DIFFERENT APPROACHES TO THE $H^p$ BOUNDEDNESS OF RIESZ TRANSFORMS

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Abstract
We give different approaches to show the $H^p$ boundedness of Riesz transforms for $0 < p \leq 1$.

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1 Introduction
Let $\phi$ be a function in $S(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. Define

$$\phi_r(x) = r^{-n} \phi(x/r), \quad r > 0, \quad x \in \mathbb{R}^n,$$

and the maximal function $f^*$ by

$$f^*(x) = \sup_{r>0} |f \ast \phi_r(x)|,$$

where the convolution operator “$\ast$” is given by

$$g \ast f(x) = \int_{\mathbb{R}^n} g(x - y)f(y) \, dy.$$

We say a tempered distributions $f \in S'(\mathbb{R}^n)$ is in the Hardy space $H^p(\mathbb{R}^n)$ if $f^*$ is in $L^p(\mathbb{R}^n)$. The quasi-norm on $H^p(\mathbb{R}^n)$ is $\|f\|_{H^p}^p \equiv \|f^*\|_{L^p}^p$, which satisfies

$$\|f + g\|_{H^p}^p \leq \|f\|_{H^p}^p + \|g\|_{H^p}^p \quad \text{for} \quad 0 < p \leq 1.$$

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When $p > 1$, $H^p$ and $L^p$ are essentially the same because of the celebrated theorem of Hardy and Littlewood

$$\|f^*\|_{L^p} \leq C_p \|f\|_{L^p} \quad \text{for} \quad p > 1;$$

however, when $p \leq 1$ the space $H^p$ is much better adapted to problems arising in the theory of harmonic analysis.

Let $R_j$, $j = 1, 2, \cdots, n$, denote the Riesz transforms in $\mathbb{R}^n$ defined by

$$R_j f(x) = \text{p.v.} K_j \ast f(x), \quad \text{where} \quad K_j(x) = \pi^{-(n+1)/2} \Gamma \left( \frac{n+1}{2} \right) \frac{x_j}{|x|^{n+1}}.$$  

For $n = 1$, the Riesz transforms reduce to the Hilbert transform

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x - y)}{y} dy.$$ 

Use $\hat{\cdot}$ and $\check{\cdot}$ to denote the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

and its inverse transform, respectively. Then

$$\check{R_j} \hat{f}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

This paper is concerned with the $H^p(\mathbb{R}^n)$ boundedness of the Riesz transforms.

Using the system of conjugate harmonic functions given by Stein and Weiss [7], we have another equivalent definition of Hardy spaces as follows. We consider $n + 1$ variables, $(X, y) = (x_1, x_2, \cdots, x_n, y)$, and suppose

$$F(X, y) = (u(X, y), v_1(X, y), v_2(X, y), \cdots, v_n(X, y))$$

is defined on the upper half space $\mathbb{R}^{n+1}_+ = \{(X, y) \in \mathbb{R}^n : y > 0\}$ satisfying the generalized Cauchy-Riemann equations,

$$\frac{\partial u}{\partial y} + \sum_{i=1}^n \frac{\partial v_i}{\partial x_i} = 0, \quad \frac{\partial u}{\partial x_i} = \frac{\partial v_i}{\partial y}, \quad i = 1, 2, \cdots, n,$$

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \quad i \neq j, \quad 1 \leq i, j \leq n.$$ 

These equations are assumed to hold on the upper half space $\mathbb{R}^{n+1}_+$. For $p > 0$, we say that $F \in H^p(\mathbb{R}^{n+1}_+)$ if

$$\|F\|_{H^p}^p := \sup_{y > 0} \int_{\mathbb{R}^n} \left( |u(X, y)|^2 + \sum_{i=1}^n |v_i(X, y)|^2 \right)^{p/2} dX < \infty.$$  

Stein and Weiss [7, page 46] showed that if $F(X, y)$ is in $H^p(\mathbb{R}^{n+1}_+)$, $p \geq (n - 1)/n$, then there exist boundary values $f(X) = F(X, 0)$ such that

$$\lim_{y \to 0} F(X, y) = f(X)$$

for almost every $X$ in $\mathbb{R}^n$. In case $p > (n - 1)/n$, $f(X)$ is also the limit in the norm of $F(X, y)$. 
On the other hand, let \( f \in H^1(\mathbb{R}^n) \) in the sense given in Introduction. We set \( f_j = R_j(f) \), \( j = 1, 2, \ldots, n \), the Riesz transforms of \( f \), and

\[
u(X, t) = P_t * f(X), \quad \nu_j(X, t) = P_t * f_j(X), \quad j = 1, 2, \ldots, n,
\]

where \( P_t \) is the Poisson kernel. Then,

\[
F(X, t) = (u(X, t), \nu_1(X, t), \nu_2(X, t), \ldots, \nu_n(X, t)) \in H^1(\mathbb{R}^{n+1}).
\]

Using the system of conjugate harmonic functions, Stein and Weiss [7] showed

**Theorem 1.1.** The Riesz transforms are bounded on \( H^1(\mathbb{R}^n) \).

By using the similar approaches indicated in [7], the Riesz transforms can be extended to \( H^p(\mathbb{R}^n) \) boundedness for \( (n-1)/n < p \leq 1 \). Later on Fefferman and Stein [3, Theorem 12] extended the result to \( 0 < p \leq 1 \) by checking the kernels of Riesz transforms

\[
K_j(x) = \pi^{-(n+1)/2} \Gamma \left( \frac{n+1}{2} \right) \frac{x_j}{|x|^{n+1}}
\]

being of class \( C^\infty \) away from the origin, and

\[
\left| \left( \frac{\partial}{\partial x} \right)^\alpha K_j(x) \right| \leq C |x|^{-n-|\alpha|} \quad \text{for all multi-indices } \alpha.
\]

**Theorem 1.2.** The Riesz transforms are bounded on \( H^p(\mathbb{R}^n), 0 < p \leq 1 \).

Next we shall give different approaches to the above result.

## 2 The Second Proof of Theorem 2

A bounded measurable function \( m \) defined on \( \mathbb{R}^n \) is said to be an \( H^p(\mathbb{R}^n) \) multiplier, \( 0 < p \leq \infty \), if \( f \in H^p \) implies \( (m \hat{f}) \in H^p \) and

\[
\| (m \hat{f}) \|_{H^p} \leq C_p \| f \|_{H^p} \quad \text{(with } C_p \text{ independent of } f \text{)}.
\]

The multiplier theorem was originally due to Hörmander [5], who considered the \( L^p(\mathbb{R}^n) \) multipliers. Later on Calderón and Torchinsky [1] extended the \( L^p(\mathbb{R}^n) \) multipliers to \( H^p(\mathbb{R}^n) \) multipliers:

**Theorem 2.1.** Let \( 0 < p \leq 1 \) and \( m \in C^k(\mathbb{R}^n \setminus \{0\}), k > n(1/p - 1/2) \). If \( m \in L^\infty(\mathbb{R}^n) \) and

\[
\sup_{R > 0} R^{2|\alpha| - n} \int_{R < |x| \leq 2R} \left| \left( \frac{\partial}{\partial x} \right)^\alpha m(x) \right|^2 dx < \infty \quad \text{for all } |\alpha| \leq k,
\]

then \( m \) is an \( H^p(\mathbb{R}^n) \) multiplier.
Another simple proof for Theorem 2 is to apply the above theorem and check the kernels of Riesz transforms satisfying the assumptions of Theorem 3.

Proof. [The second proof of Theorem 2] Write \( m_j(x) = -ix_j/|x| \). We then have

\[
\left| \left( \frac{\partial}{\partial x} \right)^\alpha m_j(x) \right| \leq C|x|^{-|\alpha|} \quad \text{for all multi-indices } \alpha.
\]

Hence,

\[
R^{2|\alpha| - n} \int_{R < |\alpha| \leq 2R} \left| \left( \frac{\partial}{\partial x} \right)^\alpha m_j(x) \right|^2 dx \leq CR^{2|\alpha| - n} \int_{R < |\alpha| \leq 2R} |x|^{-2|\alpha|} dx
\]

\[
= CR^{2|\alpha| - n} \cdot R^{n-2|\alpha|}
\]

\[
= C \quad \text{for all multi-indices } \alpha.
\]

Thus \( m_j \) satisfies the assumptions of Theorem 3, and we get the \( H^p(\mathbb{R}^n), \ 0 < p \leq 1 \), boundedness of the mapping \( f \mapsto R_j f \).

\[\blacksquare\]

3 The Third Proof of Theorem 2

Let \( \mathcal{D} = \mathcal{D}(\mathbb{R}^n) \) denote the subspace of \( C^\infty(\mathbb{R}^n) \) of compactly supported functions with their usual topology, and \( \mathcal{D}' \) denote its dual space. For \( \eta \in \mathcal{D}, \ z \in \mathbb{R}^n \) and \( t > 0 \), let

\[
\eta^{\varepsilon,t}(x) = \eta \left( \frac{x - z}{t} \right).
\]

A linear and continuous operator \( T : \mathcal{D} \mapsto \mathcal{D}' \) is said to satisfy the Weak Boundedness Property if, for each bounded subset \( B \) of \( \mathcal{D} \), there exists a positive constant \( C = C(B) \) such that for all \( \varphi, \psi \in B \), all \( z \in \mathbb{R}^n \) and all \( t > 0 \),

\[
|\langle T \varphi^{\varepsilon,t}, \psi^{\varepsilon,t} \rangle| \leq Ct^p.
\]

In [4, 8], Frazier-Torres-Weiss and Torres studied the \( T1 \) theorem and used distribution theory to prove the boundedness of Calderón-Zygmund operators on the Triebel-Lizorkin spaces \( \dot{F}^{a,d}_{p,q}(\mathbb{R}^n) \). For the special case \( \alpha = 0 \) and \( q = 2 \), we have \( \dot{F}^{a,0}_{p,2} = H^p \) for \( p \leq 1 \). Moreover, if \( k(x, y) = k(x - y) \) is the convolution type kernel, they obtained the following result.

**Theorem 3.1.** [4, page 67] and [9, page 86] Let \( 0 < p \leq 1 \) and \( [\cdot] \) denote the integer function. If \( T \) is the convolution operator with kernel \( k \) such that

(i) \( T \) satisfies the Weak Boundedness Property,

(ii) \( |D^\alpha k(x)| \leq C|x|^{-n - |\alpha|} \quad \text{for } |\alpha| \leq [n(1/p - 1)] \),

(iii) \( |D^\alpha k(x) - D^\alpha k(y)| \leq C|x - y||x|^{-n - |\alpha| - \varepsilon} \quad \text{for } |\alpha| = [n(1/p - 1)], \ 2|x - y| < |x|, \ \text{and } 1 > \varepsilon > n/p - [n/p] \),

(iv) \( T(x^\alpha) = 0 \quad \text{for } |\alpha| \leq [n(1/p - 1)] \),

then \( T \) is bounded on \( H^p(\mathbb{R}^n) \).
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**Proof.** [The third proof of Theorem 2] Write $k_j(x) = x_j / |x|^{n+1}$. Conditions (ii) and (iii) are satisfied by direct calculations. The conditions (i) and (iv) are always satisfied by principal value convolution operators (see [9, Proposition 2.2.17 and §2.3]). Hence, the mappings $f \mapsto R_j f$ are bounded on $H^p(\mathbb{R}^n)$, $0 < p \leq 1$. ■

4 The Fourth Proof of Theorem 2

Here we present one more proof of Theorem 2. The ideas and methods come from the Coifman’s atomic decomposition and Taibleson-Weiss’ molecular characterization. We first give definitions of atoms and molecules, and their related results.

**Definition 4.1 (Definition of atoms).** Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{Z}$ and $s \geq [n(1/p - 1)]$. (Such an ordered triple $(p, q, s)$ is called *admissible.*) A $(p, q, s)$-atom centered at $x_0 \in \mathbb{R}^n$ is a function $a \in L^q(\mathbb{R}^n)$, supported on a ball $B \subseteq \mathbb{R}^n$ with center $x_0$ and satisfying

(i) $\|a\|_q \leq |B|^{1/q - 1/p}$,

(ii) $\int_{\mathbb{R}^n} a(x)x^\alpha \, dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

Let $H^{p,q,s}$ denote the space consisting of tempered distributions admitting a decomposition $f = \sum \lambda_i a_i$, where $a_i$’s are $(p, q, s)$-atoms and $\sum |\lambda_i|^p < \infty$. For $f \in H^p(\mathbb{R}^n)$, we also set

$$N_{p,q,s}(f) = \inf \left\{ \left( \sum |\lambda_i|^p \right)^{1/p} : \sum \lambda_i a_i \text{ is a decomposition of } f \text{ into } (p, q, s)-\text{atoms} \right\}.$$ 

We have the following atomic decomposition for $H^p$.

**Theorem 4.2.** [2, 6] If the triple $(p, q, s)$ is admissible, then $H^p = H^{p,q,s}$. Moreover, both $\|f\|_{H^p}$ and $N_{p,q,s}(f)$ are equivalent.

If we allow an atom to have support outside of a ball and also replace its size condition, then we get a generalized atom. Such generalized atoms are called molecules and are useful in certain applications.

**Definition 4.3.** [Definition of molecules] Let $(p, q, s)$ be an admissible triple and $\varepsilon > \max\{s/n, 1/p - 1\}$. (Such a quadruple $(p, q, s, \varepsilon)$ is also called admissible.) Set $a = 1 - 1/p + \varepsilon, b = 1 - 1/q + \varepsilon$. A $(p, q, s, \varepsilon)$-molecule centered at $x_0$ is a function $M \in L^q(\mathbb{R}^n)$ satisfying

(i) $M(x) \cdot |x - x_0|^n \in L^q(\mathbb{R}^n)$,

(ii) $\|M\|_q^{a/b} \cdot \|M(x) \cdot |x - x_0|^n\|_q^{1 - a/b} \equiv \Omega(M) < \infty$,

($\Omega(M)$ is called the molecular norm of $M$.)

(iii) $\int_{\mathbb{R}^n} M(x)x^\alpha \, dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq s$.

The following molecular characterization is very useful in establishing $H^p$ boundedness of sublinear operators.
Theorem 4.4. [8] Each \((p, q, s, \varepsilon)\)-molecule \(M\) is in \(H^p\) and \(\|M\|_{H^p} \leq C \mathcal{N}(M)\), where the constant \(C\) is independent of the molecule.

Remark 1: Every \((p, q, s)\)-atom \(f\) is a \((p, q, s, \varepsilon)\)-molecule, and \(\mathcal{N}(f) \leq C\) where \(C\) is a constant independent of \(f\).

Remark 2: As a consequence of Theorems 5 and 6, for a sublinear operator \(T\) to be bounded on \(H^p\), it suffices to show that \(Tf\) is a \(p\)-molecule and \(\mathcal{N}(Tf) \leq C\) for some constant \(C\) independent of \(f\) whenever \(f\) is a \(p\)-atom.

We now prove Theorem 2 by using atom-molecule theory.

**proof.** [The fourth proof of Theorem 2] Since the Riesz transforms commute with translations, we consider atoms and molecules centered at the origin only. For \(s \in \mathbb{N}\) and \(n/(n + s) < p \leq n/(n + s - 1)\), we choose a number \(\varepsilon\) satisfying \(1/p - 1 < \varepsilon < s/n\). Then both \((p, 2, s - 1)\) and \((p, 2, s - 1, \varepsilon)\) are admissible by straightforward calculations. We shall prove that if \(f\) is a \((p, 2, s - 1)\)-atom, then \(R_j f\) are \((p, 2, s - 1, \varepsilon)\)-molecules with molecular norm \(\mathcal{N}(R_j f) \leq C\) for \(1 \leq j \leq n\).

Given a \((p, 2, s - 1)\)-atom \(f\) with \(\text{supp}(f) \subseteq \{x \in \mathbb{R}^n : |x| \leq R\}\), we have \(\|f\|_2 \leq C_n R^{n(1/2 - 1/p)}\) and \(\int f(x) x^\alpha dx = 0\) for \(0 \leq |\alpha| \leq s - 1\). Let \(a = 1 - 1/p + \varepsilon\) and \(b = 1/2 + \varepsilon\). Then

\[
\|R_j f(x) \cdot |x|^{nh}\|_2^2 = \int_{\mathbb{R}^n} |K_j * f(x)|^2 \cdot |x|^{n+2\varepsilon} dx
\]

\[
= \left( \int_{|x| \leq 2R} + \int_{|x| > 2R} \right) |K_j * f(x)|^2 \cdot |x|^{n+2\varepsilon} dx
\]

\[
\equiv I_1 + I_2.
\]

The \(L^2\) boundedness of \(R_j\) implies

\[
I_1 \leq (2R)^{n+2\varepsilon} \|K_j * f\|_2^2 \leq C_n R^{n+2\varepsilon} \|f\|_2^2 \leq C_n R^{2n\varepsilon}.
\]

To estimate \(I_2\), we use the moment condition of \(f\) to write

\[
I_2 \equiv \int_{|x| > 2R} \int_{|y| \leq R} K_j(x - y) f(y) dy \left| x \right|^{n+2\varepsilon} dx
\]

\[
= \int_{|x| > 2R} \int_{|y| \leq R} \left\{ K_j(x - y) - \sum_{|\alpha| = 0}^{s-1} \frac{1}{\alpha!} D^\alpha K_j(x)(-y)^\alpha \right\} f(y) dy \left| x \right|^{n+2\varepsilon} dx.
\]

Taylor’s theorem and Schwarz’s inequality give

\[
\left| \int_{|y| \leq R} \left\{ K_j(x - y) - \sum_{|\alpha| = 0}^{s-1} \frac{1}{\alpha!} D^\alpha K_j(x)(-y)^\alpha \right\} f(y) dy \right|^2
\]

\[
\leq C_n |x|^{-2n-2\varepsilon} \|f\|_2^2 \int_{|y| \leq R} |y|^{2\varepsilon} dy \quad \text{for } |x| \geq 2|y|,
\]

\[
\leq C_n |x|^{-2n-2\varepsilon} \|f\|_2^2 \int_{|y| \leq R} |y|^{2\varepsilon} dy \quad \text{for } |x| \geq 2|y|,
\]

\[
\leq C_n |x|^{-2n-2\varepsilon} \|f\|_2^2 \int_{|y| \leq R} |y|^{2\varepsilon} dy \quad \text{for } |x| \geq 2|y|,
\]
which implies
\[ I_2 \leq C_n \| f \|_2^2 \int_{|y| \leq R} |y|^{2s} \, dy \int_{|x| > 2|y|} |x|^{2n - n - 2s} \, dx \]
\[ \leq C_n \| f \|_2^2 \int_{|y| \leq R} |y|^{2n} \, dy \]
\[ \leq C_n \| f \|_2^2 R^{2n + 2ns} \]
\[ \leq C_n R^{2na}. \]

Thus,
\[ \| R_j f(x) \cdot |x|^{nb} \|_2 \leq C_n R^{na} \]
and
\[ \mathfrak{N}(R_j f) \equiv \| R_j f \|_{2}^{a/b} \cdot \| R_j f(x) \cdot |x|^{nb} \|_2^{1-a/b} \]
\[ \leq C_n R^{(1/2 - 1/p)a/b} \cdot R^{na(1-a/b)} \]
\[ = C_n. \]

To complete the proof, it remains to show that
\[ \int_{\mathbb{R}^n} R_j f(x) \cdot x^\alpha \, dx = 0 \quad \text{for} \quad |\alpha| \leq s - 1. \]
We first claim \( R_j f(x) \cdot x^\alpha \in L^1 \). For \(|\alpha| \leq s - 1\), since we have shown \( R_j f(x) \cdot |x|^{nb} \in L^2 \), we use Schwarz’s inequality to get
\[ \int_{|x| > 1} |R_j f(x)| \cdot |x|^{|\alpha|} \, dx \leq \| R_j f(x) \cdot |x|^{nb} \|_2 \left( \int_{|x| > 1} |x|^{2|\alpha| - 2nb} \, dx \right)^{1/2} < \infty \]
and
\[ \int_{|x| \leq 1} |R_j f(x)| \cdot |x|^{|\alpha|} \, dx \leq \| R_j f \|_2 \left( \int_{|x| \leq 1} |x|^{2|\alpha|} \, dx \right)^{1/2} < \infty. \]
We thus have \( R_j f(x) \cdot x^\alpha \in L^1(\mathbb{R}^n) \) for \(|\alpha| \leq s - 1\), and hence
\[ D^a(R_j f)(\xi) = C(R_j f(x) \cdot x^\alpha)(\xi) \quad \text{for} \quad |\alpha| \leq s - 1 \]
is continuous. Moreover, it follows from [8, Lemma 9.1] that \( \hat{f} \) is \((s - 1)\)-th order differentiable and \( \hat{f}(\xi) = O(|\xi|) \) as \( |\xi| \to 0 \). We write \( e_j \) to be the \( j \)-th standard basis vector of \( \mathbb{R}^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \) a multi-index of nonnegative integers \( \alpha_j \), \( \Delta_{he_j} g(x) = g(x) - g(x - he_j) \), \( \Delta_{he_j}^{\alpha_j} g = \Delta_{he_j}^{\alpha_j - 1} \Delta_{he_j} g \) for \( \alpha_j \geq 2 \), \( \Delta_{he_j}^{0} g = g \), and \( \Delta_{h}^{\alpha} = \Delta_{he_1}^{\alpha_1} \Delta_{he_2}^{\alpha_2} \cdots \Delta_{he_n}^{\alpha_n} \). Then the boundedness of \( m_j(x) = -ix_j/|x| \) implies
\[ \left| \int_{\mathbb{R}^n} R_j f(x) \cdot x^\alpha \, dx \right| = C_n \left| D^a(\hat{R}_j f)(0) \right| \]
\[ = C_n \lim_{h \to 0} |h|^{-|\alpha|} \Delta_{jh}^{\alpha}(m_j \hat{f})(0) \]
\[ \leq C_n \lim_{h \to 0} |h|^{s-|\alpha|} \]
\[ = 0 \quad \text{for} \quad |\alpha| \leq s - 1. \]

Thus, the proof is finished. ■
5 The $H^p$ Boundedness of Hilbert Transform

Following a similar but easier argument, we also have the following $H^p$ boundedness of Hilbert transform. We leave details to readers.

**Theorem 5.1.** The Hilbert transform is bounded on $H^p(\mathbb{R})$, $0 < p \leq 1$.

**References**


