Further Results on Dendritic Growth with Forced Oscillation and Pattern Formation

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Abstract

We first recall the results on the asymptotic theory for a mathematical model of dendritic crystal solidification. A single needle dendrite is growing from a pure melt with arbitrary under-cooling parameter ($-1 < T_\infty < 1$). The dendrite is supposed to grow under the effect of convection motion induced by an oscillating external source with magnitude $U_\infty$ and frequency $\omega$. We formulated and discussed the problem by assuming that the Reynolds number $Re$ and the frequency $\omega$ are both small. We then present the further results for the generated globally valid asymptotic expansion solutions of both the temperature field and the interface shape function in the whole physical domain. This enables us to finally explore the effect of the externally applied convection motion on the crystal growth and pattern formation.

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1. Introduction

Dendritic growth of axially symmetric needle crystal is one of the fundamental topics in the areas of solidification physics and material science in the past several decades [2]–[17]. In literature, this topic has raised great interest to the experimental results in physics of crystal growth but also the mathematical methods for problem solving [5]–[13].

In the history of studying steady dendrite growth problem, Ivantsov [1] is the first person who considered the case with zero surface tension on the interface between the solid state and liquid state; and obtained an exact similarity solution. This was the first
important contribution to the subject. The Ivantsov’s solution describes the dendrite is steady and isothermal, its interface shape is paraboloidal axisymmetric. Despite the solution does not resolve the problem of dendritic growth as it can not determine the top radius and tip velocity separately, this mathematical solution contributes a basis for further research works such as the perturbed dendritic growth with nonzero surface tension. Another significant contribution in terms of the experimental results was given by Schaefer, Glicksman and Ayers [4]. These researchers made some extensive and detailed experiments and examined the pattern selection problem for realistic dendritic growth. They found that a selection condition of tip velocity can be found even in the isotropic surface tension case.

Our paper is an investigation following the Xu’s work [7]–[13] to deal with the interaction of external convection and dendritic growth. The experimental observations have shown that convective motion in melt may have a significant effect on the instability mechanisms, and consequently, affects the micro-structure formation at the interface in dendritic solidification. Convective motion in melt can be induced by a variety of sources. But the major types of convective motions can be classified as: (i) the density change during phase transition; (ii) the buoyancy effect due to the presence of gravity field; (iii) an applied external flow or other sources.

Taking into account of convection, the problem becomes even more complicated and difficult to solve. In this case the hydrodynamics must be introduced into the system. The interaction between convection and solidification becomes the major problem. In the literature, the steady dendritic growth in an external flow was studied by a number of researchers, such as Ananth & Gill [15], Dash & Gill [16], Saville & Beaghton [17] numerically and analytically. These researchers considered the special case of zero surface tension, and obtained the similarity solutions that based on some simplified models of Navier-Stokes equations. However, their solutions which are only approximate solutions cannot be considered as good approximations in the whole physical domain, as far as the Navier-Stokes model is concerned. Moreover, the approaches adopted by these researchers neither allow the generation of the next-order approximations, nor give the numerical exploration of the solutions.

In 1993-1994, Xu introduced the method of asymptotic expansions in dendritic growth problems and has found a uniformly valid asymptotic expansion solution, for large Prandtl number [7]–[8]. In this paper we attempt to generate the uniformly valid asymptotic solutions for the temperature field and the interface shape when we consider a small Reynolds number, Re → 0. We shall consider the case with the presence of the effect of convective motion induced only by the oscillatory external flow on needle dendritic growth.

2. Mathematical Formulation

Consider a single needle dendrite which is growing into an under-cooled pure melt in the negative Z-axis direction with a constant tip velocity $V$. The growth of needle dendrite during solidification has been intensively studied by physicists and material
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scientists for several decades [1]–[6], [15]–[17]. The key problem is how to formulate a physical problem into a proper mathematical model. Our goal in this paper is to derive the uniformly valid asymptotic solutions of the temperature field and the interface shape of the dendrite in the limit of Re → 0 and make a preliminary numerical exploration for this problem.

The general mathematical formulation of the needle dendritic growth with convection induced by the effect of oscillatory external source will first be introduced. The major physical transport process in a pure melt is heat conduction. The under-cooling temperature of the melt is $T_\infty$. The melt is considered as an incompressible Newtonian fluid and is assumed to be infinite in extent. The dendrite is supposed to grow in an oscillating external flow, along the Z-axis in the far field ahead of the tip with a small amplitude $U_\infty$, with zero surface tension on the interface between liquid and solid states. Assume that the thermal diffusivity $\kappa_T$ and the heat capacity $c_p$ of the liquid state are the same as those of the solid state, the mass density of liquid state is $\rho$ and the mass density of solid state is $\rho_s$. Both the tip growth velocity $V$ and the flow velocity $U_\infty$ are measured in laboratory frame. Let $U$ be the absolute velocity field of the fluid and $T$ be the temperature field in the liquid melt.

The governing equations consisting of the mass continuity equation, the Navier-Stokes equations with Boussinesq approximation applied and the heat conduction equations for the liquid and solid states are as follows:

$$\begin{align*}
\nabla \cdot U &= 0, \\
\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times U) &= \nu \nabla^2 \Omega + \nabla \times [\beta(T - T_\infty) g e_Z], \quad \Omega = \nabla \times U, \\
\frac{\partial T}{\partial t} + U \cdot \nabla T &= \kappa_T \nabla^2 T, \\
\frac{\partial T_s}{\partial t} &= \kappa_T \nabla^2 T_s,
\end{align*}$$

where $T$ and $T_s$ denote the temperature fields, $U$ denotes the absolute velocity field of the fluid motion, $t$ denotes the time, $e_Z$ the unit vector along the Z-axis, and $\nu$ is the kinematic viscosity, $\beta$ is the thermal expansion coefficient, $P$ is the reduce pressure, $g$ is the acceleration of gravity.

The boundary conditions are given as follows.

(1) At far field: $U = U_\infty e_Z$, and $T = T_\infty$.

(2) At the interface tip: $U$, $T$, $T_s$ are regular.

(3) At the interface, we assume the system is in the local thermodynamical equilibrium state. Thus we have

(a) continuity condition of temperature;

(b) Gibbs-Thomson condition (the temperature at the interface depends on the local interface curvature);
(c) enthalpy conservation condition;
(d) mass conservation condition;
(e) continuity condition of tangential component of velocity.

The above PDE system will be non-dimensionalized in order to reduce the complexity of the formulation. We adopt the parabolic cylindrical coordinate system \((\xi, \eta, \theta)\) moving with the constant velocity \(V\) as follows:

\[
\begin{align*}
\frac{r}{\eta^2_0} &= \xi \eta, \\
\frac{z}{\eta^2_0} &= \frac{1}{2}(\xi^2 - \eta^2).
\end{align*}
\]

In the above, the parameter \(\eta^2_0\) is to be determined by locating the tip of dendrite at \(\eta = 1\) and it will be seen that this parameter is needed to normalize the interface shape function.

In this paper, we attempt to derive and numerically compute the approximation of the leading order asymptotic expansion solutions of the temperature field and the interface shape function. We shall consider the case with the presence of the effect of convective motion induced only by the oscillatory external flow on needle dendritic growth, in the far field ahead of the tip with a small magnitude \(U_\infty\). In particular, we shall investigate the effect on the interface shape change and the temperature gradient change on the dendrite’s interface. We assume that gravity is taken to be negligible, the surface tension is assumed to be zero, so the dendrite should be axisymmetric and convection is only induced by the oscillatory external source. In the following we shall first express the non-dimensionalized governing equations and the boundary conditions in terms of the paraboloidal coordinates. The PDE system which gives rise to a free boundary problem is our target model to be solved analytically using the asymptotic expansion method.

**Kinematic equation**

\[
D_1^2 \Psi = -\eta_0^4 (\xi^2 + \eta^2) \xi, \tag{2.1}
\]

**Vorticity equation**

\[
\frac{1}{\Re} D_1^2 \xi = \eta_0^4 (\xi^2 + \eta^2) \frac{\partial \xi}{\partial t} + \frac{2\xi}{\eta_0^2 \xi^2 \eta^2} \frac{\partial (\Psi, \xi \eta)}{\partial (\xi, \eta)} - \frac{1}{\eta_0^2 \xi \eta} \frac{\partial (\Psi, \xi \eta)}{\partial (\xi, \eta)}, \tag{2.2}
\]

**Energy equation for the liquid state**

\[
\nabla_1^2 T = \eta_0^4 (\xi^2 + \eta^2) \frac{\partial T}{\partial t} + \frac{1}{\eta_0^2 \xi \eta} \left( \frac{\partial \Psi}{\partial \eta} \frac{\partial T}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \frac{\partial T}{\partial \eta} \right), \tag{2.3}
\]

where

\[
D_1^2 = \begin{bmatrix}
\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\eta} \frac{\partial}{\partial \eta}
\end{bmatrix},
\]

\[
\nabla_1^2 = \begin{bmatrix}
\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{1}{\eta} \frac{\partial}{\partial \eta}
\end{bmatrix}.
\]

The boundary conditions are:
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1. The far-field conditions: as \( \eta \to \infty \),
\[
\Psi \to \frac{1}{2} \eta_0^4 (1 + U_\infty \exp(i\omega t)) \xi^2 \eta^2, \quad \xi \to 0, \quad T \to T_\infty. \tag{2.4}
\]

2. The smooth tip condition: at \( \xi = 0 \),
\[
\eta'_s(0,t) = 0, \quad \eta_s(0,0) = 1. \tag{2.5}
\]

3. On the interface \( \eta = \eta_s(\xi, t) \), we must have

(a) thermodynamical equilibrium condition:
\[
T = T_s = 0, \tag{2.6}
\]

(b) enthalpy conservation condition:
\[
\left( \frac{\partial T}{\partial \eta} - \eta_s \frac{\partial T}{\partial \xi} \right) + \eta_0^4 (\xi \eta_s)' + \eta_0^4 (\xi^2 + \eta_s^2) \frac{\partial \eta_s}{\partial t} = 0, \tag{2.7}
\]

(c) mass conservation condition:
\[
\left( \frac{\partial \Psi}{\partial \xi} + \eta_s \frac{\partial \Psi}{\partial \eta} \right) = \eta_0^4 (\xi \eta_s)(\xi \eta_s)', \tag{2.8}
\]

(d) continuity condition of tangential component of velocity:
\[
\left( \frac{\partial \Psi}{\partial \eta} - \eta_s \frac{\partial \Psi}{\partial \xi} \right) + \eta_0^4 (\xi \eta_s)(\eta_s \eta_s' - \xi) = 0. \tag{2.9}
\]

We succeed in applying the method of matched asymptotic expansions to find the uniformly valid asymptotic solution for the flow field \( \Psi \) in [18]. We shall borrow the analytical results of the flow field and assume further that the dendrite’s temperature field as well as the interface shape function will have the same uniformly valid asymptotic expansion solutions. Below let us first review the method of matched asymptotic expansions. To study the asymptotic expansion solution of flow field for the small Reynolds number \( \text{Re} = Vl_T/\nu \) which is equivalent to the case, when the Prandtl number \( \text{Pr} = \nu/\kappa_T \) is large. The entire physical space is divided into two regions, the inner region near the dendrite and the outer region apart from the dendrite. The asymptotic solution which is valid in the inner region is called the inner solution, while which is valid in the outer region is called the outer solution. The inner solution is found to have a different asymptotic expansion from the outer solution, and the two independent asymptotic solutions can be matched in an intermediate region. The matched solution for the flow field \( \Psi \) is a globally valid asymptotic solution in the whole physical domain.
3. Asymptotic Theory For Dendritic Evolution of Needle Crystal

The famous Ivantsov (1947) exact solution represents that the dendrite is isothermal and its interface shape is paraboloidal. This similarity solution with zero surface tension is a particular solution of eqns. (2.1)–(2.3):

\[
\begin{align*}
T^* &= T_\infty + \frac{1}{2} \eta_0^2 \exp\left(\frac{1}{2} \eta_0^2 \eta^2\right) E_1\left(\frac{1}{2} \eta_0^2 \eta^2\right); \quad T_{S^*} = 0; \\
\eta_* &= 1; \quad \zeta_* = 0; \quad \Psi_* = \frac{1}{2} \eta_0^4 \xi^2 \eta^2 \\
E_1(x) &= \int_x^\infty \frac{\exp(-t)}{t} dt,
\end{align*}
\]

Equation (3.1)

The parameter \( \eta_0^2 \) is determined by the under-cooling temperature \( T_\infty \) because \( T^* = 0 \) when \( \eta = 1 \) is assumed in (3.1).

Based on the Ivantsov solution we can generate the uniformly valid expansion solutions of the perturbed boundary value problem (2.1)–(2.3) in the limit as \( \text{Re} \to 0 \) by the method of matched asymptotic expansion solutions [18]. To hunt for the asymptotic expansion solutions around the Ivantsov solution in the limit as \( \text{Re} \to 0 \) we assumed that

\[
\begin{align*}
\Psi - \Psi_* &= \epsilon_0(\text{Re}) \Psi_0(\xi, \eta, t) + \epsilon_1(\text{Re}) \Psi_1(\xi, \eta, t) + \cdots, \\
\xi - \xi_* &= \epsilon_0(\text{Re}) \xi_0(\xi, \eta, t) + \epsilon_1(\text{Re}) \xi_1(\xi, \eta, t) + \cdots, \\
T - T_* &= \epsilon_0(\text{Re}) T_0(\xi, \eta, t) + \epsilon_1(\text{Re}) T_1(\xi, \eta, t) + \cdots, \\
\etaS - 1 &= \epsilon_0(\text{Re}) h_0(\xi, t) + \epsilon_1(\text{Re}) h_1(\xi, t) + \cdots,
\end{align*}
\]

Equation (3.2)

where the asymptotic sequence \( \{\epsilon_0(\text{Re}), \epsilon_1(\text{Re}), \epsilon_2(\text{Re}), \cdots\} \) can be computed by matching different terms between different orders of the inner region solutions and the outer region solutions. The matching procedure involves a tricky continuation cycle. Nevertheless, the matched asymptotic solution for the flow field can be written as

\[
\Psi(\xi, \eta, t) = \frac{1}{2} \eta_0^4 \xi^2 \eta^2 + \frac{U_\infty}{\ln(1/\text{Re}) - \lambda} \Psi_0 + \frac{\text{Re}}{\ln(1/\text{Re})} \Psi_1 + \frac{\text{Re}^2}{\ln(1/\text{Re})} \Psi_2 + \cdots,
\]

Equation (3.3)

where

\[
\lambda := \gamma + \ln\left(\frac{1}{2} \eta_0^2 (1 + U_\infty)\right)
\]

Equation (3.4)

(\( \gamma \) is the Euler’s constant) and

\[
\begin{align*}
\Psi_0 &= \exp(i\omega t) \psi(\xi, \eta), \\
\psi(\xi, \eta) &= \frac{1}{2} \eta_0^4 \left\{ \left(\frac{1}{2} \eta^4 - \eta^2 + \frac{1}{2}\right) - \xi^2 \left[ -\eta^2 \ln \eta^2 + \eta^2 - 1 \right] \right\}.
\end{align*}
\]
Asymptotic Expansions for Temperature Field and Interface Shape

We are in a position to extend the analytical result to the temperature field and the interface shape function in the presence of oscillatory external flow. With reference to (3.2)–(3.5), we may assume that the temperature field and the interface shape function have the following asymptotic expansion solutions in the limit of Re → 0:

\[
T(\xi, \eta, t) = T^*(\eta) + \frac{U_\infty}{\ln(1/\text{Re}) - \lambda} T_0 + \frac{\text{Re}}{\ln(1/\text{Re})} T_1 + \frac{\text{Re}^2}{\ln(1/\text{Re})} T_2 + \cdots,
\]

(3.6)

and

\[
\eta_s(\xi, t) = 1 + \frac{U_\infty}{\ln(1/\text{Re}) - \lambda} h_0 + \frac{\text{Re}}{\ln(1/\text{Re})} h_1 + \frac{\text{Re}^2}{\ln(1/\text{Re})} h_2 + \cdots,
\]

(3.7)

respectively. In order to investigate the effect on the temperature field and the interface shape of the needle crystal growth, we shall first deduce the zero-order asymptotic solutions \(T_0(\xi, \eta, t)\) and \(h_0(\xi, t)\) in the expansions (3.6)–(3.7).

The governing equation for the temperature field in liquid state is given by eqn. (2.3). We substitute the asymptotic expansions of \(\Psi_1\) and \(T\) respectively in (3.3) and (3.6) into eqn. (2.3) and equate coefficients of each order, we could first derive the governing equation for \(T_0\):

\[
\nabla_1^2 T_0 = \eta_0^4 (\xi^2 + \eta^2) \frac{\partial T_0}{\partial t} + \frac{1}{\eta_0^2 \xi \eta} \left( \frac{\partial \Psi_1}{\partial \eta} \frac{\partial T_0}{\partial \xi} - \frac{\partial \Psi_1}{\partial \xi} \frac{\partial T_0}{\partial \eta} - \frac{\partial \Psi_1}{\partial \xi} \frac{\partial T_0}{\partial \eta} \right).
\]

(3.8)

The boundary conditions are given by

(1) as \(\eta \to \infty\) (far-field condition),

\[T_0 \to 0\] (exponentially);

(3.9)

(2) as \(\xi \to 0\) (regularity conditions at the interface tip),

\[T_0\] is regular of \(\xi\),

\[h_0(0, t) = 0,\]

(3.10)

\[h_0(0, 0) = 0;\]

(3.11)

(3) as \(\xi \to \infty\) (near the dendrite's root),

\[T_0\] grows algebraically;

(3.12)

(4) at \(\eta = 1\) (interface conditions),

\[T_0(\xi, 1, t) = \eta_0^2 h_0(\xi, t),\]

(3.13)

\[T_0'(\xi, 1, t) = -\eta_0^2 (\eta_0^2 + 2) h_0(\xi, t) - \eta_0^4 (\xi^2 + 1) \frac{\partial h_0}{\partial t} - \eta_0^2 \xi \frac{\partial h_0}{\partial \xi}.
\]

(3.14)
By (3.5), $\Psi_0 = \psi \exp(i\omega t)$ and hence the last term of eqn. (3.8) which involves the first derivative of $\Psi_0$ contains the factor $\exp(i\omega t)$. We then consider the solutions of the form (time-independent functions in non-italic symbols)

\[
\begin{align*}
T_0(\xi, \eta, t) &= T_0(\xi, \eta) \exp(i\omega t), \\
h_0(\xi, t) &= h_0(\xi) \exp(i\omega t).
\end{align*}
\]

Simplifying the equation (3.8) gives rise to the governing equation for $T_0(\xi, \eta)$:

\[
\frac{\partial^2 T_0}{\partial \xi^2} + \frac{\partial^2 T_0}{\partial \eta^2} + \left(\frac{1}{\xi} - \eta_0^2 \xi\right) \frac{\partial T_0}{\partial \xi} + \left(\frac{1}{\eta} + \eta_0^2 \eta\right) \frac{\partial T_0}{\partial \eta} = \omega A_0(\xi, \eta) T_0 + B_0(\eta),
\]

where (by definition)

\[
A_0(\xi, \eta) := i \eta_0^4 (\xi^2 + \eta^2),
\]

\[
B_0(\eta) := \eta_0^4 \left(2 \ln \eta - 1 + \frac{1}{\eta^2}\right) \exp\left(\frac{1}{2} \eta_0^2 (1 - \eta^2)\right).
\]

It should be noted that eqn. (3.16) is a complex-valued equation. In principle, after solving the equation, we have to take the real part of the solution as our final solution. However, eqn. (3.16) is yet technically uneasy to solve. To overcome this difficulty let us make an additional assumption that the needle crystal is formed under a periodic external flow of a small frequency $\omega$. We attempt to further study the series solutions of $T_0(\xi, \eta)$ and $h_0(\xi)$ in the limit of $\omega \to 0$ and formally write

\[
\begin{align*}
T_0(\xi, \eta) &= T_{0,0}(\xi, \eta) + \omega T_{0,1}(\xi, \eta) + \omega^2 T_{0,2}(\xi, \eta) + \cdots, \\
h_0(\xi) &= h_{0,0}(\xi) + \omega h_{0,1}(\xi) + \omega^2 h_{0,2}(\xi) + \cdots.
\end{align*}
\]

Substituting the above series solution of $T_0$ in eqn. (3.16) and equating each power of $\omega$ deduces the recursive equations of each order. The recursive equations are

\[
\tilde{\nabla}_1^2 \tilde{T}_{0,k} = A_0(\xi, \eta) \tilde{T}_{0,k}, \quad k = 0, 1, 2, \cdots,
\]

\[
B_0(\eta).
\]

where $A_0, B_0$ are defined in (3.17) and

\[
\tilde{\nabla}_1^2 = \left\{\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \left(\frac{1}{\xi} - \eta_0^2 \xi\right) \frac{\partial}{\partial \xi} + \left(\frac{1}{\eta} + \eta_0^2 \eta\right) \frac{\partial}{\partial \eta}\right\}.
\]

**Construction of Series Solutions for the Leading Behaviour**
We shall investigate the leading behaviour of the zero-order temperature solution, that
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is $T_{0,0}(\xi, \eta)$, when $\omega$ is considered as a small parameter. The solution can be derived by using the method of separation of variables. Substituting

$$T_{0,0}(\xi, \eta) = H(\xi) K(\eta)$$

into the first equation of (3.19) gives

$$K(\eta) \left[ H''(\xi) + \left( \frac{1}{\xi} - \eta_0^2 \xi \right) H'(\xi) \right] + H(\xi) \left[ K''(\eta) + \left( \frac{1}{\eta} + \eta_0^2 \eta \right) K'(\eta) \right] = B_0(\eta).$$

(3.20)

We may assume that $H(\xi) \equiv \hat{H}_0$ in order to find a particular solution of the non-homogeneous eqn. (3.20), where $\hat{H}_0$ is a nonzero constant. The first term of eqn. (3.20) thus vanishes and a solution $K = K_p(\eta)$ satisfies

$$K''(\eta) + \left( \frac{1}{\eta} + \eta_0^2 \eta \right) K'(\eta) = \frac{1}{\hat{H}_0} B_0(\eta).$$

The far field boundary condition (3.9) imposes a boundary condition on $K_p(\eta)$ as $\eta \to \infty$,

$$K_p \to 0 \text{ (exponentially)}. $$

The boundary value problem for $K_p(\eta)$ can easily be solved by the method of reduction of order. Using integration twice, we could obtain a particular solution of eqn. (3.20)

$$\hat{H}_0 K_p(\eta) = \eta_0^4 \exp \left( \frac{1}{2} \eta_0^2 \right) \left[ \hat{k}_0 I_0(\eta) + I_1(\eta) \right],$$

(3.21)

where $\hat{k}_0$ is the constant of integration to be determined (whereas another constant of integration vanishes because the solution $K_p(\eta)$ must satisfy the far field boundary condition) and the exponential integrals are defined as

$$\left\{ 
\begin{align*}
I_0(\eta) &:= \int_\eta^\infty \frac{1}{t} \exp \left( -\frac{1}{2} \eta_0^2 t^2 \right) \, dt, \\
I_1(\eta) &:= \int_\eta^\infty \left( t - t \ln t - \frac{1}{t} \ln t \right) \exp \left( -\frac{1}{2} \eta_0^2 t^2 \right) \, dt.
\end{align*}
\right\}$$

(3.22)

To solve eqn. (3.20) we next find the general solution of the associated homogeneous equation for (3.20) so that we can add this solution together with the particular solution in (3.21) to get

$$T_{0,0}(\xi, \eta) = \hat{H}_0 K_p(\eta) + H_h(\xi) K_h(\eta).$$

(3.23)

Based on the associated homogeneous equation of (3.20) one could easily see that $H_h(\xi)$ and $K_h(\eta)$ must satisfy

$$H''(\xi) + \left( \frac{1}{\xi} - \eta_0^2 \xi \right) H'(\xi) = \sigma H(\xi),$$

(3.24)

$$K''(\eta) + \left( \frac{1}{\eta} + \eta_0^2 \eta \right) K'(\eta) = -\sigma K(\eta),$$

(3.25)
respectively. The proportional constant $\sigma$ is to be determined. To solve eqn. (3.24) we introduce the new variable

$$x = \frac{1}{2} \eta_0^2 \xi^2$$

and the equation can be transformed into the confluent hypergeometric differential equation

$$xH'' + (1 - x)H' - \frac{\sigma}{2\eta_0} H = 0$$

whose independent solutions are

$$M\left(\frac{\sigma}{2\eta_0}, 1, x\right) \quad \text{and} \quad U\left(\frac{\sigma}{2\eta_0}, 1, x\right),$$

the so-called confluent hypergeometric functions of the first and second kinds. The functions are generally defined as follows:

$$M(a, b, z) = 1 + \frac{a}{b} z + \frac{a(a + 1)}{b(b + 1)} \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_n z^k}{(b)_n k!},$$

$$U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} M(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} M(a - b + 1, 2 - b, z),$$

where

$$(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1), \quad (a)_0 = 1 \quad \text{(Pochhammer symbols)},$$

and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad \text{(Gamma function for Re}(z) > 0).$$

Notice that

$$M\left(\frac{\sigma}{2\eta_0}, 1, 0\right) = 1,$$

$$U\left(\frac{\sigma}{2\eta_0}, 1, x\right) \approx -\frac{1}{\Gamma(a)} \left[ \ln |x| + \frac{\Gamma'(a)}{\Gamma(a)} \right] + O\left(\ln x\right), \quad \text{as } x \to 0.$$

By the tip-regularity condition (3.10) we choose

$$H(x) = M\left(\frac{\sigma}{2\eta_0}, 1, x\right)$$

because $U(a, b, x)$ has a logarithmic singularity at $x = \xi = 0$. Based on the fact that the function $M(a, b, x)$ is a polynomial of degree $n$ in $x$ when $b \neq -m$ and $a = -n$ ($m$, $n$ are positive integers), it follows that as $x \to \infty$ ($\xi \to \infty$),

$$M\left(\frac{\sigma}{2\eta_0}, 1, x\right) \begin{cases} \text{grows algebraically when } \frac{\sigma}{2\eta_0} = -n = 0, -1, -2, \cdots, \\ \text{grows exponentially, otherwise.} \end{cases}$$
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It is necessary to require that the solution does not grow too fast near the dendrite’s root, therefore from the condition (3.13) we set
\[ \frac{\sigma^2}{2\eta_0^2} = -n. \]
It then follows that the fundamental solutions of eqn. (3.24) are given by
\[ H(\xi) = M\left(-n, 1, \frac{1}{2} \eta_0^2 \xi^2\right) = L_n\left(\frac{1}{2} \eta_0^2 \xi^2\right), \quad (3.26) \]
where \( L_n \) is called the Laguerre polynomial of degree \( n \).

Next we want to solve eqn. (3.25). Introducing the transformation
\[ K(\eta) = \frac{1}{\sqrt{y}} \exp\left(-\frac{1}{2} y\right) Z(y), \quad y = \frac{1}{2} \eta_0^2 \eta^2, \]
transforms the equation into the Whittaker’s equation
\[ Z'' + \left(-\frac{1}{4} + \frac{\kappa}{y} + \frac{1}{4y^2}\right) Z = 0, \]
where
\[ \kappa = \frac{\sigma^2}{2\eta_0^2} - \frac{1}{2} = -(n + \frac{1}{2}). \]

The two independent solutions of the Whittaker’s equation are given by
\[
Z(y) = \begin{cases} 
M_{\kappa,0}(y) &= \sqrt{y} \exp\left(-\frac{1}{2} y\right) \cdot M\left(\frac{1}{2} - \kappa, 1, y\right), \\
W_{\kappa,0}(y) &= \sqrt{y} \exp\left(-\frac{1}{2} y\right) \cdot U\left(\frac{1}{2} - \kappa, 1, y\right).
\end{cases}
\]

It follows that
\[
K(\eta) = \begin{cases} 
\exp\left(-\frac{1}{2} \eta_0^2 \eta^2\right) \cdot M\left(n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2\right), \\
\exp\left(-\frac{1}{2} \eta_0^2 \eta^2\right) \cdot U\left(n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2\right),
\end{cases}
\]
where \( M \) and \( U \) are the confluent hypergeometric functions defined in the above. The asymptotic behaviour as \( \eta \to \infty \),
\[
\begin{align*}
M\left(n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2\right) &\approx \frac{\Gamma(1)}{\Gamma(n + 1)} \left(\frac{1}{2} \eta_0^2 \eta^2\right)^n \exp\left(\frac{1}{2} \eta_0^2 \eta^2\right) \left[1 + \mathcal{O}\left(\frac{1}{\eta_0^2 \eta^2}\right)\right], \\
U\left(n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2\right) &\approx \left(\frac{1}{2} \eta_0^2 \eta^2\right)^{-\left(n+1\right)} \left[1 + \mathcal{O}\left(\frac{1}{\eta_0^2 \eta^2}\right)\right].
\end{align*}
\]
Accordingly and respectively, when \( \eta \to \infty \),

\[
K(\eta) \approx \begin{cases} 
\frac{\Gamma(1)}{\Gamma(n + 1)} \left( \frac{1}{2} \eta_0^2 \eta^2 \right)^n \left[ 1 + \mathcal{O} \left( \frac{1}{2} \eta_0^2 \eta^2 \right) \right] \\
\left( \frac{1}{2} \eta_0^2 \eta^2 \right)^{-(n+1)} \exp \left( -\frac{1}{2} \eta_0^2 \eta^2 \right) \left[ 1 + \mathcal{O} \left( \frac{1}{2} \eta_0^2 \eta^2 \right) \right]
\end{cases}
\]

(grows algebraically).

\[
\left( \frac{1}{2} \eta_0^2 \eta^2 \right)^{(n+1)} \exp \left( -\frac{1}{2} \eta_0^2 \eta^2 \right) \left[ 1 + \mathcal{O} \left( \frac{1}{2} \eta_0^2 \eta^2 \right) \right]
\]

(vanishes exponentially).

The far-field condition (3.9) finally determines the fundamental solutions for eqn. (3.25),

\[
K(\eta) = \exp \left( -\frac{1}{2} \eta_0^2 \eta^2 \right) \cdot U \left( n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2 \right). \tag{3.27}
\]

By combining (3.21), (3.23), (3.26) and (3.27), the explicit analytical solution for the zero-order temperature field in liquid state is given by

\[
T_{0,0}(\xi, \eta) = \eta_0^4 \exp \left( \frac{1}{2} \eta_0^2 \right) \left[ \hat{k}_0 I_0(\eta) + I_1(\eta) \right] \\
+ \exp \left( \frac{1}{2} \eta_0^2 \eta^2 \right) \cdot \sum_{n=0}^{\infty} \hat{\alpha}_n L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right) \frac{U \left( n + 1, 1, \frac{1}{2} \eta_0^2 \eta^2 \right)}{U \left( n + 1, 1, \frac{1}{2} \eta_0^2 \right)} 
\]

where \( \hat{k}_0, \hat{\alpha}_n (n \geq 0) \) are constants to be determined, \( I_0(\eta) \) and \( I_1(\eta) \) are exponential integrals defined by (3.22), \( L_n \) is the Laguerre polynomial of degree \( n \), \( U \) is the confluent hypergeometric function of the second kind. The solution (3.28) satisfies the boundary conditions (3.9), (3.10) and (3.13). The interface condition (3.14) determines the interface shape function \( h_{0,0}(\xi) \):

\[
h_{0,0}(\xi) = \eta_0^2 \exp \left( \frac{1}{2} \eta_0^2 \right) \left[ \hat{k}_0 I_0(1) + I_1(1) \right] + \sum_{n=0}^{\infty} \frac{\hat{\alpha}_n}{\eta_0^2} L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right). \tag{3.29}
\]

The tip-regularity condition (3.11) is automatically satisfied for the solution (3.29).

The solutions (3.28) and (3.29) are dependent of the parameter \( \eta_0^2 \). Given an undercooling \( T_\infty \), \( \eta_0^2 \) can be determined by the Ivantsov solution as

\[
-T_\infty = \eta_0^2 \exp \left( \frac{1}{2} \eta_0^2 \right) \int_1^{\infty} e^{-\frac{1}{2} \eta_0^2 u^2} \frac{1}{u} \, du.
\]

Practically, the undercooling \( -1 < T_\infty < 0 \) is often regarded as the experimental parameter under control. Figure 1 shows the variations of the undercooling \( T_\infty \) against \( \eta_0^2 \). It is observed that \( T_\infty \) is monotonic decreasing and \( T_\infty \to -1 \) when \( \eta_0^2 \) gets large.
enough. For a fixed small under-cooling $T_\infty$, it is often to introduce a small value of $\eta_0^2$. We note that $T_\infty = -0.1297$ at $\eta_0^2 = 0.1$, $T_\infty = -0.0803$ at $\eta_0^2 = 0.05$ and $T_\infty = -0.0238$ at $\eta_0^2 = 0.01$ for examples.

**Finding Undetermined Constants by the Boundary Conditions**

So far we have constructed the analytical series solutions of the zero-order temperature field in (3.28) as well as the zero-order interface shape function in (3.29). However, the solutions contain the undetermined constants $\hat{k}_0$ and $\hat{\alpha}_n$ that need to be fixed based on the boundary conditions (3.12) and (3.15). Differentiating (3.28) and (3.29) give

$$\frac{\partial T_{0,0}}{\partial \eta}(\xi, 1) = -\eta_0^4 (1 + \hat{k}_0) + \sum_{n=0}^{\infty} \hat{\alpha}_n c_n L_n\left(\frac{1}{2} \eta_0^2 \xi^2\right)$$

(3.30)

and

$$\xi \frac{d h_{0,0}}{d \xi} = \sum_{n=1}^{\infty} \frac{\hat{\alpha}_n}{\eta_0^2} (2n) \left[ L_n\left(\frac{1}{2} \eta_0^2 \xi^2\right) - L_{n-1}\left(\frac{1}{2} \eta_0^2 \xi^2\right) \right],$$

(3.31)

where

$$c_n = -\left(\eta_0^2 + 2(n + 1)\right) + 2(n + 1)^2 \cdot \frac{U\left(n + 2, 1, \frac{1}{2} \eta_0^2\right)}{U\left(n + 1, 1, \frac{1}{2} \eta_0^2\right)}$$

(3.32)

is a temporary constant. The results in (3.30)–(3.32) can easily be obtained by the following formulas:

$$\begin{align*}
L_n(z) &= M(-n, 1, z), \\
M'(a, b, z) &= \frac{a}{b} M(a + 1, b + 1, z), \\
U'(a, b, z) &= -a U(a + 1, b + 1, z), \\
bM(a, b, z) - bM(a - 1, b, z) - zM(a, b + 1, z) &= 0, \\
(b - a) U(a, b, z) + U(a - 1, b, z) - zU(a, b + 1, z) &= 0.
\end{align*}$$
Use the series in (3.18) to expand the interface condition in (3.15) in the orders of \( \omega \) and apply the derivatives in (3.30)–(3.31) to the zero-order expansion,

\[
\begin{align*}
-\eta_0^4 (1 + \hat{k}_0) + \sum_{n=0}^{\infty} \hat{\alpha}_n c_n L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right) \\
&= -\eta_0^2 (\eta_0^2 + 2) \left\{ \eta_0^2 \exp \left( \frac{1}{2} \eta_0^2 \right) \right\} + \sum_{n=0}^{\infty} \frac{\hat{\alpha}_n}{\eta_0^2} L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right) \bigg\}
\end{align*}
\]

\[
+ 2 \sum_{n=0}^{\infty} (n + 1) \hat{\alpha}_{n+1} L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right) - 2 \sum_{n=0}^{\infty} n \hat{\alpha}_n L_n \left( \frac{1}{2} \eta_0^2 \xi^2 \right).
\]

Comparing the both sides deduces the two relations:

\[
1 + \hat{k}_0 = (\eta_0^2 + 2) \exp \left( \frac{1}{2} \eta_0^2 \right) \left[ \hat{k}_0 I_0(1) + I_1(1) \right],
\]

\[
c_n \hat{\alpha}_n = -\left( \eta_0^2 + 2 \right) \hat{\alpha}_n + 2(n + 1) \hat{\alpha}_{n+1} - 2n \hat{\alpha}_n.
\]

It follows that the constants \( \hat{k}_0 \) and \( \hat{\alpha}_n \) can be determined by the constructions,

\[
\hat{k}_0 = \frac{(\eta_0^2 + 2) I_1(1) - \exp \left( -\frac{1}{2} \eta_0^2 \right)}{\exp \left( -\frac{1}{2} \eta_0^2 \right) - (\eta_0^2 + 2) I_0(1)} \tag{3.33}
\]

and

\[
\hat{\alpha}_{n+1} = \left[ 1 + \frac{c_n + \eta_0^2}{2(n + 1)} \right] \hat{\alpha}_n, \quad n = 0, 1, 2, \ldots. \tag{3.34}
\]

In the expression of (3.33), \( I_0(1) \) and \( I_1(1) \) are two improper integrals defined in (3.22) which are convergent. The limits are regarded as parameters which depend on the values of \( \eta_0^2 \). Theoretically, the values of \( \eta_0^2 \) can be determined by the under-cooling \( T_\infty \). In view of this, \( \hat{k}_0 \) may simply be regarded as a constant as long as the under-cooling temperature is fixed. On the other hand, (3.34) is a recurrence formula for the coefficients \( \hat{\alpha}_n \) provided that the information of the value of \( \hat{\alpha}_0 \) is known. We may rewrite (3.34) as \( \hat{\alpha}_{n+1} = d_{n+1} \hat{\alpha}_n \), where

\[
d_{n+1} = 1 + \frac{c_n + \eta_0^2}{2(n + 1)} = (n + 1) \cdot \frac{U \left( n + 2, 1, \frac{1}{2} \eta_0^2 \right)}{U \left( n + 1, 1, \frac{1}{2} \eta_0^2 \right)}
\]

of which the equality follows from (3.32). Again we may rewrite (3.34) as \( \hat{\alpha}_{n+1} = (d_{n+1} d_n \cdots d_1) \hat{\alpha}_0 \) for further reductions. The value of \( \hat{\alpha}_0 \) is uniquely determined by
applying the tip-regularity condition (3.12) to the solution (3.29), i.e., by setting $h_{0,0}(0) = 0$. After simplifying all the expressions, we write explicitly

\[
\begin{align*}
\hat{\alpha}_n &= \Gamma_n \hat{\alpha}_0, \\
\hat{\alpha}_0 &= \frac{-\eta_0^4 \exp\left(\frac{1}{2} \eta_0^2 \right) \left[ \hat{k}_0 I_0(1) + I_1(1) \right]}{1 + \sum_{k=1}^{\infty} \Gamma_k}, \\
\Gamma_n &:= (n!) \frac{U\left(n + 1, 1, \frac{1}{2} \eta_0^2 \right)}{U\left(1, 1, \frac{1}{2} \eta_0^2 \right)}, \quad n = 1, 2, \cdots.
\end{align*}
\] (3.35)

Our numerical results reveal that the series in the above formula converges for any fixed value of $\eta_0^2$ (small values of $\eta_0^2$ are assumed). In summary, once the undercooling temperature in melt, $T_\infty$, is fixed, the parameter $\eta_0^2$ can be determined by the Ivantsov similarity solution (3.1). Hence, the parameter $\hat{k}_0$ can be derived using (3.33). It follows that $\hat{\alpha}_0$ can be calculated using the explicit formula (3.35) in which the standard confluent hypergeometric function of the second kind, $U(n, 1, x)$. As a consequence, all the coefficients $\hat{\alpha}_n (n = 1, 2, \cdots)$ can be generated by the formula in (3.35).

4. Summary and Conclusions

In this paper, the globally valid asymptotic expansion solutions in the limit of $Re \to 0$ for the temperature field and the interface shape functions have been derived. The explicit formulas are expressions in terms of the standard confluent hypergeometric functions of the second kind. These further results reveal that when $U_\infty$ is not small, the effect of the convective motion may be large even when the Reynolds number is very small. In other words, when we apply a strong external flow, the convective effect would be large even for a viscous fluid. The nature of the phenomenon occurring in dendritic growth interacting with oscillatory external source has been identified with asymptotic theory recently developed.

References


