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Perturbation and Linear Delay Equation Results on $\omega-$Order Reversing Partial Contraction Mapping in Semigroup of Linear Operator

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Abstract

In this paper, perturbation results were established by proving a sufficient condition in order for the sum of an infinitesimal generator of a C_0-Semigroup of contraction and also showed that C_0-Semigroups generated by symmetric operators in Hilbert spaces are differentiable by obtaining results on linear delay equations.

Keywords: ω-$ORCP_n$, Perturbation, Differentiable semigroup, and C_0-Semigroup.

1. INTRODUCTION

Perturbation is considered an important aspect of pure and applied mathematics because of its well-posed nature. Let X be a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega-ORCP_n$ be ω-order-reversing partial contraction mapping which is an example of C_0-semigroup, $\omega - ORCP_n \subseteq ORCP_n$ (Order Reversing Partial Contraction Mapping). Let $Mm(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ the bounded linear operator in X, P_n, the partial transformation semigroup, $\rho(A)$ a resolvent of A, where A is the generator of a semigroup of linear operator. In this paper, condition for the sum of an infinitesimal generator of a C_0-Semigroup of contraction with a densely defined linear operator and linear delay equation will be investigated. Akinyele *et al.* [1], introduced some dissipative properties of ω-order-preserving partial contraction mapping in semigroup of linear operator. Engel and Nagel [3], obtained one-parameter semigroup for linear evolution equations. Feller [4], deduced parabolic differential equations and associated semigroups of transformation. Gutman [5], introduced some compact perturbations of m-accretive operators in general Banach spaces. Hale [6], established functional differential equations in applied mathematics. Neerven [7], established adjoint of a semigroup of linear operators. Pazy [8], proved perturbation theorem for linear m-dissipative operators. Thieme [9] showed some positive perturbation of operator semigroups. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established its properties, also in [11], Rauf *et al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [12], characterized new generator of differentiable semigroups and also in [13], Vrabie deduced some results of C_0-semigroup and its applications. Yosida [14], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. PRELIMINARIES

Definition 2.1 ($C_0 - Semigroup$) [13]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (Differentiable Semigroup) [13]
A C_0-Semigroup is called:
(i) differentiable at $\tau \geq 0$, if, for each $x \in X$, the function $t \to T(t)x$ is differentiable at τ;
(ii) differentiable, if it differentiable at each $\tau \in (0, +\infty)$; and
(iii) eventually differentiable if there exists $\theta > 0$ such that $t \to T(t)x$ is differentiable at each $\tau \in (\theta, +\infty)$.
Definition 2.3 (Perturbation) [3]
Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A+B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A is perturbed by operator B or that B is a perturbation of A.
Definition 2.4 (ω-$ORCP_n$) [10]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in$Dom$\alpha : x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t + s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

A C_0-semigroup $\{T(t); t \geq 0\}$ is called of type (M, ω) with $M \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have

$\|T(t)\|_{L(X)} \leq Me^{t\omega}$.

A C_0-semigroup $\{T(t); t \geq 0\}$ is called a C_0-semigroup of contraction or non expansive operator, if it is of type $(1, 0)$, that is, if for each $t \geq 0$, we have

$\|T(t)\|_{L(X)} \leq 1$.

Example 1

3×3 matrix $[M_m(\mathbb{C})]$, we have

for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose we have

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{3t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{3t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2

Suppose $A : D(A) \subseteq X \to X$ is an unbounded generator of a strongly continuous semigroup and take an isomorphism $S \in L(X)$ such that $D(A) \cap S(D(A)) = \{0\}$. Then $B = SAS^{-1}$ is a generator as well, but $A + B$ is defined only on $D(A + B) = D(A) \cap D(B) = D(A) \cap S(D(A)) = \{0\}$.

A concrete example for this situation is given on $X = C_0(\mathbb{R}_+)$ by $Af = f'$ with its canonical domain $D(A) = C_0^{'}(\mathbb{R}_+)$ and $Sf = q.f$ for some continuous, positive function q such that q and q^{-1} are bounded and nowhere differentiable. Defining the operator B as $Bf = q.(q^1.f)'$ on $D(B) = \{f \in X : q^{-1}.f \in D(A)\}$, we obtain that the sum $A + B$ is defined only on $\{0\}$.

Example 3

Let $r > 0$ and $X = C([-r, 0]; \mathbb{R}^n)$ which, endowed with the sup-norm, is Banach space if $x : [-r, +\infty) \to \mathbb{R}^n$ is continuous, then for each $t \geq 0$, the function $x_t : [-r, 0) \to \mathbb{R}^n$ by $x_t(\theta) = x(t + \theta)$, for each $\theta \in [-r, 0]$, belongs to X. Let $L : X \to \mathbb{R}^n$ be a linear continuous operator $\varphi \in X$ and consider the Cauchy problem for the linear delay equation

$$\begin{cases} x^{'}(t) = Ax_1 \;\; t \geq 0 \\ x(s) = \varphi(s) \;\; s \in [-r, 0]. \end{cases}$$

By a solution of this problem e mean a function $x \in C([-r, +\infty); \mathbb{R}^n)$ with the property $x_{|[0,+\infty)} \in C^{'}([0, +\infty); \mathbb{R}^n)$ and satisfying $x(s) = \varphi(s)$, for each $s \in [-r, 0]$ and $x^{'}(t) = Lx_t$ for each $t \geq 0$.

Theorem 2.1 [13]
Let $n \in \mathbb{N}^*$. Then for each $\xi \in D(A^n)$ and $t \geq 0$, we have $T(t)\xi \in D(A^n)$, the function $u : [0, +\infty) \to X$, $u(t) = T(t)\xi$ is of class C^n and is a solution of Cauchy problem

$$\begin{cases} u^n(t) = A^n u(t) \;\; t \geq 0 \\ u^k(0) = A^k \xi \;\; k = 0, 1, ..., n-1. \end{cases}$$

3. MAIN RESULTS

In this section, perturbation and linear delay equation results on ω-$ORCP_n$ in semigroup of linear operator (C_0-semigroup) were established:

Theorem 3.1
Suppose $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ are two linear operators with $D(A) \subseteq D(B)$, and such that

$\|\lambda x - (A + tB)x\| \geq \lambda \|x\|$
for all $\lambda > 0$, $t \in [0, 1]$, $x \in D(A)$ and $A, B \in \omega$-$ORCP_n$, and

$\|Bx\| \leq \alpha \|Ax\| + \beta \|x\|$
for each $x \in D(A)$, where $\alpha \in [0, 1)$ and $\beta > 0$. If there exists $s \in [0, 1]$ such that $A + sB$ is closed, and $(0, +\infty) \subseteq \rho(A + sB)$, then we have $(0, +\infty) \subseteq \rho(A + tB)$ for each $t \in [0, 1]$. In this case for each $t \in [0, 1]$, $A + tB$ is the infinitesimal generator of a C_0-Semigroup of contractions.

Proof :
Let $s \in [0, 1]$ be such that, for each $\lambda > 0$, $\lambda I - (A + sB)$ is invertible, and $(\lambda I - (A + sB))^{-1} \in L(X)$. To complete the proof, it suffices to show that there exists $\delta > 0$ such that, for each $t \in [0, 1]$ with $|t - s| \leq \delta$, we have $(0, +\infty) \subseteq \rho(A + tB)$. From

$$\|\lambda x - (A + tB)x\| \geq \lambda \|x\|, \tag{3.1}$$

it follows that

$$\|R(\lambda; A + sB)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{3.2}$$

We shall show next that

$BR(\lambda; A+sB) \in L(X)$. From

$$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|, \tag{3.3}$$

it follows that

$$\begin{aligned} \|Bx\| \leq \alpha\|Ax\| + \beta\|x\| &\leq \alpha\|(A+sB)x\| + \alpha s\|Bx\| + \beta\|x\| \\ &\leq \alpha\|(A+sB)x\| + \alpha\|Bx\| + \beta\|x\| \end{aligned} \tag{3.4}$$

for each $x \in D(A)$ and $A, B \in \omega\text{-}ORCP_n$. Accordingly

$$\|Bx\| \leq \frac{\alpha}{1-\alpha}\|(A+sB)x\| + \frac{\beta}{1-\alpha}\|x\|. \tag{3.5}$$

Since $R(\lambda; A+sB) : X \to D(A)$ and

$$(A+sB)R(\lambda; A+sB) = \lambda R(\lambda; A+sB) - I, \tag{3.6}$$

from (3.5), it follows that

$$\begin{aligned} \|BR(\lambda; A+sB)x\| \leq \frac{\alpha}{1-\alpha}\|\lambda R(\lambda; A+sB) - Ix\| + \frac{\beta}{1-\alpha}\|R(\lambda; A+sB)x\| \\ \leq \frac{2\lambda\alpha+\beta}{\lambda(1-\alpha)}\|x\|, \end{aligned} \tag{3.7}$$

for each $x \in X$. Consequently $BR(\lambda; A+sB) \in L(X)$, and

$$\|BR(\lambda; A+sB)\|_{L(X)} \leq \frac{2\lambda\alpha+\beta}{\lambda(1-\alpha)}. \tag{3.8}$$

At this point let us observe that

$$\lambda I-(A+tB) = \lambda I-(A+sB)+(s-t)B = (I+(s-t)BR(\lambda; A+sB))(\lambda I-(A+sB)). \tag{3.9}$$

Then $\lambda I - (A+tB)$ is invertible if and only if $I + (s-t)BR(\lambda; A+sB)$ enjoys the same property. But the latter is invertible for each $t \in [0,1]$ with

$|t-s| < \lambda(1-\alpha)(2\alpha\lambda+\beta)^{-1} \leq \|BR(\lambda; A+sB)\|^{-1}_{L(X)}$. Taking $\delta < \lambda(1-\alpha)(2\alpha\lambda+\beta)^{-1}$, let us observe that $\lambda I - (A+tB)$ is invertible for each $t \in [0,1]$ with $|t-s| \leq \delta$. Since $[0,1]$ can be covered by a finite union of intervals whose lenght is less than δ, then the proof is complete.

Theorem 3.2
Let $A : D(A) \subseteq H \to H$ be the infinitesimal generator of a C_0-semigroup of contraction $\{T(t); t \leq 0\}$, such that $A \in \omega$-$ORCP_n$. Let $\xi \in H$, and $u(t) = T(t)\xi$ for $t \geq 0$. Suppose A is self-adjoint, then
(i) $u \in C([0, +\infty); H \cap C((0, +\infty); D(A)) \cap C^{'}(0, +\infty)H)$ and u is the unique solution of the Cauchy problem

$$\begin{cases} u^{'}(t) = Au \\ u(0) = \xi \end{cases}$$

in this space;
(ii) $\|Au(t)\| \leq \frac{1}{t\sqrt{2}}\|\xi\|$ for each $t > 0$;
(iii) the function $t \to \sqrt{t}\|Au(t)\|$ belongs to $L^2(0, +\infty)$ and
$\int_0^\infty s\|Au(s)\|^2 ds \leq \frac{1}{4}\|\xi\|^2$;
(iv) if $\xi \in D(A)$, then $\|Au(t)\|^2 \leq \frac{1}{2t} < -A\xi, \xi >$ for each $t > 0$; and
(v) $t \to Au(t) \in L^2(0, +\infty)$ and $\int_0^t \|Au(s)\|^2 ds \leq \frac{1}{2} < -A\xi, \xi >$.
Proof :
It is suffices to show a prove of necessity, let call it lemma 3.3 before we can proceed on the proof of this theorem.

Lemma 3.3
If $\{T(t); t \geq 0\}$ is a C_0-semigroup, where (A,D(A)) is the infinitesimal generator of the C_0-semigroup and $A \in \omega$-$ORCP_n$, then the mapping $(t, x) \to T(t)$ is jointly continuous from $[0, +\infty)xX \to X$.
Proof :
Let $x, y \in D(A)$, $A \in \omega$-$ORCP_n$, $t \geq 0$ and $h \in \mathbb{R}$ with $t + h \geq 0$. We distinguish between two cases : $h > 0$ or $h < 0$. If $h > 0$, we have

$$\begin{aligned} \|T(t+h)y - T(t)x\| &\leq \|T(t+h)y - T(t+h)x\| + \|T(t+h)x - T(t)x\| \\ &\leq \|T(t+h)\|_{L(X)}\|y - x\| + \|T(t+h)x - T(t)x\| \\ &\leq Me^{(t+h)\omega}\|x - y\| + \|T(t)\|_{L(X)}\|T(h)x - x\| \end{aligned} \tag{3.10}$$

for each $M \geq 1$ and $\omega \in \mathbb{R}$, which shows that

$$\lim_{(\tau,y)\to(t+0,x)} T(\tau)y = T(t)x. \tag{3.11}$$

If $h < 0$ and by definition (2.5), we deduced

$$\begin{aligned} \|T(t+h)y - T(t)x\| &= \|T(t+h)y - T(t+h)T(-h)x\| \\ &\leq \|T(t+h)\|_{L(X)}\|y - T(-h)x\| \\ &\leq Me^{(t+h)\omega}(\|y - x\| + \|T(-h)x - x\|) \end{aligned} \tag{3.12}$$

for each $M \geq 1$ and $\omega \in \mathbb{R}$, which implies that

$$\lim_{(\tau,y)\to(t-0,x)} T(\tau)y = T(t)x. \tag{3.13}$$

Now, back to the proof of the theorem. Suppose $\xi \in D(A^2)$ and by virtue of Lemma (3.3) above, it follows that the problem

$$\begin{cases} u^{'}(t) = Au \\ u(0) = \xi \end{cases}$$

has a unique solution u of class C^2 on $\mathbb{R}_+$ which satisfies $u^{''}(t) = Au^{'}(t)$ for each $t \in \mathbb{R}_+$. Taking the inner product of both sides in the problem above by $u^{'}(t)$, integrating from s to t, with $0 \leq s \leq t$, and using dissipative property (see []$< Ax, x >\leq 0$, we deduced

$$\frac{1}{2}\|u^{'}(t)\|^2 - \frac{1}{2}\|u^{'}(s)\|^2 = \int_s^t < Au^{'}(\tau), u^{'}(\tau) > d\tau \leq 0, \tag{3.14}$$

which shows that function $t \to \|u^{'}(t)\|$ is non-increasing. We also have that

$$\frac{d}{dt}\|u(t)\|^2 = 2 < Au(t), u(t) >, \tag{3.15}$$

and

$$\frac{d}{dt} < Au(t), u(t) >= 2 < Au(t), u^{'}(t) >= 2\|u^{'}(t)\|^2 \geq 0. \tag{3.16}$$

Accordingly, $t \to< Au(t), u(t) >$ is non-decreasing. Integrating (3.15) from 0 to t, we successively obtained

$$\|u(t)\|^2 - \|\xi\|^2 = 2\int_0^t < Au(s), u(s) > ds \leq 2t < Au(t), u(t) > \tag{3.17}$$

and

$$\begin{aligned} -t < Au(t), u(t) >\leq & -\int_0^t < Au(s), u(s) > ds \\ & = -\frac{1}{2}\|u(t)\|^2 + \|\frac{1}{2}\|\xi\|^2 \\ & \leq \frac{1}{2}\|\xi\|^2. \end{aligned} \tag{3.18}$$

Integrating (3.16) from 0 to t and recalling that the function $t \to \|u^{'}(t)\|$ is non-increasing, we then obtained

$$< Au(t), u(t) > - < A\xi, \xi >= 2\int_0^t \|u^{'}(s)\|^2 ds \geq 2t\|u^{'}(t)\|^2. \tag{3.19}$$

Since $< Au(t), u(t) >\leq 0$, it follow that

$$2t\|u^{'}(t)\|^2 \ \leq \ < -A\xi, \xi > . \tag{3.20}$$

Multiplying both sides in (3.16) by t and integrating, we obtained

$$\begin{aligned} 2t^2\|u^{'}(t)\|^2 \leq \int_0^t s < Au(s), u^{'}(s) > ds = \int_0^t s\frac{d}{ds} < Au(s), u(s) > ds \\ = t < Au(t), u(t) > - \int_0^t < Au(s), u(s) > ds. \end{aligned} \tag{3.21}$$

Since $t < Au(t), u(t) > \ \leq \ 0$, from (3.15) and (3.21), we deduced

$$2t^2\|u^{'}(t)\|^2 \leq \|\xi\|^2. \tag{3.22}$$

Accordingly,

$$\|Au(t)\| \leq \frac{1}{t\sqrt{2}}\|\xi\| \tag{3.23}$$

for each $\xi \in D(A^2)$, $A \in \omega\text{-}0RCP_n$ and $t \geq 0$. Since $D(A^2)$ is dense in H and the mapping $\xi \to u(t)$ is non-expansive, by (3.23) above we obtained (ii). Since (i) and (iii) follow from (ii), while (iv) and (v) are consequences of (3.20), and we need to note that all the items in conclusion of the theorem holds also if A is symmetric and generates a C_0-semigroup of contractions on H. Hence, the prove is complete.

Theorem 3.4
Let $A : D(A) \subseteq H \rightarrow H$ be a self-adjoint and negatively defined operator, if $\{T(t); t \geq 0\}$ is C_0-semigroup of contractions generated by A, where $A \in \omega\text{-}0RCP_n$, $\xi \in H$ and $u(t) = T(t)\xi$ for $t \geq 0$. Suppose $(x_n)_{n\in\mathbb{N}}$ is sequence of spaces, then, for each $n \in \mathbb{N}^*$, $u \in C^{\infty}((0, +\infty); X_n)$ and
$\|A^n u(t)\| \leq (\frac{n}{t\sqrt{2}})^n\|x\|$
for each $t > 0$.
Proof :
Let X be a Banach space endowed with the graph-norm $|.|_{D(A)}$ and X_1 is a Hibert space with respect to the inner product $< ., . >_1$, defined by

$$< x, y >_1 \ = \ < x - Ax, y - Ay > \tag{3.24}$$

for each $x, y \in X_1$ and $A \in \omega\text{-}0RCP_n$. In addition, $A_{(1)}$ is self-adjoint on X_1. Indeed, we have

$$\begin{aligned} < A_{(1)}x, y >_1 \ &= \ < Ax - A^2x, y - Ay > \\ = \ < Ax, y > - < Ax, Ay > - < A^2x, y > + < A^2x, Ay > \\ = \ < x, Ay > - < x, A^2y > - < Ax, Ay > + < Ax, A^2y > \\ &= \ < x, A_{(1)}y >_1 \end{aligned} \tag{3.25}$$

for each $x, y \in D(A_{(1)})$ and $A_{(1)} \in \omega$-$0RCP_n$. Thus $A_{(1)}$ is symmetric. Since its generates a C_0-semigroup of contractions on X_1, it follows that $(I - A_{(1)})^{-1} \in L(X)$. by [6]. it follows that $A_{(1)}$ is self-adjoint. A simple inductive argument shows that, for $n \in \mathbb{N}^*$, X_n is a Hilbert space and A_n is self-adjoint. Let $t > 0$ and $n \in \mathbb{N}^*$. From (ii) in Theorem (3.2), we deduced that

$$\|Au(\frac{t}{n})\| \leq \frac{n}{t\sqrt{2}}\|x\|. \tag{3.26}$$

Assume $u(2t/n) = T(t/n)u(t/n)x \in X_2$, applying once again in (ii) of Theorem (3.2), this time to operator $A_{(1)}$, we obtained

$$\|A^2u(\frac{2t}{n})\| \leq (\frac{n}{t\sqrt{2}})^2\|x\|. \tag{3.27}$$

By induction, we conclude that

$$\|A^nu(t)\| \leq (\frac{n}{t\sqrt{2}})^n\|x\|. \tag{3.28}$$

To complete the proof, we need to show that, for each $n \in \mathbb{N}^*$, $u \in C^\infty((0, +\infty); X_n)$. But this follows from the simple remark that, for each $x \in H$ and $t > 0$, we have $u(t) \in \cap_{n\geq 0}X_n$, combined with Theorem (2.1), hence the proof is achieved.

Theorem 3.5

Assume $(A, D(A))$ is the infinitesimal generator of C_0-semigroup $\{T(t); t \geq 0\}$, where $A \in \omega$-$0RCP_n$ such that for each $\varphi \in X$, the Cauchy problem for the linear delay equation

$$\begin{cases} x^{'}(t) = Ax_t \;\; t \geq 0 \\ x(t) = \varphi(t) \end{cases}$$

has a unique solution $x : [-r, +\infty) \to \mathbb{R}^n_+$.

Proof :

Let us observe that this is equivalent with the delay integral equation

$$x(t) = \begin{cases} \varphi(t) & for \;\; t \in [-r, 0] \\ \varphi(0) + \int_0^t Ax_s ds & for \;\; t \in (0, +\infty) \end{cases} \tag{3.29}$$

Therefore, to complete the proof, it suffices to show that, for each $\varphi \in X$, $A \in \omega$-$0RCP_n$ and each $T > 0$, then (3.29) has a unique solution $x : [-r, T] \to \mathbb{R}^n$. In order to show this, let $Y = C([-r, T]; \mathbb{R}^n)$, which, endowed with sup-norm $\|.\|_Y$, is a Banach space and let $Q : Y \to Y$ be defined by

$$(Qy)(t) = \begin{cases} \varphi(t) & for \;\; t \in [-r, 0] \\ \varphi(0) + \int_0^t Ay_s ds & for \;\; t \in (0, T] \end{cases} \tag{3.30}$$

Let us observe that $x \in Y$ is a solution of the delay integral equation (3.30) if and only if x is a fixed point of Q. So, in that regards, we shall prove that Q has a unique fixed point. A simple inductive argument shows that, for each $k \in \mathbb{N}^*$ and each $t \in [-r, T]$, we have

$$\|(Q^k y)(t) - (Q^k z)(t)\| \le \frac{\|A\|^k t^k}{k!}\|x - y\|_Y. \tag{3.31}$$

Clearly, (3.31) implies

$$\|Q^k y - Q^k z\|_Y \le \frac{\|A\|^k T^k}{k!}\|x - y\|_Y. \tag{3.32}$$

Therefore, for sufficiently large k, Q^k is a strict contraction, which by the Banach fixed point theorem, has a unique fixed point $x \in Y$. Assume we have

$$\|Qx - x\| = \|Q^k Q^x - Q^k x\| \le q\|Qx - x\| \tag{3.33}$$

with $a \in (0, 1)$, it follows that x is a fixed point of Q. In addition, this is unique because each one of its fixed points is necessarily a fixed point of Q^k and this complete the proof.

Theorem 3.6
The family $\{T(t); t \ge 0\}$ is a semigroup of class C_0 in X whose generator $A : D(A) \subseteq X \in X$, is given by $D(A) = \{\varphi \in C^{'}([-r, 0]); \mathbb{R}^n) : \varphi^{'}(0) = L\varphi\}$, where $A \in \omega\text{-}0RCP_n$ and

$$(A\varphi)(\theta) = \begin{cases} \varphi^{'}(\theta) & for \ \ \theta \in [-r, 0) \\ L\varphi & for \ \ \theta = 0 \end{cases}$$

In addition, for each $t \ge r$, $T(t)$ is a compact operator and the semigroup is differentiable on $[r, +\infty)$

Proof :
Let us observe that, for each $t \ge 0$, and each $\theta \in [-r, 0]$, we have

$$[T(t)\varphi](\theta) = \begin{cases} \varphi(t + \theta) & for \ \ t + \theta \le 0 \\ \varphi(0) + \displaystyle\int_0^{t+\theta} L(T(\tau)\varphi)d\tau & for \ \ t + \theta > 0 \end{cases} \tag{3.34}$$

From this representation formula, after a simple calculation and by Theorem (3.5), we deduced the expression of A. From this, it follows that $T(t)$ is compact for each $t \ge r$ and the semigroup is differentiable on $[r, +\infty)$. Hence the proof is complete.

4. CONCLUSION

In this paper, perturbation and linear delay equation on $\omega - ORCP_n$ were obtained and also we were able to established that generator A is self-adjoint, a symmetric A also generates a C_0-semigroup of contraction and a semigroup generated by A is differentiable.

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