

On Quantile Approach of Generalized Quasi-Log Probabilistic Theoretic Measure with Edge Detection and Object Pixel Threshold Grey Values

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Abstract

The study looks at a novel probabilistic entropy for a probability distribution $P = \{p_1, p_2, \dots, p_n\}$ and its characteristics. Using a non-Shannon measure of entropy, we additionally establish the noisy image's threshold value.

Introduction

Entropy is a term used to describe the degree of disorder in a physical system. Shannon reformulated the Boltzmann/Gibbs entropy notion as a measure of uncertainty about the mathematical content of a system in [15]. He proposed a formula for calculating the quantity of information a process generates in terms of numbers.

According to this definition, a random event A is said to contain $I(A) = \ln[1/P(A)] = -\ln[P(A)]$ units of information if it occurs with probability $P(A)$. $I(A)$ is referred to as the self-information of event A . The relationship between an event's self-information and probability is inverse. $I(A) = 0$ and no information is given to it if $P(A) = 1$. In

this case, the event's range of uncertainty is zero. Therefore, if the event constantly happens, says that it has happened would not transfer any information. By saying that the event has occurred, some information can be explained if $P(A) = 0.8$ [14].

The level of uncertainty there is in a signal or in a random occurrence is essential to the information theory concept of entropy. Consider the amount of information that the signal can carry as a different point of view [8]. Randomness can be measured by entropy.

Let $p_1, p_2, \dots, p_k$ represent the discrete source's probability distribution. Consequently, $0 \leq p_i \leq 1, i = 1, 2, \dots, k$ and $\sum_{i=1}^k p_i = 1$ where k is the total number of states. The probability distribution is frequently used to determine the entropy of a discrete source. The definition of the Shannon entropy is

$$H(P) = - \sum_{i=1}^k p_i \ln(p_i)$$

It has been established that this formalism is limited to the Boltzmann-Gibbs-Shannon (BGS) statistics' valid domain. When useful microscopic interactions and microscopic memory are short-ranged, these statistics appear to describe nature. Extensive systems are usually referred to as systems that follow BGS statistics. If we consider that a physical system can be divided into two statistically independent subsystems A and B , the probability of the composite system [23] is $p^{A+B} = p^A \cdot p^B$, it has been proven that the Shannon entropy has the extended property (additive):

$$H(A + B) = H(A) + H(B) \quad (1.1)$$

and

$$H_\alpha(A + B) = H_\alpha(A) + H_\alpha(B) + \varphi(\alpha) \cdot H_\alpha(A) \cdot H_\alpha(B), \quad (1.2)$$

Where $\varphi(\alpha)$ is depends on the entropy index. In Shannon entropy $\varphi(\alpha) = 1$. The generalised distribution's Rènyi entropy[12] can be expressed as follows:

$$H_\alpha^R(P) = \frac{1}{1-\alpha} \ln \sum_{i=1}^k (p_i)^\alpha, \alpha > 0, \quad (1.3)$$

When Havrda and Charvat [5] originally developed non-additive entropy, it was as follows:

$$H_{\alpha}(P) = \frac{[\sum_{i=1}^n p_i^{\alpha}]^{-1}}{2^{1-\alpha}-1}, \alpha \neq 1, \alpha > 0 \quad (1.4)$$

Renyi's [12] measure was further generalised by Kapur [7] to generate an entropy measure of order α and β is

$$H_{\alpha,\beta}(P) = \frac{1}{1-\alpha} \ln \left(\frac{\sum_{i=1}^n p_i^{\alpha+\beta-1}}{\sum_{i=1}^n p_i^{\beta}} \right), \alpha \neq 1, \alpha > 0, \beta > 0, \alpha + \beta - 1 > 0 \quad (1.5)$$

Brissaud [1], Chakrabarti [2], Chen [3], Garbaczewski [4], Herremoes [6], Lavenda [9], Nanda and Paul [10], Rao, Yunmei and Wang [11], Sergio [13], Sharma and Taneja [16], Verma [17, 18, 19] and many others have analysed and produced numerous alternative probabilistic measures of entropy. Applications of the findings from different authors have been made to different mathematical science domains. We present a new generalised probabilistic information theoretic measure in this study.

2. Our Work

2.1 Based on Probability Distributions, a Novel Generalised Information Theoretic Tool

This section examines the key characteristics and desirable attributes of a new generalised information measure we propose for a probability distribution $P = \{(p_1, p_2, \dots, p_n), p_i \geq 0, \sum_{i=1}^n p_i = 1\}$. This generalised entropy is determined by the following mathematical expression, which depends on n real parameters $\alpha_1, \alpha_2, \dots, \alpha_n$.

$$H_{\alpha, \alpha_1, \dots, \alpha_n}^n(P) = \frac{\ln \sum_{i=1}^n p_i^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \quad (2.1.1)$$

Where

$$\alpha + \sum_{i=1}^n \alpha_i \neq 1, \alpha + \sum_{i=1}^n \alpha_i > 1 \text{ and } \alpha \neq 1, \alpha > 0, \alpha_i \geq 0 \quad (2.1.2)$$

If $\sum_{i=1}^n \alpha_i = 0$, then $\alpha \neq 1, \alpha > 0$.

Thus, The recommended measure (1.1) is evidently transformed as

$$H_{\alpha}^n(P) = \frac{\ln \sum_{i=1}^n p_i^{\alpha}}{1-\alpha} \quad (2.1.3)$$

It represents the entropy of order as measured by Verma [18] and Renyi's entropies [11]. We then look at a few important features of this generalised measure. The measure (2.1.1) satisfies the criteria listed below:

- (i) It is a continuous function of $p_1, p_2, \dots, p_n$, meaning that minor changes in $p_1, p_2, \dots, p_n$ cause a small change in it.
- (ii) It is a permutationally symmetric function of $p_1, p_2, \dots, p_n$, meaning that it remains unchanged when $p_1, p_2, \dots, p_n$ are permuted among one another.

$$(iii) \quad H_{\alpha, \alpha_1, \dots, \alpha_n}^n(P) \geq 0$$

$$(iv) \quad H_{\alpha, \alpha_1, \dots, \alpha_n}^{n+1}(p_1, p_2, \dots, p_n, 0) = \frac{\ln \sum_{i=1}^n p_i^{\alpha+\alpha_1+\dots+\alpha_n}}{1-\alpha-\alpha_1-\dots-\alpha_n} = H_{\alpha, \alpha_1, \dots, \alpha_n}^n(P)$$

According to this property, the inclusion of an impossible event with probability zero has no effect on entropy.

- (v) $H_{\alpha, \alpha_1, \dots, \alpha_n}^n(P)$ must reach its maximum value because it is an entropy measure.

We go about finding the largest value as follows:

Let

$$\frac{\ln \sum_{i=1}^n p_i^{\alpha+\alpha_1+\dots+\alpha_n}}{1-\alpha-\alpha_1-\dots-\alpha_n} - \lambda \left(\ln \sum_{i=1}^n p_i \right)$$

Then, we have

$$\frac{\partial f}{\partial p_1} = \frac{1}{p_1} \left(\frac{\alpha + \alpha_1 + \dots + \alpha_n}{1 - \alpha - \alpha_1 - \dots - \alpha_n} - \lambda \right)$$

$$\frac{\partial f}{\partial p_2} = \frac{1}{p_2} \left(\frac{\alpha + \alpha_1 + \dots + \alpha_n}{1 - \alpha - \alpha_1 - \dots - \alpha_n} - \lambda \right)$$

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$$\frac{\partial f}{\partial p_n} = \frac{1}{p_n} \left(\frac{\alpha + \alpha_1 + \dots + \alpha_n}{1 - \alpha - \alpha_1 - \dots - \alpha_n} - \lambda \right)$$

For maximum value, we take

$$\frac{\partial f}{\partial p_1} = \frac{\partial f}{\partial p_2} = \dots = \frac{\partial f}{\partial p_n} = 0$$

Thus

$$\sum_{i=1}^n p_i = 1 \text{ gives } p_1 = p = p_n = \frac{1}{n}$$

In light of this, we can see that the generalised entropy measure (2.1.1) has a maximum value, which is subject to the natural constraint that $\sum_{i=1}^n p_i = 1$ and obtains when $p_1 = p_2 = \dots = p_n = \frac{1}{n}$. This outcome is the ideal one.

(vi) An increasing function of n determines the maximum value. To prove this conclusion, we have,

$$f(n) = \frac{n^{1-\alpha-\alpha_1-\dots-\alpha_n}}{1-\alpha-\alpha_1-\dots-\alpha_n}$$

Thus

$$f'(n) = \frac{1}{n^{\alpha+\alpha_1+\dots+\alpha_n}} > 0$$

Since

$$\alpha + \sum_{i=1}^n \alpha_i \neq 1, \alpha + \sum_{i=1}^n \alpha_i > 1$$

Therefore, the maximum value is an increasing function of n .

(vi) Additive property: To show that the measure (2.1.1) is additive, we consider

$$\begin{aligned}
H_{\alpha, \alpha_1, \dots, \alpha_n}^{n,m}(P \cup Q) &= \frac{\ln \sum_{i=1}^n \sum_{j=1}^m (p_i q_j)^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \\
&= \left(\frac{\ln \sum_{i=1}^n p_i^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \left(\ln \sum_{j=1}^m q_j^{\alpha + \alpha_1 + \dots + \alpha_n} \right) + \left(\frac{\ln \sum_{i=1}^n p_i^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \\
&\quad + \left(\frac{\ln \sum_{j=1}^m q_j^{\alpha - \alpha_1 - \dots - \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \\
&= (1 - \alpha - \alpha_1 - \dots - \alpha_n) \left(\frac{\ln \sum_{i=1}^n p_i^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \left(\ln \sum_{j=1}^m \frac{q_j^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \\
&\quad + \left(\frac{\ln \sum_{i=1}^n p_i^{\alpha + \alpha_1 + \dots + \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) + \left(\frac{\ln \sum_{j=1}^m q_j^{\alpha - \alpha_1 - \dots - \alpha_n}}{1 - \alpha - \alpha_1 - \dots - \alpha_n} \right) \\
&= (1 - \alpha - \alpha_1 - \dots - \alpha_n) H^n(P) \cdot H^m(Q) + H^n(P) + H^m(Q) \tag{2.1.4}
\end{aligned}$$

which shows that the generalized entropy (2.1.1) is additive.

2.2 RENYI ENTROPY-BASED THRESHOLD VALUE SELECTION

A new image is generated by thresholding based on the previous image represented by f . It effectively makes a new thresholded image by using a different function, $g(x,y)$. For every pixel value, a threshold is identified. The new value of the current pixel is calculated by comparing this threshold to the original image. The following equation [7] can be used to express g .

$$g(x,y) = \begin{cases} 0 & \text{if } f(x,y) \leq t \\ 1, & \text{if } f(x,y) > t \end{cases}, t \text{ is thresholding value.}$$

Entropy evaluates the normality of the particular grey level distribution of an image when it is used in image processing techniques. The Renyi entropy as defined in (2.1.1) will show how much the intensity distribution is normal when the entire image is taken into account. The concept can be applied to picture segmentation, which is the division

of an image's foreground and background parts. Entropy is calculated for both regions, and the outcome indicates the way the segmentation occurred. For each region, two priori equations [20, 21, 22] are required in this situation.

Every grey level value is a potential candidate to be the threshold value when performing maximum entropy thresholding on images. Each value will be used to separate the pixels into two groups based on their affinity, as less or larger than the threshold value (t), and their respective grey levels.

Let h_t be the grey level histogram and p_t be the normalised histogram, where $p_t = h_t / (M \times N)$ is the normalised histogram and $p_1, p_2, \dots, p_t, p_{t+1}, \dots, p_k$ is the probability distribution for an image with k grey levels. Two probability distributions, one for the object itself (Type A) and the other for the background (Type B), can be derived from this distribution, and they are represented as follows:

$$p_A: \frac{p_1}{P_A}, \frac{p_2}{P_A}, \dots, \frac{p_t}{P_A}, \text{ and } p_B: \frac{p_{t+1}}{P_B}, \frac{p_{t+2}}{P_B}, \dots, \frac{p_k}{P_B}, \quad (2.2.1)$$

Where

$$P_A = \sum_{i=1}^t p_i, P_B = \sum_{i=t+1}^k p_i, t \text{ is the threshold value.} \quad (2.2.2)$$

The entropy of object pixels and the entropy of background pixels can be described as follows in terms of the definition of Renyi entropy of order α :

$$H_\alpha^A(t) = \frac{1}{1-\alpha} \ln \sum_{i=1}^t \left(\frac{p_i}{P_A} \right)^\alpha \quad \alpha > 0 \quad (2.2.3)$$

$$H_\alpha^B(t) = \frac{1}{1-\alpha} \ln \sum_{i=t+1}^k \left(\frac{p_i}{P_B} \right)^\alpha \quad \alpha > 0 \quad (2.2.4)$$

Renyi's entropy The threshold value t for the object and background determines how entropy, providing the additive property for statistically independent systems. Between the two classes (object and background), we aim to maximise the information measure. The brightness level t that maximises the function is thought of as the ideal threshold value when $H_\alpha(t)$ is maximised.

$$t^{opt} = \text{Arg max}[(1 - \alpha)H_\alpha^A(t). H_\alpha^B(t) + H_\alpha^A(t) + H_\alpha^B(t)] \quad (2.2.5)$$

The threshold value in (1.4) is equal to the value determined by Shannon entropy when $\alpha \rightarrow 1$. Shannon's method is thus included as a particular case in this suggested solution. For the best threshold at $\alpha \rightarrow 1$, the following expression can be employed as a criterion function.

$$t_{sh}^{opt} = Arg \max[H_{\alpha}^A(t) + H_{\alpha}^B(t)] \quad (2.2.6)$$

The Renyi Threshold algorithm, which chooses an appropriate threshold value, may now be explained as follows:

Step 1: Input: A noisy image I of size $M \times N$.

Step 2: Let $f(x, y)$ represent the pixel's initial grey value at the coordinates (x, y) and calculate the probability distribution p_i .

Step 3: Calculate P_A, P_B, p_A , and p_B from equation (2.2.1) and (2.2.2).

Step 4: If $0 < \alpha < 1$ then use Equation (2.2.5) to determine the ideal threshold value t^{opt} otherwise use Equation (2.2.6) to determine the ideal threshold value t_{sh}^{opt} .

Step 5: Output: An appropriate threshold value t^{opt} of I for $\alpha > 0$.

Conclusion:

The proposed function, with the exception of additivity and recursivity, satisfies all the significant features that Shannon's measure of entropy satisfies, as shown by this study. This work presents an effective method for detecting edges in grayscale images using the Renyi entropy. The recommended method stands out with standard edge detectors.

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