Necessary and sufficient conditions for Local optimality via weak subdifferentials

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Abstract

In this paper necessary and sufficient conditions for local optimality in finite dimensional normed spaces in terms of weak subdifferential are studied. It is based on newly introduced local sigma supporting function that describes the structure of given set at a particular point by using the contingent cone. In this way, the local optimality versions of related results in [8] are established.

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1. Introduction

In this paper the problem of necessary and sufficient conditions of optimality in non-convex optimization problems in a finite dimensional normed space is considered. As a classic technique, Fréchet differentiability helps to derive necessary conditions for optimization problems, however, it can not be applied for deriving sufficient conditions.
Investigations of normal cones to convex sets can be traced back to Minkowski [11]. Later, the concept of global normal cone was treated by Fenchel [4] as the outward normals to supporting half spaces to the set. Deriving necessary and sufficient conditions for Fréchet differentiable functions can be obtained by using Fréchet differentiability and normal cone defined for convex sets.

As the functions are not always Fréchet differentiable, the main approach to derive necessary conditions of optimality for nonsmooth and non-convex problems, is to generalize the notion of differentiability and normal cones. For the first time Moreau and Rockafellar [15] come up with the concept of subdifferentials as a generalization of the concept of ordinary derivative to deal with optimization problems involving convex nonsmooth (nondifferentiable) functions.

For different classes of Nonsmooth and non-convex optimization problems, the variety of different subdifferentials and normal cones have been introduced. Some of the major concepts related to this paper are mentioned here in brief. These concepts (normal cones, subdifferentials) depend on the characterizations of the objective function as well as properties of the variable space. Fréchet subdifferentials were first introduced for finite dimensions in [2] (under the name “lower semidifferentials”).

Clarke [3] introduced one of the important generalization of normal cones beyond convexity of functions and sets based on the generalization of the ordinary directional derivative. The locally Lipschitz functions are investigated by Clarke's directional derivative and its related subdifferential. We mention some of the attractive generalizations of directional derivative: generalized directional derivative (upper subderivative) introduced by Rockafellar [16, 17], lower semiderivative introduced by Penot [13], lower Dini (or Dini- Hadamard) directional derivative introduced by Ioffe [5], and subderivative introduced by Rockafellar and Wets [14]. Mordukhovich and Kruger [12] come up with the concept of the non-convex limiting Fréchet normal cone in finite dimensional spaces, and then the concept is extended to infinite dimensional spaces ([10]).

Augmented normal cones and weak subdifferentials are one of the most useful nonlinear global concepts introduced by Azimov and Gasimov [1]. Recently, Kasimbeyli and Mammadov [7, 8] considered "necessary" and "sufficient" conditions of optimality for a wide range of non-convex and nonsmooth problems in Euclidean space. This is the first generalization obtained in the form of a necessary and sufficient condition for global non-convex optimization problems.

In this paper, we consider local optimality conditions for non-convex and nonsmooth optimization problems by applying augmented normal cones and weak subdifferentials similar to global optimality conditions introduced in [8]. Together with this, we also establish the analogies of these results for a broader class of finite dimensional normed spaces.

2. **Local supporting function**

Throughout the paper we assume that $\mathbf{X}$ is a finite dimensional space with norm $\| \cdot \|$, $\Omega \subset \mathbf{X}$, $\bar{x} \in \Omega$ and $\mathbf{K} = \text{cl} (\text{cone}(\Omega - \bar{x}))$ where “cl” stands for the closure of a set.
“cone(\textbf{A})” for a given set \textbf{A} stands for
\[
\text{cone}(\textbf{A}) = \{ \lambda x : \lambda \geq 0, \ x \in \textbf{A} \}.
\]

The classical or Bouligand tangent cone, also called the contingent cone, for the set \( \Omega \) at the point \( \bar{x} \) is denoted by \( T_\Omega(\bar{x}) \) and defined as follows:
\[
T_\Omega(\bar{x}) = \{ w \in \textbf{X} : \exists x^v \to \bar{x} \text{ where } x^v \in \Omega \text{ and } \tau^v \downarrow 0 \text{ such that } \frac{x^v - \bar{x}}{\tau^v} \to w \}.
\]

The unit sphere and the unit ball of \( \textbf{X} \) are denoted by \( \textbf{U} \) and \( \textbf{B} \), respectively:
\[
\textbf{U} = \{ x \in \textbf{X} : \|x\| = 1 \}, \quad \textbf{B} = \{ x \in \textbf{X} : \|x\| \leq 1 \}.
\]

The norm of dual space \( \textbf{X}^* \) is denoted by \( \| \cdot \|_* \) where
\[
\| \cdot \|_* := \max\{ \langle \cdot, x \rangle : x \in \textbf{U} \}
\]
and \( \langle \cdot, \cdot \rangle \) is the scalar product. Note that we use maximum in definition of \( \| \cdot \|_* \) as the space \( \textbf{X} \) is finite dimensional space. The unit sphere and unit ball of dual space of \( \textbf{X} \) are denoted by \( \textbf{U}^* \) and \( \textbf{B}^* \), respectively. The notation \( B(x; \varepsilon) \) stands for the ball at center \( x \) with radius \( \varepsilon > 0 \); that is,
\[
B(x; \varepsilon) := \{ y \in \textbf{X} : \|y - x\| \leq \varepsilon \}.
\]

We will also denote by “int”, “bd” and “co” the interior, the boundary and the convex hull of a set.

**Definition 2.1.** Given point \( \bar{x} \in \Omega \), function \( \sigma_L(\cdot; \bar{x}) : \textbf{U}^* \to \mathbb{R} \) defined by
\[
\sigma_L(x^*; \bar{x}) := \max_{y \in T_\Omega(\bar{x}) \cap \textbf{U}^*} \langle x^*, y \rangle.
\]

will be called a local supporting function.

Clearly the maximum in (2.1) is attained. Moreover, it is not difficult to verify that \( \sigma_L(\cdot; \bar{x}) \) is a continuous and convex function defined on \( \textbf{X}^* \). The following lemma is a characterization for non-convex sets by the mean of function \( \sigma_L(x^*; \bar{x}) \).

**Lemma 2.2.** Assume that the following relation holds
\[
0 < \min_{x^* \in \textbf{U}^*} \sigma_L(x^*; \bar{x}) < 1.
\]

Then \( T_\Omega(\bar{x}) \) is non-convex and given any \( y \in \textbf{X} \setminus T_\Omega(\bar{x}) \) there exists \( \varepsilon, \delta > 0 \) such that
\[
\text{int}(\text{co} [B(\delta y; \varepsilon) \cup \{0\}]) \subset (\textbf{X} \setminus T_\Omega(\bar{x})) \setminus (\Omega - \{\bar{x}\}).
\]
Proof. Let $0 < \min_{x^* \in U^*} \sigma_L(x^*; \bar{x}) < 1$ and on the contrary assume that $T_\Omega(\bar{x})$ is convex. If $T_\Omega(\bar{x}) \neq X$ then there exists a closed hyperplane such that

$$T_\Omega(\bar{x}) \subseteq \{ x \in X : \langle x^*, x \rangle \leq 0 \}$$

for some linear function $x^* \in X^*$. Then

$$\sigma_L(x^*; \bar{x}) = \max\{ \langle x^*, y \rangle : y \in T_\Omega(\bar{x}) \cap U \} \leq 0,$$

which is a contradiction. On the other hand, if $T_\Omega(\bar{x}) = X$ then $\sigma_L(x^*; \bar{x}) = 1$ for all $x^* \in U^*$ which is again a contradiction. Therefore, $T_\Omega(\bar{x})$ is non-convex.

Now we show the second assertion of the lemma. Take any $y \in X \setminus T_\Omega(\bar{x})$. (2.4)

First we note that there is a sufficiently small number $\delta > 0$ such that

$$\lambda y /\in T_\Omega(\bar{x}), \quad \forall \lambda \in (0, \delta].$$

Indeed if $\lambda_k y \in T_\Omega(\bar{x})$ for some sequence $\lambda_k \to 0$, then for the sequence $x_k := \lambda_k y + \bar{x}$ we have $[x_k - \bar{x}] /\lambda_k \to y$ or $y \in T_\Omega(\bar{x})$ that contradicts (2.4).

Denote $z = \delta y$. Since $z \in X \setminus T_\Omega(\bar{x})$ and $X \setminus T_\Omega(\bar{x})$ is an open cone, there exists a small number $\epsilon > 0$ such that

$$\text{int}(\text{co} [B(z; \epsilon) \cup \{0\}]) \subset X \setminus T_\Omega(\bar{x}).$$

Next we show that the number $\epsilon > 0$ can be chosen so small that the relation

$$\text{int}(\text{co} [B(z; \epsilon) \cup \{0\}]) \cap (\Omega - \{\bar{x}\}) = \emptyset$$

(2.6)

is also satisfied. This will lead to (2.3) and complete the proof of the lemma.

On the contrary assume that (2.6) is not true. Then there are a sequence $\epsilon_k \to 0$ and a sequence of points $y_k \in \Omega - \{\bar{x}\}$ such that

$$y_k \in \text{int}(\text{co} [B(z; \epsilon_k) \cup \{0\}]), \quad \forall k.$$ 

Since $y_k$ is bounded, for the sake of simplicity we can assume that $y_k \to y^*$; and clearly $y^* \in \Omega$.

The above relation implies that $y_k$ can be represented in the form

$$y_k = \lambda_k z_k + (1 - \lambda_k) 0 = \lambda_k z_k$$

where $\lambda_k \in (0, 1]$ and $z_k \in B(z; \epsilon_k)$. Clearly, $z_k \to z$ and reminding that $y_k \to y^*$, the sequence $\lambda_k$ also converges to some number $\lambda^* \in [0, 1]$; that is, $\lambda_k \to \lambda^*$.

Now if $\lambda^* > 0$ then we have $\lambda^* z = \lambda^* \delta y \in \Omega - \{\bar{x}\}$ that contradicts (2.5). Thus, $\lambda^* = 0$ and consequently $\lambda_k \to 0$ and $y_k \to 0$. Denoting $x_k = y_k + \bar{x}$ we obtain $\lim[x_k - \bar{x}] /\lambda_k = \lim z_k = z$ which means that $z \in T_\Omega(\bar{x})$. This again contradicts (2.4).
Lemma is proved.

The following example shows that the inverse of Lemma 2.2 is not true; that is, the non-convexity of $T_{\Omega}(\bar{x})$ does not necessarily mean that $\min_{x^* \in U^*} \sigma_L(x^*; \bar{x}) = 1$.

**Example 2.3.** Let the set $R^2$ is equipped with $L_{\infty}$ (i.e. for any $x \in R^2$, $\|x\|_{\infty} = \max\{|x_1|, |x_2|\}$). It is clear that the dual norm of $R^2$ is $L_1$ (i.e. for any $x \in R^2$, $\|x\|_1 = |x_1| + |x_2|$). Let $\Omega = \{(x, y) \in R^2 : y \geq -x \text{ or } y \leq x\}$ and $\bar{x} = (0, 0)$. In this example, $\Omega = T_{\Omega}(\bar{x})$. It is not difficult to observe that there is no $x^* \in U^*$ with $\sigma_L(x^*; \bar{x}) < 1$ while $T_{\Omega}(\bar{x})$ is non-convex.

Below we show that the inverse of Lemma 2.2 is true in strictly convex spaces.

Before we prove the related lemma, we present the definition of strictly convex spaces and some properties.

**Definition 2.4.** (page 112, [18]) Normed space $X$ is called strictly convex if its unit ball is a strictly convex set; i.e., if $x \neq y$, $x, y \in U$ and $h = \frac{1}{2}(x + y)$ then $\|h\| < 1$.

Let $x' \in U$. By Theorem 5.20 in [18], there exists $x^* \in U^*$ such that

$$\langle x^*, x' \rangle = \max_{x \in U} \langle x^*, x \rangle = 1. \quad (2.7)$$

We also need the following property of strictly convex spaces.

**Proposition 2.5.** [6] Let $X$ be strictly convex space and $x^* \in X^*$. Then the maximum of $x^*$ on unit sphere $U$ is unique.

The following is a characterization of non-convex sets by applying local supporting function $\sigma_L(x^*; \bar{x})$ in strictly convex spaces which is not true in any normed space as shown in Example 2.3.

**Lemma 2.6.** Let $X$ be strictly convex space and $T_{\Omega}(\bar{x}) \neq X$. Then the relation (2.2) holds if and only if $T_{\Omega}(\bar{x})$ doesn’t belong to any half space; that is, there is no $z^* \in X^*$ such that $\langle z^*, x \rangle \leq 0$ for all $x \in T_{\Omega}(\bar{x})$.

**Proof.** In the proof of Lemma (2.2) it is shown that if (2.2) satisfy then $T_{\Omega}(\bar{x})$ doesn’t contain in a half space. That is why we consider only the inverse proving that (2.2) holds.

Assume that $\min_{x^* \in U^*} \sigma_L(x^*; \bar{x}) = 1$. Then $\sigma_L(x^*; \bar{x}) = 1$ for all $x^* \in U^*$. Take any $x \in U$. By equation 2.7, there exists $z^* \in X$ such that

$$\sigma_L(z^*; \bar{x}) = \max_{y \in T_{\Omega}(\bar{x}) \cap U} \langle z^*, y \rangle = 1 = \langle z^*, x \rangle = \max_{y \in U} \langle z^*, y \rangle.$$ 

By proposition 2.5, maximum $z^*$ on $U$ is unique and consequently $x \in T_{\Omega}(\bar{x}) \cap U$. Therefore $T_{\Omega}(\bar{x}) = X$ which is a contradiction. Thus we conclude that there is $x^* \in U^*$ such that $\sigma_L(x^*; \bar{x}) < 1$. 

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This content is a detailed explanation of the necessary and sufficient conditions for local optimality in the context of convex analysis and optimization. It involves proving the inverse of a lemma and providing an example to show the non-convexity of a set in a certain space. The definition of strictly convex spaces and the properties of these spaces are also discussed, along with a proposition that characterizes non-convex sets using local supporting functions in strictly convex spaces.
Now assume that there exists \( z^* \in U^* \) such that
\[
\sigma_L(z^*; \bar{x}) = \max_{y \in T_\Omega(\bar{x}) \cap U} \langle z^*, y \rangle \leq 0.
\]
Then, \( \langle z^*, y \rangle \leq 0 \) for any \( y \in T_\Omega(\bar{x}) \cap U \) which means \( T_\Omega(\bar{x}) \) contains in a half space. This contradicts the assumption of the lemma.

Lemma is proved. \( \blacksquare \)

In the last part of this section, we consider a relation between the separation property introduced in [9] and local supporting function \( \sigma_L(z^*; \bar{x}) \). We start with the definition of separation property.

**Definition 2.7.** ([9]) Let \( C \) and \( K \) be closed cones of a normed space \( X \). Let \( \tilde{C} \) and \( \tilde{K}^\partial \) be the closure of the sets \( \text{co}(C \cap U) \) and \( \text{co}((\text{bd}(K) \cap U) \cup \{0_X\}) \). The cones \( C \) and \( K \) are said to have the separation property with respect to the norm \( \| \cdot \| \) if
\[
\tilde{C} \cap \tilde{K}^\partial = \emptyset. \tag{2.8}
\]

Take any positive number \( \beta < 1 \) and \( x^* \in U^* \). Let
\[
C = \text{cone}\{x \in U : \langle x^*, x \rangle \geq \beta\}. \tag{2.9}
\]

In the following theorem we show that under some conditions on the local supporting function, the cones \( C \) and \( T_\Omega(\bar{x}) \) satisfy the separation property.

**Theorem 2.8.** Let there exists \( x^* \in U^* \) such that \( \sigma_L(x^*; \bar{x}) < 1 \). Then given any positive number \( \beta \in (\sigma_L(x^*; \bar{x}), 1) \), cones \( C \) and \( T_\Omega(\bar{x}) \) satisfy the separation property.

**Proof.** By the assumption of the theorem
\[
\max_{y \in T_\Omega(\bar{x}) \cap U} \langle x^*, y \rangle = \sigma_L(x^*; \bar{x}) < 1 = \|x^*\|_* = \max_{x \in U} \langle x^*, x \rangle. \tag{2.10}
\]

Denote \( \alpha = \sigma_L(x^*; \bar{x}) \) and take any \( \beta > 0 \) such that
\[
\alpha = \max_{y \in T_\Omega(\bar{x}) \cap U} \langle x^*, y \rangle = \sigma_L(x^*; \bar{x}) < \beta < 1. \tag{2.11}
\]

Since \( U \) is closed, there exists \( a \in U \) such that \( \langle x^*, a \rangle = \|x^*\|_* = 1 \). Then \( a \in C \) and \( C \neq \emptyset \).

Denote \( \tilde{C} = \text{cl}(\text{co}(C \cap U)) \) and \( \tilde{T_\Omega(\bar{x})}^\partial = \text{cl}(\text{co}((\text{bd}(T_\Omega(\bar{x})) \cap U) \cup \{0_X\})) \). We need to prove \( \tilde{C} \cap \tilde{T_\Omega(\bar{x})}^\partial = \emptyset \).

First we show that for any \( x \in \tilde{C} \) the inequality \( \langle x^*, x \rangle \geq \beta \) holds. Let \( x \in \text{co}(C \cap U) \).

Then the following representation is true \( x = \sum_{i=1}^{n+1} \alpha_i x_i \); where \( x_i \in C \cap U \) and \( \sum_{i=1}^{n+1} \alpha_i = 1 \). As \( x_i \in C \cap U \), from (2.9) we have
\[
\langle x^*, x \rangle = \sum_{i=1}^{n+1} \alpha_i \langle x^*, x_i \rangle \geq \beta. \tag{2.12}
\]
From continuity of \((x^*, \cdot)\) and (2.12), for any \(x \in \text{cl}(\text{co}(C \cap U))\), we have \((x^*, x) \geq \beta\).

It is clear from (2.11) that for any \(y \in T_\Omega(\bar{x}) \cap U\), the relation \((x^*, y) \leq \alpha < \beta\) holds. Since \(\beta > 0\), we have \((x^*, 0) = 0 < \beta\). Thus \((x^*, y) \leq \max\{\alpha, 0\} < \beta\) for any \(y \in T_\Omega(\bar{x})\). Therefore \(\overline{C} \cap T_\Omega(\bar{x}) = \emptyset\).

\[\text{■}\]

3. Necessary and sufficient conditions of optimality

In ([8]), necessary and sufficient condition of global optimality for a class of non-convex and nonsmooth optimization problems are considered by applying weak subdifferential and augmented normal cone. Below we give the definition of weak subdifferential and augmented normal cone introduced in [8]. Let \(f: \Omega \to \mathbb{R}\) be a single-valued function.

The weak subdifferential of \(f\) at \(\bar{x}\) on \(\Omega\) is defined as

\[
\partial_w f(\bar{x}) = \{(x^*, \alpha) \in X^* \times \mathbb{R} : f(x) - f(\bar{x}) \geq (x^*, x - \bar{x}) + \alpha \|x - \bar{x}\|, \forall x \in \Omega\}.
\]

The set

\[
N^A(\bar{x}; \Omega) = \{(x^*, \alpha) \in X^* \times \mathbb{R} : (x^*, x - \bar{x}) + \alpha \|x - \bar{x}\| \leq 0, \forall x \in \Omega\},
\]

is called an augmented normal cone to \(\Omega\) at \(\bar{x}\). These are global concepts and consequently the global optimality condition considered in [8] is

\[
(0, 0) \in \partial_w f(\bar{x}) + N^A(\bar{x}; \Omega).
\]

Below we consider the local versions of these definitions, where the set \(\Omega\) is replaced by

\[
\Omega_T(\bar{x}) := (T_\Omega(\bar{x}) + \bar{x}) \cap \Omega.
\]

Accordingly, we call corresponding sets \(\partial_w f(\bar{x})\) and \(N^A(\bar{x}; \Omega)\) a local weak subdifferential and a local augmented normal cone, respectively:

\[
\partial_w f(\bar{x}) = \{(x^*, \alpha) \in X^* \times \mathbb{R} : \exists \varepsilon > 0, f(x) - f(\bar{x}) \geq (x^*, x - \bar{x}) + \alpha \|x - \bar{x}\|, \forall x \in \Omega_T(\bar{x}) \cap B(\bar{x}, \varepsilon)\};
\]

\[
N^A(\bar{x}; \Omega) = \{(x^*, \alpha) \in X^* \times \mathbb{R} : (x^*, x - \bar{x}) + \alpha \|x - \bar{x}\| \leq 0, \forall x \in \Omega_T(\bar{x})\}.
\]

Clearly,

\[
\partial_w f(\bar{x}) \supset \partial_w f(\bar{x}) \supset \partial_w f(\bar{x}) \text{ and } N^A(\bar{x}; \Omega) = N^A(\bar{x}; \Omega_T(\bar{x})).
\]

In terms of these definitions, the necessary and sufficient conditions of local optimality can be established in the form of (3.1); that is,

\[
(0, 0) \in \partial_w f(\bar{x}) + N^A(\bar{x}; \Omega).
\]

Clearly, if \(\bar{x}\) is a global optimal solution then it is also a local optimal solution and the optimality condition (3.2) is satisfied if (3.1) holds. Naturally, in a convex case, these conditions coincide.
Classical directional derivative of function $f$ at $\bar{x}$ on direction $h$ is defined as follows:

$$f'(\bar{x}; h) := \lim_{t \downarrow 0} \frac{f(\bar{x} + th) - f(\bar{x})}{t}.$$  

We will require the following assumptions hold.

**Assumption A:**

A1: $f'(\bar{x}; h)$ is defined for all $h \in T_{\Omega}(\bar{x})$ and is lower semicontinuous in $h$;

A2: There exist $\varepsilon, \delta > 0$ such that

$$f(x) - f(\bar{x}) \geq \delta f'(\bar{x}; x - \bar{x}), \; \forall x \in \Omega_T(\bar{x}) \cap B(\bar{x}, \varepsilon). \quad (3.3)$$

The next theorem describes a necessary condition of optimality in the form (3.2) that generalizes Theorem 5 in [8] to any normed spaces by assuming an additional condition $\sigma_L(x^*; \bar{x}) < 1$ and provides the local optimality version of that theorem.

**Theorem 3.1.** Let $\bar{x} \in \Omega \subset X$ be a local minimizer of $f$. Assume that Assumption A holds, there exists $x^* \in U^*$ such that $\sigma_L(x^*; \bar{x}) < 1$, $\Omega \setminus \{\bar{x}\} \neq \emptyset$ and

$$\bar{\beta} := \inf \{ f'(\bar{x}; h) : h \in T_{\Omega}(\bar{x}) \cap U \} > 0. \quad (3.4)$$

Then, there exists a nontrivial solution to (3.2); namely, there is $(z^*, \alpha) \in \partial_{lw} \Omega_{f}(\bar{x})$ such that

$$\langle -z^*, x - \bar{x} \rangle + \alpha \|x - \bar{x}\| \geq 0, \; \forall x \in \Omega_T(\bar{x}), \quad (3.5)$$

and

$$\langle z^*, z - \bar{x} \rangle + \alpha \|z - \bar{x}\| < 0, \; \text{for some } z \notin \Omega_T(\bar{x}). \quad (3.6)$$

The rest of proof is the same as in Theorem 4 in [8] to show that there exists $z^* \neq 0$ and $\alpha \geq 0$ such that $(z^*, \alpha) \in \partial_{lw} f(\bar{x})$ and

$$\langle z^*, x - \bar{x} \rangle + \alpha \|x - \bar{x}\| \geq 0, \; \forall x \in \Omega_T(\bar{x}), \quad (3.7)$$

Multiplying both sides of (3.5) by $-1$, we obtain

$$\langle -z^*, x - \bar{x} \rangle - \alpha \|x - \bar{x}\| \leq 0 \; \forall x \in \Omega_T(\bar{x}),$$

that means $(-z^*, -\alpha) \in N_{\bar{x}}^L(\bar{x}; \Omega)$. Thus, (3.2) is satisfied.
Now we show that \((z^*, \alpha)\) is a nontrivial solution; that is, \(-\alpha > -\|z^*\|_*\) or \(\alpha < \|z^*\|_*\). By contradiction let \(\alpha \geq \|z^*\|_*\). Then from the Cauchy-Schwarz inequality it follows that
\[
(z^*, x - \bar{x}) + \alpha \|x - \bar{x}\| \geq (z^*, x - \bar{x}) + \|z^*\|_* \cdot \|x - \bar{x}\| \geq 0, \quad \forall x \in \Omega_T(\bar{x}).
\]
This contradicts (3.6). Theorem 3.1 is proved.

Theorem 6 from [8] states that condition (3.1) is also a sufficient condition of optimality. This result is straightforward from the definitions of \(\partial w/\Omega f(\bar{x})\) and \(\text{N}(\bar{x}; \Omega)\) and does not require any additional assumptions.

In our case, condition (3.2) may not be a sufficient condition of local optimality if the set \(\Omega\) is not convex around \(\bar{x}\). The next theorem investigates this problem. It shows that if function \(f\) is Lipschitz continuous then the sufficiency of condition (3.2) can be established. Note that this theorem is in any normed spaces, while Theorem 6 from [8] considers the Euclidean norm.

**Theorem 3.2.** Let (3.2) has a solution, \(f\) is Lipschitz continuous on \(\Omega\) and
\[
\inf \{f'(\bar{x}; h) : h \in \text{bd}(T_\Omega(\bar{x}))\} \geq \delta > 0. \tag{3.7}
\]
Then \(\bar{x} \in \Omega\) is a local minimizer of function \(f\) on \(\Omega\).

**Proof.** By assumption, there is \((z^*, \alpha) \in \partial w/\Omega f(\bar{x})\) such that \((-z^*, -\alpha) \in \text{N}(\bar{x}; \Omega)\).

Then there is \(\varepsilon > 0\) such that the following holds
\[
f(x) - f(\bar{x}) \geq (z^*, x - \bar{x}) + \alpha \|x - \bar{x}\| \geq 0; \quad \forall x \in \Omega_T(\bar{x}) \cap B(\bar{x}, \varepsilon).
\]
We need to show that there is a sufficiently small number \(\varepsilon' \leq \varepsilon\) such that this inequality also holds for all \(x \in \Omega \cap B(\bar{x}, \varepsilon')\); that is,
\[
f(x) - f(\bar{x}) \geq 0; \quad \forall x \in \Omega \cap B(\bar{x}, \varepsilon'). \tag{3.8}
\]
On the contrary, assume that (3.8) does not hold. Then for any \(n \in N\) satisfying \(n \varepsilon > 1\) there exists \(x_n \in B\left(\bar{x}; \frac{1}{n}\right) \cap \Omega\) such that
\[
f(x_n) - f(\bar{x}) < 0
\]
holds.

Clearly \(x_n \notin \Omega_T(\bar{x})\) and \(\{x_n - \bar{x}\}_{n \in N}\) approaches 0. Moreover, there exists a convergent subsequence of \(z_n := \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\) as \(z_n \in U\) is bounded. For the sake of simplicity let \(z_n \to z\). By definition of \(T_\Omega(\bar{x})\) it follows that \(z \in T_\Omega(\bar{x})\). On the other hand since \(x_n \notin \Omega_T(\bar{x})\) we have \(x_n - \bar{x} \notin T_\Omega(\bar{x})\) which means that \(z \notin \text{bd} T_\Omega(\bar{x}) \cap U\). By assumption (3.7)
\[
f'(\bar{x}; z) \geq \delta > 0. \tag{3.10}
\]
Denote $\lambda_n = \|x_n - \bar{x}\|$. Since function $f$ is Lipshchitz, there is $K > 0$ such that for any $n$ the inequality

$$
\|f(\bar{x} + \lambda_n z_n) - f(\bar{x} + \lambda_n z)\| \leq \lambda_n K \|z_n - z\|
$$

holds and it implies

$$
f(\bar{x} + \lambda_n z_n) - f(\bar{x} + \lambda_n z) \geq -\lambda_n K \|z_n - z\|.
$$

(3.11)

Now, from (3.10) we have

$$
f(\bar{x} + \lambda_n z) = f(\bar{x}) + \lambda_n f'(\bar{x}; z) + o(\lambda_n)
$$

where $\frac{o(\lambda_n)}{\lambda_n} \to 0$ as $\lambda_n \to 0$. This together with (3.11) leads to

$$
f(\bar{x} + \lambda_n z_n) \geq f(\bar{x}) + \lambda_n \delta + o(\lambda_n) - \lambda_n K \|z_n - z\|
$$

or

$$
f(\bar{x} + \lambda_n z_n) \geq f(\bar{x}) + \lambda_n \left( \delta + \frac{o(\lambda_n)}{\lambda_n} - K \|z_n - z\| \right).
$$

Since $\delta > 0$, $\frac{o(\lambda_n)}{\lambda_n} \to 0$ and $\|z_n - z\| \to 0$, we obtain that the inequality $f(\bar{x} + \lambda_n z_n) \geq f(\bar{x})$ holds for sufficiently large $n$. Taking into account the notations $\lambda_n = \|x_n - \bar{x}\|$ and $z_n = \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}$ we have $\bar{x} + \lambda_n z_n = x_n$ and therefore $f(x_n) \geq f(\bar{x})$ which contradicts (3.9).

Theorem 3.2 is proved.

The following example shows that if $f$ is not Lipshchitz then Theorem 3.2 may not be true even if condition (3.7) still holds.

**Example 3.3.** Let $X = \mathbb{R}^2$ and $\Omega = \{ (x_1, x_2) : x_1 \geq 0, x_2 \leq 2x_1^2 \}$. Function $f$ is given by

$$
f(x_1, x_2) = \begin{cases} 
  x_1 - x_2, & \text{if } x_2 \leq 0; \\
  x_1 - \frac{x_2}{x_1}, & \text{if } x_1 \neq 0, x_2 \in (0, 2x_1^2); \\
  -x_1, & \text{if } x_2 \geq 2x_1^2.
\end{cases}
$$

(3.12)

It is not difficult to observe that $f$ is continuous on $\Omega$. We have $T\Omega(\bar{x}) = \{ (x_1, x_2) : x_1 \geq 0, x_2 \leq 0 \}$ with two boundary directions $\hat{h} = (1, 0)$ and $\tilde{h} = (0, -1)$. The directional derivatives at these directions can be easily calculated to obtain

$$
f'(\bar{x}; \hat{h}) = 1 > 0, \quad f'(\bar{x}; \tilde{h}) = 1 > 0.
$$

Thus, condition (3.7) holds. We show that function $f$ is not Lipshchitz continuous.
Take any $\epsilon > 0$ and $h = (0, 1)$. Calculate the directional derivative at point $y^\epsilon = (\epsilon, 0)$. We have

$$
 f'(y^\epsilon; h) = \lim_{t \downarrow 0 \atop |t| < 2\epsilon^2} \frac{f(y^\epsilon + t h) - f(y^\epsilon)}{t} = -\frac{1}{\epsilon}.
$$

Then, $f'(y^\epsilon; h) \to -\infty$ as $\epsilon \downarrow 0$.

Therefore in this example condition (3.7) is satisfied but function $f$ is not Lipschitz continuous. As a result, Theorem 3.2 is not true; that is, $x = (0, 0) \in \Omega$ is not a local minimizer of $f$. Indeed, for $x^\epsilon = (\epsilon, \epsilon^2) \in \Omega$ we have $f(x^\epsilon) = -\epsilon < 0 = f(x)$ and $x^\epsilon \to x$ as $\epsilon \downarrow 0$.

4. Conclusions

In this paper, we introduce local supporting function $\sigma_L(x^*; x)$ and apply it to characterize non-convex sets at a particular points. The necessary and sufficient conditions of local optimality are derived in terms of three concepts weak subdifferentials, augmented normal cones and the function $\sigma_L(x^*; x)$. Similar optimality conditions for global optimization are obtained in [8] for Euclidean spaces. This paper generalizes these conditions to local optimization problems and to any finite dimensional normed spaces by applying function $\sigma_L(x^*; x)$.

References


