^2}\right) = z\cdot\left(\frac{1}{z-\rho} + \frac{\rho}{(z-\rho)^2}\right)$$
$$= z\left(\frac{1}{z-\rho} + \rho\frac{\partial}{\partial\rho}\frac{1}{z-\rho}\right).$$

Lemmas 3.1 and 3.2 yield:

$$(\rho_1^j)_{j=0}^{+\infty} * (\rho_2^j)_{j=0}^{+\infty} = \frac{\rho_1}{\rho_1-\rho_2}(\rho_1^j)_{j=0}^{+\infty} + \frac{\rho_2}{\rho_2-\rho_1}(\rho_2^j)_{j=0}^{+\infty} \tag{29}$$

and

$$(\rho^j)_{j=0}^{+\infty} * (\rho^j)_{j=0}^{+\infty} = (\rho^j)_{j=0}^{+\infty} + \rho\frac{\partial}{\partial\rho}(\rho^j)_{j=0}^{+\infty} = \rho(j\rho^{j-1})_{j=0}^{+\infty}. \tag{30}$$

Thus, Gauss-Ostrogradsky theorem yields a relationship:

$$
\begin{aligned}
&Z\left(\left(\binom{j}{m}\rho_1^{j-m}\right)_{j=0}^{\infty} * \left(\binom{j}{n}\rho_2^{j-n}\right)_{j=0}^{\infty}\right)\\
=\;& z\cdot\left(\frac{z}{(z-\rho_1)^{m+1}\cdot(z-\rho_2)^{n+1}}\right)\\
=\;& \sum_{k=0}^{m}\alpha_k\,\frac{z}{(z-\rho_1)^{k+1}}+\sum_{l=0}^{n}\beta_l\,\frac{z}{(z-\rho_2)^{l+1}}
\end{aligned}
\qquad (31)
$$

which does hold true when $\rho_1 \neq \rho_2$ (complex numbers α_0, α_1, …, α_m, β_0, β_1, …, β_n and γ_0, γ_1, …, γ_{m+n} in general case depend on ρ_1, ρ_2, m and n). Analogously,

$$
\begin{aligned}
&Z\left(z\left|\left(\binom{j}{m}\rho^{j-m}\right)_{j=0}^{\infty} * \left(\binom{j}{n}\rho^{j-n}\right)_{j=0}^{\infty}\right.\right)\\
=\;& z\cdot\left(\frac{z}{(z-\rho)^{2+m+n}}\right)=\sum_{k=0}^{1+m+n}\gamma_k\,\frac{z}{(z-\rho)^{1+k}}
\end{aligned}
\qquad (32)
$$

when $\rho_1=\rho_2=\rho$.

Corollary 1. Eq. (31) yields the following relationship:

$$
\begin{aligned}
&\left(\binom{j}{m}\rho_1^{j-m}\right)_{j=0}^{\infty} * \left(\binom{j}{n}\rho_2^{j-k}\right)_{j=0}^{\infty}\\
=\;& \sum_{k=0}^{m}\alpha_k\left(\binom{j}{k}\rho_1^{j-k}\right)_{j=0}^{\infty}+\sum_{l=0}^{n}\beta_l\left(\binom{j}{l}\rho_2^{j-l}\right)_{j=0}^{\infty}
\end{aligned}
\qquad (33)
$$

for all $m, n = 0, 1, 2, \ldots$ where $\alpha, \beta, \gamma \in \mathrm{C}$.

Analogously Eq. (32) yields:

$$
\left(\binom{j}{m}\rho^{j-m}\right)_{j=0}^{\infty} * \left(\binom{j}{n}\rho^{j-n}\right)_{j=0}^{\infty} = \sum_{k=0}^{1+m+n} C_k\left(\binom{j}{k}\rho^{j-k}\right)_{j=0}^{\infty}, \qquad (34)
$$

4.The algebraic structure of the Ho-Kalman model

Let $U := \left(u(0), u(1), \ldots\right)$ and $Y := \left(y(0), y(1), y(2), \ldots\right)$ are input and output sequences. Let us introduce a sequence of linear operators $K := \left(K_0, K_1, K_2, \ldots\right)$ where $K_j = H\Phi^j G$; $j = 0,1,2,\ldots$. Then Eq. (3) yields the following relationship:

$$\begin{aligned} Y &= P\left(K_0u(0), K_0u(1)+K_1u(0), K_0u(2)+K_1u(1)+K_2u(0), \ldots\right) \\ &= P(K*U) = (PK)*U = K*(PU). \end{aligned} \tag{35}$$

Corollary 2. The equality:

$$P(K*(aU_1+bU_2)) = a(P(K*U_1)) + b(P(K*U_2)) \tag{36}$$

holds true for all $a,b \in \mathbb{C}$ and input sequences U_1 and U_2. The proof follows from Eq. (19) and properties of the convolution operator.

Without the loss of generality we do assume that $p,q,m = 2$. The evolution of the output sequences is now described by the following iterative equation:

$$\begin{aligned} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} &= \sum_{j=0}^{n} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}^{j} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} u_1(n-j) \\ u_2(n-j) \end{bmatrix}, \\ n &= 0,1,2,\ldots. \end{aligned} \tag{37}$$

4.1 Types of the output sequence

Two different types of the output sequence ($\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}$, $n = 0,1,2,\ldots$) can be produced.

Type 1 output sequence. If the matrix Φ can be decomposed into a pair of conjugate idempotents D and $\overline{D}$ ($\Phi = \lambda_1 D + \lambda_2 \overline{D}$), then Eq. (3) yields the expression of the output sequence:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} &= \sum_{j=0}^{n} \left(\lambda_1^j A + \lambda_2^j B\right) \begin{bmatrix} u_1(n-j) \\ u_2(n-j) \end{bmatrix} \\ &= A \begin{bmatrix} \sum_{j=0}^{n} \lambda_1^j \cdot u_1(n-j) \\ \sum_{j=0}^{n} \lambda_1^j \cdot u_2(n-j) \end{bmatrix} + B \begin{bmatrix} \sum_{j=0}^{n} \lambda_2^j \cdot u_1(n-j) \\ \sum_{j=0}^{n} \lambda_2^j \cdot u_2(n-j) \end{bmatrix}; \\ n &= 0,1,2,\ldots \end{aligned} \tag{38}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ are two different eigenvalues of Φ; $A := HDG$ and $B := H\overline{D}G$. Thus the output sequence reads:

$$\left(\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}, n = 0,1,2,\dots\right)$$
$$= \quad A\begin{bmatrix} (\lambda_1^j)_{j=0}^{\infty} * (u_1(j))_{j=0}^{\infty} \\ (\lambda_1^j)_{j=0}^{\infty} * (u_2(j))_{j=0}^{\infty} \end{bmatrix} + B\begin{bmatrix} (\lambda_2^j)_{j=0}^{\infty} * (u_1(j))_{j=0}^{\infty} \\ (\lambda_2^j)_{j=0}^{\infty} * (u_2(j))_{j=0}^{\infty} \end{bmatrix} \tag{39}$$

Type 2 output sequence. If the matrix Φ has a single recurrent eigenvalue $\lambda_0 \in \mathbb{C}$:

$$\Phi = \lambda_0 I + N, \tag{40}$$

then Eq. (3) yields the expression of the output sequence:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} = \sum_{j=0}^{n} (\lambda_0^j C + j\lambda_0^{j-1}E) \begin{bmatrix} u_1(n-j) \\ u_2(n-j) \end{bmatrix}$$
$$= \quad C\begin{bmatrix} \sum_{j=0}^{n} \lambda_0^j \cdot u_1(n-j) \\ \sum_{j=0}^{n} \lambda_0^j \cdot u_2(n-j) \end{bmatrix} + E\begin{bmatrix} \sum_{j=0}^{n} j\lambda_0^{j-1} \cdot u_1(n-j) \\ \sum_{j=0}^{n} j\lambda_0^{j-1} \cdot u_2(n-j) \end{bmatrix},$$
$$n = \quad 0,1,2,\dots \tag{41}$$

where N is the nilpotent of Φ $(N \neq O)$; $C := HG$ and $E := HNG$. Then the output sequence reads:

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}_{n=1}^{\infty} = C\begin{bmatrix} P(\lambda_0^j)_{j=0}^{\infty} * (u_1(j))_{j=0}^{\infty} \\ P(\lambda_0^j)_{j=0}^{\infty} * (u_2(j))_{j=0}^{\infty} \end{bmatrix} + E\begin{bmatrix} P\dfrac{d(\lambda_0^j)_{j=0}^{\infty}}{d\lambda_0} * (u_1(j))_{j=0}^{\infty} \\ P\dfrac{(\lambda_0^j)_{j=0}^{\infty}}{d\lambda_0} * (u_2(j))_{j=0}^{\infty} \end{bmatrix} \tag{42}$$

Note that the scalar matrix

$$\Phi := \lambda_0 I. \tag{43}$$

can be produced from the Type 1 matrix at $\lambda_1 = \lambda_2 := \lambda_0$ or from the Type 2 matrix at $N = O$. Then:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1(n+1) \\ y_2(n+1) \end{bmatrix} = C \begin{bmatrix} P\sum_{j=0}^{n} \lambda_0^j \cdot u_1(n-j) \\ P\sum_{j=0}^{n} \lambda_0^j \cdot u_2(n-j) \end{bmatrix}. \tag{44}$$

The expression of the output sequence then reads:

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}_{n=0}^{\infty} = C \begin{bmatrix} P\left(\lambda_0^j\right)_{j=0}^{\infty} * \left(u_1(j)\right)_{j=0}^{\infty} \\ P\left(\lambda_0^j\right)_{j=0}^{\infty} * \left(u_2(j)\right)_{j=0}^{\infty} \end{bmatrix}. \tag{45}$$

Theorem 4.1 Equations (9), (38) yield the equality $Y = A\left(P\left(\Lambda_1 * U\right)\right) + B\left(P\left(\Lambda_2 * U\right)\right)$ when $\Lambda_k := \left(\lambda_k^j\right)_{j=0}^{\infty}$; $k = 1,2$. Analogously, equations (40), (42) yield the equality $Y = C\left(P\left(\Lambda_0 * U\right)\right) + E\left(P\left(\Lambda_0' * U\right)\right)$ when $\Lambda_0 := \left(\lambda_0^j\right)_{j=0}^{\infty}$ and $\Lambda_0' := \left(j\lambda_0^{j-1}\right)_{j=0}^{\infty}$. Equations (43), (45) yield $Y = C\left(P\left(\Lambda_0 * U\right)\right)$.

5. Algebraic Expressions Of The Output Sequence Produced By The Ho-Kalman Model

Let the input sequence U is an algebraic progression:

$$U := \left(\sum_{k=1}^{m} \sum_{r=0}^{n_k - 1} U_j(k,r) \right)_{j=0}^{\infty}; \tag{46}$$

where $U_j(k,r) := \begin{bmatrix} a_{kr}^{(1)} \binom{j}{r} \rho_k^{j-r} \\ a_{kr}^{(2)} \binom{j}{r} \rho_k^{j-r} \end{bmatrix}$; coefficients $a_{kr}^{(1)}, a_{kr}^{(2)} \in C$; $\left|a_{kn_k-1}^{(1)}\right| + \left|a_{kn_k-1}^{(2)}\right| > 0$ for $k = 1,2,\ldots,m$. Note that Eq. (46) does not require that roots ρ_k, $k = 1,2,\ldots,m$ should be the same for both input scalar sequences – one of the coefficients $a_{kn_k-1}^{(1)}$ or $a_{kn_k-1}^{(2)}$ could be equal to 0 (then the root ρ_k would be associated only with one of the two input sequences).

Five different cases can be singled out in respect to the relationship among roots ρ_k ($k = 1,2,\ldots,m$) and eigenvalues λ_l ($l = 1,2$).

The following Theorem can be formulated now.

Theorem 5.1 If the sequence of input signals into the Ho-Kalman model at $p,q,m = 2$ is a linear recurrent sequence, then the output sequence is also a linear recurrent sequence. Moreover, the set of roots of the output signals are comprised from the set of roots of the input signals plus eigenvalues of the matrix Φ.

The proof follows from the previous derivations.

6.Computational Experiments

A number of computational experiments will be used to illustrate different possible relationships among the eigenvalues of matrix Φ and the roots of the characteristic equation for the input sequences. Let us consider two input sequences of real numbers: $S^{(1)} = \left(x_k^{(1)}\right)_{k=0}^{+\infty}$ and $S^{(2)} = \left(x_k^{(2)}\right)_{k=0}^{+\infty}$. Then the input of the system reads: $u = \left(\begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix}, \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \end{bmatrix}, \begin{bmatrix} x_2^{(1)} \\ x_2^{(2)} \end{bmatrix}, \ldots\right)$. The same input sequences are used for all examples. Moreover, we assume that both sequences $S^{(1)}$ and $S^{(2)}$ are period-3 sequences. Then, roots of the characteristic equations for $S^{(1)}$ and $S^{(2)}$ read [10]:

$$\rho_1 = 1, \rho_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \rho_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

Note that $|\rho_1| = |\rho_2| = |\rho_3| = 1$; $\arg(\rho_1) = 0$; $\arg(\rho_2) = \frac{2\pi}{3}$; $\arg(\rho_2) = \frac{4\pi}{3}$. The elements of $S^{(1)}$ and $S^{(2)}$ are formed by choosing different coefficients $\mu_{10}^{(1)} = 1$; $\mu_{20}^{(1)} = 2$; $\mu_{30}^{(1)} = 2$ (the superscript denotes the number of the input sequence) are used for $S^{(1)}$ and $\mu_{10}^{(2)} = 2$; $\mu_{20}^{(2)} = -1$; $\mu_{30}^{(2)} = -1$ are used for $S^{(2)}$:

$$x_j^{(k)} = \mu_1^{(k)} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^j \mu_2^{(k)} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^j \mu_3^{(k)}, \; j = 0,1,2,\ldots, k = 1,2.$$

Thus, input of the system reads: $u = \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \ldots\right)$; the orders of $S^{(1)}$ and $S^{(2)}$ are: $HrS^{(1)} = HrS^{(2)} = 3$.

Also, we fix $G = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

6.1 $\lambda_1 \neq \lambda_2 \neq \rho_k$

Let $\Phi = \begin{bmatrix} -\frac{\sqrt{3}}{2} - \frac{5}{2} & -1 \\ \frac{13}{2} & \frac{5}{2} - \frac{\sqrt{3}}{2} \end{bmatrix}$; $\lambda_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$; $\lambda_2 = -\frac{\sqrt{3}}{2} - \frac{1}{2} i$ (Φ is an idempotent matrix); $|\lambda_1| = |\lambda_2| = 1$; $\arg\lambda_1 = \frac{5\pi}{6}$; $\arg\lambda_2 = \frac{7\pi}{6}$. Note that according to Eq. (20) $\Phi^n = \lambda_1^n D_1 + \lambda_2^n D_2$ where $D_1 = \begin{bmatrix} 0.5 + 2.5i & i \\ -6.5i & 0.5 - 2.5i \end{bmatrix}$; $D_1 = \begin{bmatrix} 0.5 - 2.5i & -i \\ 6.5i & 0.5 + 2.5i \end{bmatrix}$.

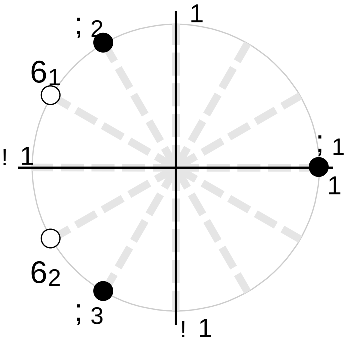


Figure 1: The set of roots of the characteristic equation for the output sequences; the greatest comon divisor is $\frac{\pi}{6}$.

According to Theorem 5.1, the number of roots of the characteristic equation for the output sequence is 5: ρ_1, ρ_2, ρ_3, λ_1 and λ_2. Note that all roots are different; the greatest common divisor of $arg(\rho_1)$, $arg(\rho_2)$, ... and $arg(\lambda_2)$ is $\frac{\pi}{6}$. Thus the analytical expression of the output sequence reads:

$$y_k^{(1)} = \eta_{10}^{(1)}\rho_1^k + \eta_{20}^{(1)}\rho_2^k + \eta_{30}^{(1)}\rho_3^k + \eta_{40}^{(1)}\lambda_1^k + \eta_{50}^{(1)}\lambda_2^k,$$
$$y_k^{(2)} = \eta_{10}^{(2)}\rho_1^k + \eta_{20}^{(2)}\rho_2^k + \eta_{30}^{(2)}\rho_3^k + \eta_{40}^{(2)}\lambda_1^k + \eta_{50}^{(2)}\lambda_2^k$$

where η_{s0}^l; $s = \overline{1,5}$, $l = 1,2$, are scalar coefficients of the expansion and the period of the output sequence is 12. Computational simulation of the Ho-Kalman model confirms this analytical prediction (Fig. 2).

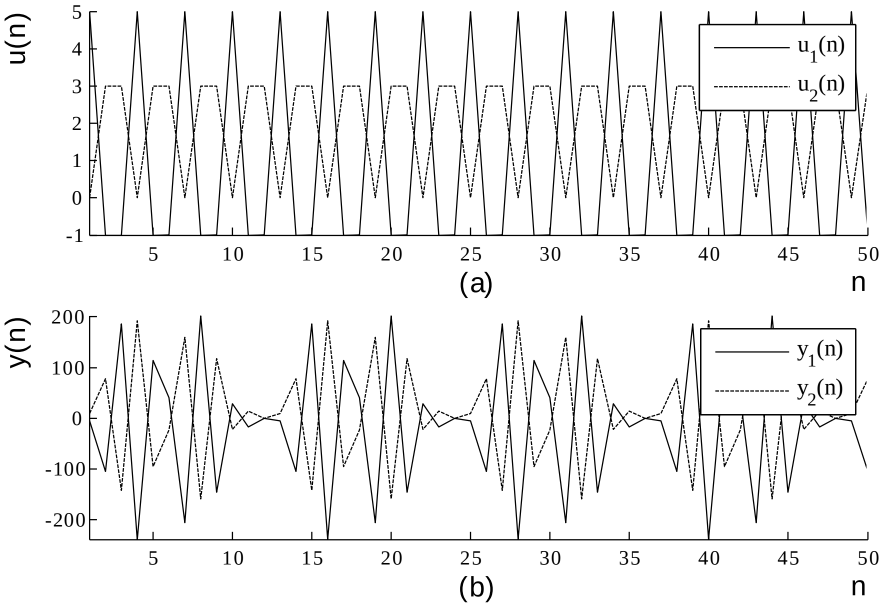


Figure 2: Input (part A) and output (part B) sequences of the Ho-Kalman model at

$u = \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \dots \right)$; $\Phi = \begin{bmatrix} -\frac{\sqrt{3}}{2} - \frac{5}{2} & -1 \\ \frac{13}{2} & \frac{5}{2} - \frac{\sqrt{3}}{2} \end{bmatrix}$; $G = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. The periodicity of the input sequences is 3; the periodicity of the output sequences is 12.

6.2 $\lambda_1 \neq \lambda_2; \lambda_1 = \rho_2; \lambda_2 = \rho_3$

As mentioned previously, the input of the system is the same as before. But now we set $\Phi = \begin{bmatrix} -\frac{5\sqrt{3}}{2} - \frac{1}{2} & -\sqrt{3} \\ \frac{13\sqrt{3}}{2} & \frac{5\sqrt{3}}{2} - \frac{1}{2} \end{bmatrix}$; $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$; $|\lambda_1| = |\lambda_2| = 1$. The greatest common divider of $\arg(\rho_1)$; $\arg(\rho_2)$; $\arg(\rho_2)$; $\arg \lambda_1$ and $\arg \lambda_2$ is $\frac{\pi}{3}$.

Now, according to Theorem 5.1, there exist 5 roots of the output signal – but only 3 of these roots are different:

$$\gamma_1 = \rho_1 = 1; \gamma_{2,3} = \rho_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i; \gamma_{4,5} = \rho_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

All modulus of the output roots are equal to 1. But since two roots are multiple, the output signal is not periodic but diverges as the number of steps tends to infinity (note that elements of Φ^j do generate periodic sequences):

$$y_k^{(1)} = \eta_{10}^{(1)} \rho_1^k + \eta_{20}^{(1)} \rho_2^k + k \eta_{11}^{(1)} \rho_2^{k-1} + \eta_{20}^{(1)} \rho_3^k + k \eta_{21}^{(1)} \rho_3^{k-1},$$
$$y_k^{(2)} = \eta_{10}^{(2)} \rho_1^k + \eta_{20}^{(2)} \rho_2^k + k \eta_{11}^{(2)} \rho_2^{k-1} + \eta_{20}^{(2)} \rho_3^k + k \eta_{21}^{(2)} \rho_3^{k-1}$$

when η_{sr}^l; $s = 1,2,3$, $l = 1,2$, $r = 0,1$ are scalar coefficients of the expansion.

Computational simulation of the Ho-Kalman model confirms this analytical prediction (Fig. 3).

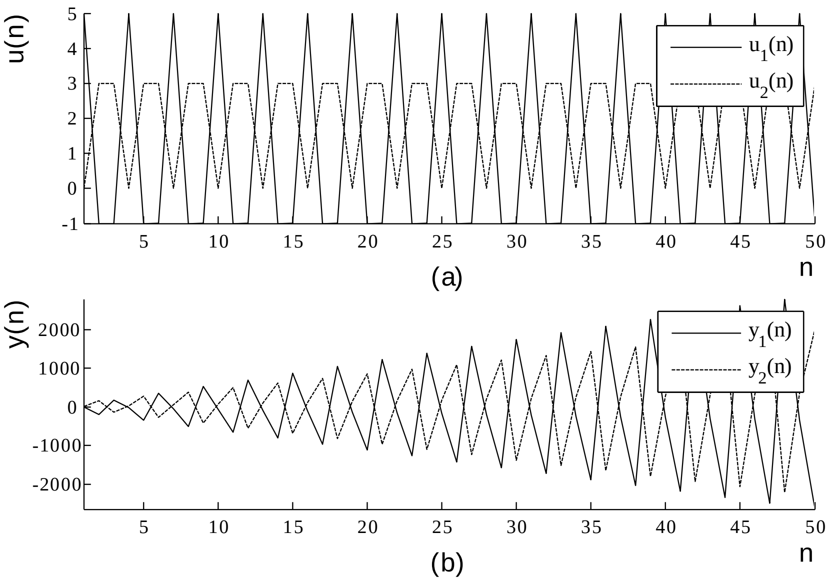


Figure 3: Input (part A) and output (part B) sequences of the Ho-Kalman model at

$$u=\left(\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\ldots\right);\quad \Phi=\begin{bmatrix}-\frac{5\sqrt{3}}{2}-\frac{1}{2} & -\sqrt{3}\\ \frac{13\sqrt{3}}{2} & \frac{5\sqrt{3}}{2}-\frac{1}{2}\end{bmatrix};\quad G=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\ \text{and}\ H=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}.$$

Eigenvalues of Φ coincide with two roots of the input sequences. The output signal diverges but the eigenvalues of Φ^n stay on the unit circle in the complex plane.

6.3 $\lambda_1 \neq \lambda_2; \arg(\lambda_1) = \arg(\varphi_2); \arg(\lambda_2) = \arg(\varphi_3)$

Now we assume $\Phi=\begin{bmatrix}-\frac{5\sqrt{3}}{3}-\frac{1}{3} & -\frac{2\sqrt{3}}{3}\\ \frac{13\sqrt{3}}{3} & \frac{5\sqrt{3}}{3}-\frac{1}{3}\end{bmatrix}$. The matrix is idempotent, but $\lambda_{1,2}=-\frac{1}{3}\pm\frac{\sqrt{3}}{3}i$ are inside the unit circle: $|\lambda_1|=|\lambda_2|<1$. According to Eq. (20) eigenvalues of matrix Φ^n vanish as n tends to infinity. The greatest common divider of $\arg(\varphi_1)$; $\arg(\varphi_2)$; $\arg(\varphi_2)$; $\arg\lambda_1$ and $\arg\lambda_2$ is $\frac{\pi}{3}$. The output sequences become periodic period-3 sequences after a short transient process (Fig. 4).

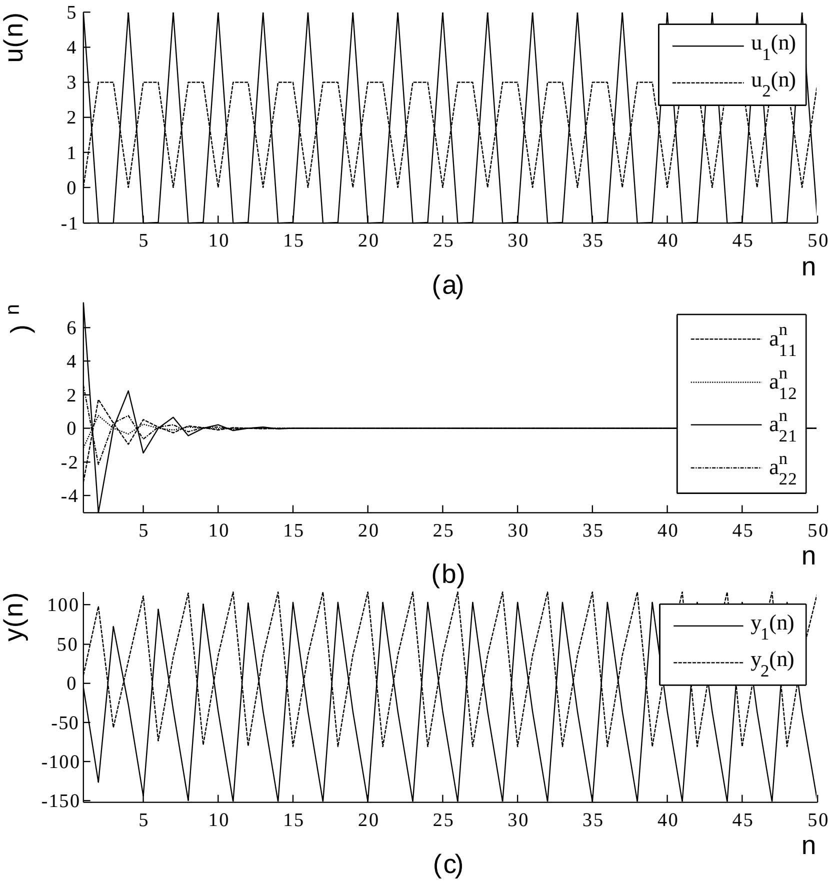


Figure 4: Input (part A) and output (part B) sequences of the Ho-Kalman model at

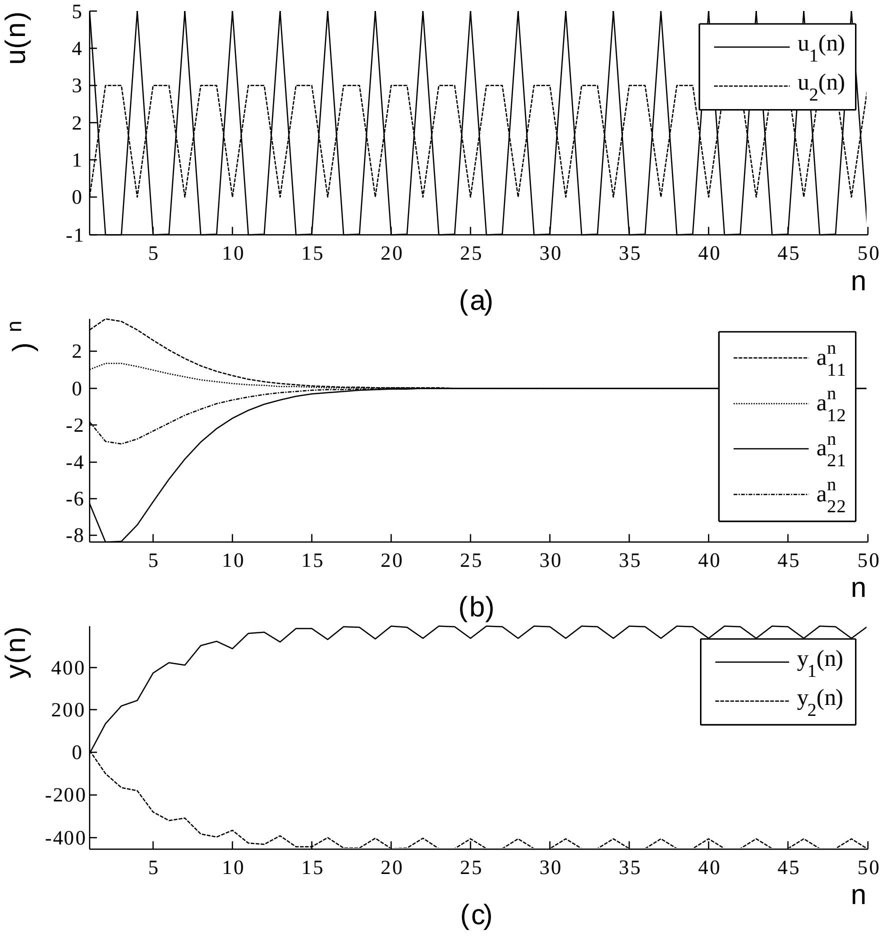
$$u=\left(\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\ldots\right);\quad \Phi=\begin{bmatrix}-\frac{5\sqrt{3}}{3}-\frac{1}{3} & -\frac{2\sqrt{3}}{3}\\ \frac{13\sqrt{3}}{3} & \frac{5\sqrt{3}}{3}-\frac{1}{3}\end{bmatrix};\quad G=\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}\ \text{and}\ H=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}.$$

The output sequences become period-3 sequences after a short transient.

6.4 $\lambda_1=\lambda_2\neq\rho_k$

Let $\Phi=\begin{bmatrix}\frac{19}{6} & 1\\ -\frac{25}{4} & -\frac{11}{6}\end{bmatrix}$; $\lambda_1=\lambda_2=\frac{2}{3}=\lambda_0$; $|\lambda_0|<1$; $\arg\lambda_0=0$.

Note that according to Eq. (20) $\Phi^n=\lambda_0^n I+n\lambda_0^{n-1}N$, where $N=\begin{bmatrix}2.5 & 1\\-6.25 & -2.5\end{bmatrix}$. Elements of matrix Φ^n vanish after some transient process ($\lim_{n\to\infty} n\left(\frac{2}{3}\right)^{n-1}=0$) and the output sequences become periodic period-3 sequences (Fig. 5).



(a)

(b)

(c)

Figure 5: Input (part A) and output (part B) sequences of the Ho-Kalman model at

$u = \left(\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\dots\right)$; $\Phi = \begin{bmatrix}\frac{19}{6} & 1\\ -\frac{25}{4} & -\frac{11}{6}\end{bmatrix}$; $G = \begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$ and $H = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$. Output sequences become period-3 sequences after a transient process.

6.5 $\lambda_1 = \lambda_2 = \rho_1$

Let $\Phi = \begin{bmatrix}\frac{7}{2} & 1\\ -\frac{25}{4} & -\frac{3}{2}\end{bmatrix}$. Then, $\lambda_1 = \lambda_2 = 1 = \lambda_0$; $|\lambda_0| = 1$; $\arg\lambda_0 = 0$. In this example the elements of matrix Φ^n tend to infinity as $\lim_{n\to\infty} n = \infty$.

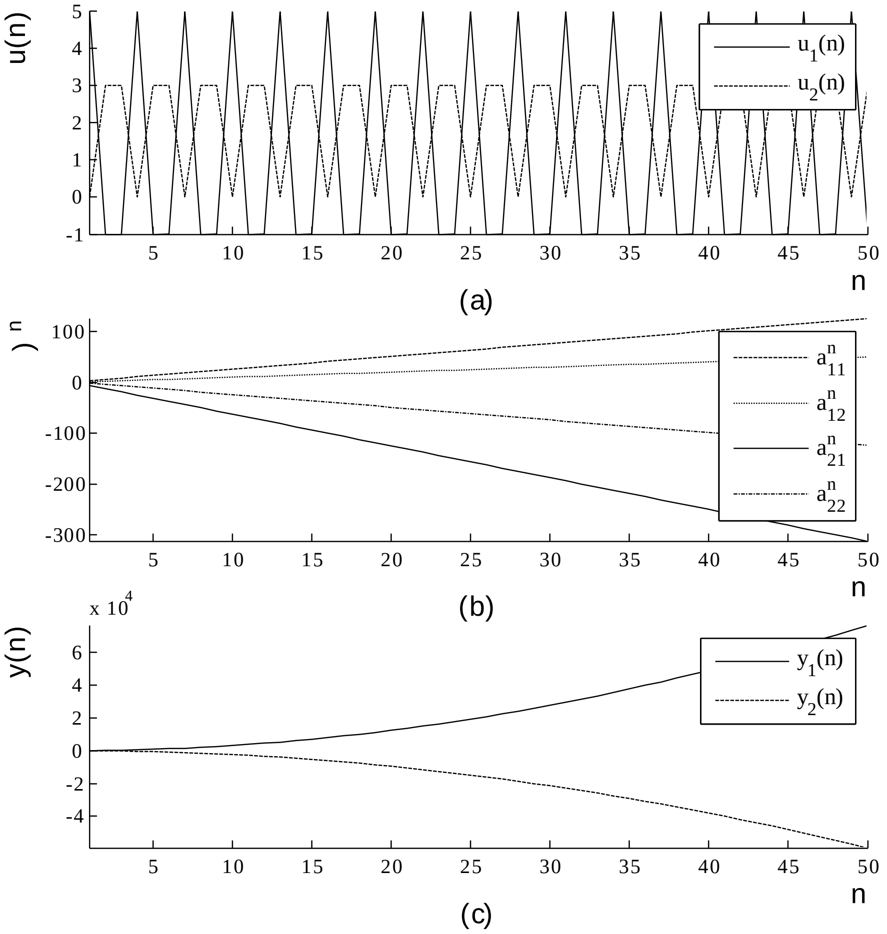


Figure 6: Input (part A) and output (part B) sequences of the Ho-Kalman model at

$u=\left(\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\dots\right)$; $\Phi=\begin{bmatrix}\frac{7}{2} & 1\\ -\frac{25}{4} & -\frac{3}{2}\end{bmatrix}$; $G=\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$ and $H=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$. Eigenvalues of Φ are equal to ρ_1. The elements of Φ tends to infinity and the output sequences also diverges at quadratic speed.

6.6 $\lambda_1=\lambda_2\neq\rho_1$

Let $\Phi=\begin{bmatrix}4 & 1\\ -\frac{25}{4} & -1\end{bmatrix}$. Then, $\lambda_1=\lambda_2=\frac{3}{2}=\lambda_0$; $|\lambda_0|>1$; $\arg\lambda_0=0$. Now the elements of matrix Φ^n tend to infinity according to the exponential law.

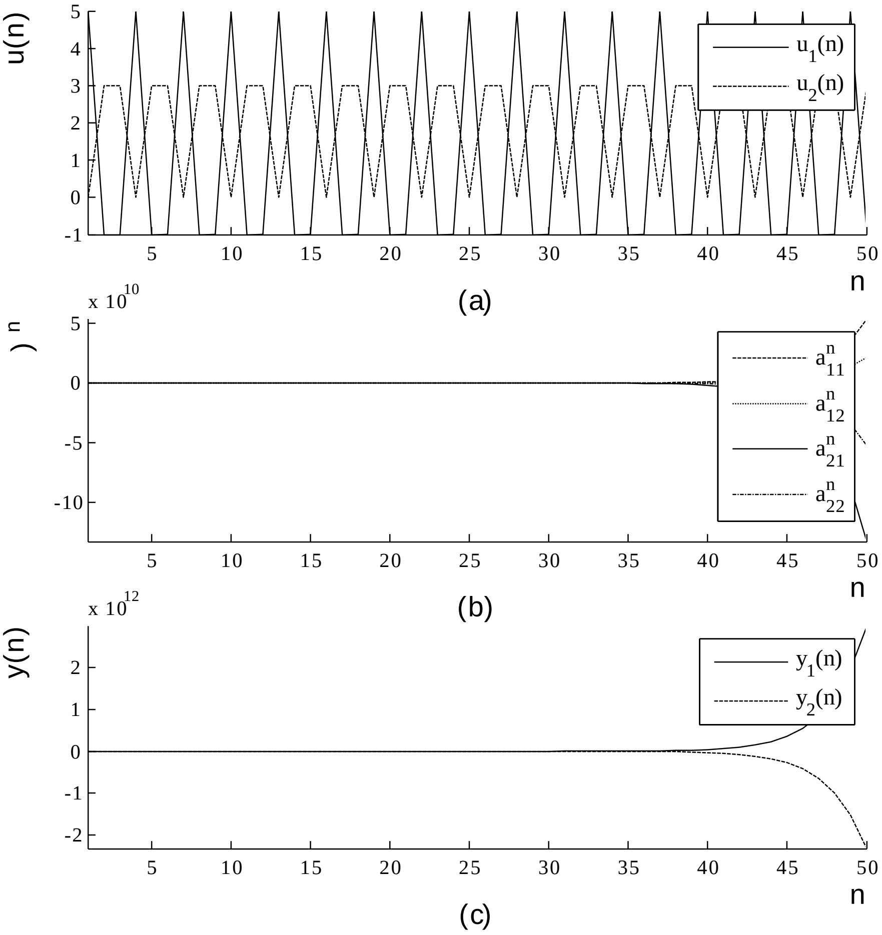


Figure 7: Input (part A) and output (part B) sequences of the Ho-Kalman model at

$u=\left(\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix},\dots\right)$; $\Phi=\begin{bmatrix}4 & 1\\ -\frac{25}{4} & -1\end{bmatrix}$; $G=\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$ and $H=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$. Eigenvalues of Φ are outside the unit circle. The elements of Φ tends to infinity and the output sequences also diverges at the same exponential speed.

7.Conclusions

The main objective of this paper is to demonstrate that the Ho-Kalman model is a transformer of linear recurring sequences – what is far from being trivial. Algebraic expressions of the output sequence produced by the Ho-Kalman model are derived in an explicit form; different scenarios among the roots of the characteristic equation of the input signal and the eigenvalues of the system matrix Φ are considered. These results provide the insight in the complexity of processes governed by the Ho-Kalman model. A number of computational experiments are used to illustrate the theoretical results.

All derivations and computations are performed when the dimensions of the input, the output signals and the system matrices are equal to 2. These results could be extrapolated to higher dimensions without losing the generality - though the applicability of the discussed properties of the Ho-Kalman model for complex real-world systems remains a definite topic of future research.

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