Analytical Solution of Magnetohydrodynamic Flow with Varying Viscosity in an Annular Channel

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Abstract

Flow of a viscous, incompressible, electrically conducting fluid with varying viscosity through an annular pipe in the presence of a radial magnetic field is considered. Here we take viscosity as a function of radial distance. Exact solutions for velocity, rate of volume flow and stress on the walls of channel are obtained. In limiting case these solutions reduce to the classical case flow when viscosity is constant and magnetic field is zero. Obtained results are exhibited graphically.

AMS Subject Classification:
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1. Introduction

Generally scientists, engineers and mathematicians assume that the viscosity is constant to simplify the fluid flow problems but sometimes this assumption does not conform to the real situation. In many applications viscosity of fluid is variable; naturally it depends on temperature and concentration. It may also change due to suspending solid particles in the form of ash or soot or as a result of corrosion. Considerable amount of work has been done on fluid flow problems with temperature dependent viscosity; to cite a few cases, we may mention the papers by D.D. Joseph (1964), Dai (1992), Saikrishnan (2002), Hazema and Eswara (2004). But very little literature is available on the concentration dependent viscosity problems. Varying viscosity fluid flow problems involving stratified fluids have been taken up amongst others by Malik and Hooper (2005) and Pasa and Titoud (2005.) Payr et al. (2005) considered the flow problem with viscosity as exponential function.
of concentration and solved the problem for superimposed upper and lower layers of fluids of different viscosities. Since it may be seen that the concentration is spatially varying quantity it is reasonable to assume that viscosity may be taken as a function of space coordinates. This type of problems appear where changing concentration of solute, particularly mixing of two fluids, occurs. Thiele (1999), Haber (2006) and J.B. Shukla (1980) studied fluid flow problems involving spatially varying viscosity. Hooper et al. (1982) studied the flow of fluid of non-uniform viscosity in converging and diverging channels. He considered viscosity as a function of angular coordinate and studied the effect of viscosity variation on the velocity profiles.

Magnetohydrodynamic flow through channel and pipes finds wide applications in industry. Magnetic field can be used to control the motion of electrically conducting fluids. Considerable research studies have been carried out to investigate the flow through pipes.

The steady flow of a viscous, incompressible, electrically conducting fluid along a channel through a magnetic field has been considered by a number of investigators. The earliest known published work in this field was done by Hartmann (1937) and Hartmann and Lazarus (1937). Hartmann, in his well-known paper considered the flow between two parallel, non-conducting walls in the presence of magnetic field normal to the walls. Shercliff (1953) solved the corresponding problem of the flow in a pipe of rectangular cross-section and obtained exact solution for fluid velocity and magnetic field. Later in 1962 Shercliff considered the flow of a conducting liquid at high Hartmann number in non-conducting pipes of arbitrary cross-section under uniform transverse magnetic field. Gold (1962) obtained exact solution to the problem of the steady one-dimensional flow of an incompressible, viscous, electrically conducting fluid through a circular pipe in the presence of transverse uniform magnetic field, valid for all values of Hartmann number. Globe (1959) solved the problem of steady flow of an electrically conducting, incompressible fluid in the annular space between two infinitely long circular cylinders, under a radially impressed magnetic field. He found exact solution for velocity and magnetic field. Molokovt and Allen (1992) investigated the flow of a viscous incompressible fluid between two non-conducting cylinders under strong radial magnetic field applied parallel to the free surface of the liquid. The flow region was divided into various subregions and the asymptotic solutions for each subregion were obtained for $M \to \infty$ where $M$ is the Hartmann number.

However, the present authors were unable to locate any theoretical or experimental work dealing with the magnetohydrodynamic flow in channels and pipes having non-uniform viscosity distribution. Thus, there is a need for investigation of such problems. Hence, the objective of present work is to investigate the influence of viscosity variation on the flow of an electrically conducting fluid through annular channel in the presence of magnetic field.

In this chapter we consider the flow of a viscous, incompressible and electrically conducting fluid in an annular pipe in the presence of radial magnetic field when viscosity is a function of radial distance. Here we consider two particular interesting cases of small viscosity variation; case I, when the viscosity variation is linear, and case II, when the
viscosity variation is quadratic. Exact solutions for velocity and rate of volume flow, in both cases, have been obtained in terms of hypergeometric functions. In limiting case these solutions reduce to the classical case flow when viscosity is constant and magnetic field is zero. Obtained results are presented graphically.

2. Formulation of the problem

![Figure 1: Annular channel with radial magnetic field.](image)

Consider the fully developed, steady axial flow of a viscous, incompressible, electrically conducting fluid in an annular cylindrical pipe of inner radius \( r_1 \) and outer radius \( r_2 \). A radial magnetic field \( H_0 = \frac{\omega}{r} \), \( \omega \) a constant, is applied. Flow is governed by magnetohydrodynamic equations with Lorentz force as the external force, subject to relevant boundary conditions which will be specified in the sequel for the concerned problem. Simplified version, of magnetohydrodynamic equations for the steady case appropriate for the present problem as considered by Globe (1959) are given below

\[
\nabla \times \mathbf{H} = 4\pi \mathbf{j} \tag{2.1}
\]

\[
\nabla \times \mathbf{E} = -\mu_m \frac{\partial \mathbf{H}}{\partial t} = 0 \tag{2.2}
\]

\[
\nabla \cdot \mathbf{H} = 0 \tag{2.3}
\]

\[
\mathbf{j} = \sigma (\mathbf{E} + \mu_m \mathbf{V} \times \mathbf{H}) \tag{2.4}
\]

\[
\nabla \cdot \mathbf{V} = 0 \tag{2.5}
\]

\[
\rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mu \nabla^2 \mathbf{V} + 2(\nabla \mu \cdot \nabla)\mathbf{V} + (\nabla \mu) \times (\nabla \times \mathbf{V}) + \mu_m \mathbf{j} \times \mathbf{H} \tag{2.6}
\]

Where \( \mu \) is the viscosity of fluid, \( \mu_m \) the magnetic permeability, \( \mathbf{V} \) the fluid velocity, \( \mathbf{H} \) the magnetic field, \( \mathbf{E} \) the electric field, \( \mathbf{j} \) the current density vector, \( \rho \) the density, \( p \) the pressure. It may be noted that quantities having bar on the top are dimensional quantities...
and equation (2.6) is written for varying viscosity. To get the coupled equation for \( \mathbf{V} \) and \( \mathbf{H} \) we eliminate \( j \) and \( \mathbf{E} \) amongst equations (2.1), (2.2) and (2.4), and \( j \) between equations (2.1) and (2.6). Making use of equations (2.3) and (2.5), we obtained the resulting equations in the form.

\[
\lambda \nabla^2 \mathbf{H} = - (\mathbf{H} \cdot \nabla) \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{H} \quad (2.7)
\]

\[
\rho (\mathbf{V} \cdot \nabla) \mathbf{V} = - \nabla \left( p + \frac{\mu_m}{8\pi} |\mathbf{H}|^2 \right) + \mu \nabla^2 \mathbf{V} + 2 (\nabla \mu \cdot \nabla) \mathbf{V} + (\nabla \mu) \times (\nabla \times \mathbf{V}) + \frac{\mu_m}{4\pi} (\mathbf{H} \cdot \nabla) \mathbf{H} \quad (2.8)
\]

Where \( \lambda = \frac{1}{4\pi \mu m \sigma} \), is the magnetic viscosity. Now we use following conditions to simplify the above equations (2.7) and (2.8):

- We assume symmetry around the axis so that \( \frac{\partial}{\partial \phi} = 0 \).
- Direction of flow is only along axis of the cylinder, so velocity components \( V_r = V_\phi = 0 \) and from equation (2.5) \( \frac{\partial V_z}{\partial z} = 0 \). Thus \( V_z \) is a function of \( r \) only.
- Since direction of fluid velocity \( \mathbf{V} \) and applied magnetic field \( \mathbf{H}_o \) are along the direction of \( z \) and \( r \) thus \( \mathbf{V} \times \mathbf{H} \) is the direction of \( \phi \). By equation (2.4), it is clear that, in the absence of free charges in the flow and an applied electric field \( \mathbf{E} \) current \( j \) has only \( j_\phi \) component. Since \( H_\phi \) can arise only from an \( r \) and \( z \) components of currents in the fluid, thus \( H_\phi \) vanishes everywhere in the channel.
- It is compatible with the governing equations and boundary conditions to take component \( H_r \) as \( H_o \), applied magnetic field.

- With \( H_r = H_o = \frac{\omega}{r} \) and \( H_\phi = 0 \), we have from equation (2.3) \( \frac{\partial H_z}{\partial z} = 0 \).

When these simplifications are introduced, the equations corresponding to (2.7) and (2.8) in cylindrical polar co-ordinates reduce to following equations,

\[
\mu \left( \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right) + \frac{\partial \mu}{\partial r} \frac{\partial V_z}{\partial r} + \frac{\mu_m \omega}{4\pi} \frac{\partial H_z}{\partial r} = \frac{\partial p}{\partial z} \quad (2.9)
\]

\[
- \frac{\mu_m}{4\pi} H_z \frac{\partial H_z}{\partial r} = \frac{\partial p}{\partial r} \quad (2.10)
\]

\[
\lambda \left( \frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} \right) = - \frac{\omega}{r} \frac{\partial V_z}{\partial r} \quad (2.11)
\]
Boundary conditions required for the solutions are
\[ V_z (r_1) = 0, \quad V_z (r_2) = 0 \]  
\[ H_z (r_2) = 0, \]  
\[ \frac{dH_z}{dr} \bigg|_{r=r_2} = 0. \]  

Boundary condition (2.12) are obvious no fluid slip condition at the walls. The condition (2.13) is justified by the fact that \( \mathbf{j} \) has a \( \phi \) component only, so that the currents in the annular channel are like those in an infinite solenoid. These current will therefore produce no field for \( r > r_2 \), and since there is no impressed field in the \( \bar{z} \) direction, continuity of the tangential component of \( \mathbf{H} \) requires (2.13) to be true. Now from equation (2.1), we have
\[ 4\pi j_\phi = \frac{\partial H_z}{\partial r} \]  
Since walls are nonconducting hence current vanish at \( r = r_2 \). Therefore by above equation \( \frac{\partial H_z}{\partial r} \) also vanish there. This justifies condition (2.14).

Now equation (2.11) is
\[
\frac{\lambda}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H_z}{\partial r} \right) = -\frac{\omega}{r} \frac{\partial V_z}{\partial r} \tag{2.16}
\]
Integral of this equation gives us
\[
\lambda \left( r \frac{\partial H_z}{\partial r} \right) = -\omega V_z + K \tag{2.17}
\]
Using boundary condition (2.12) and (2.14) at \( r = r_2 \), we get \( K = 0 \). Thus above equation reduce to
\[
\lambda \left( r \frac{\partial H_z}{\partial r} \right) = -\omega V_z \tag{2.18}
\]
Replacing \( H_z \) in equation (2.9) with the use above equation, we obtain
\[
\mu \frac{d^2 V_z}{dr^2} + \left( \frac{\mu}{r} + \frac{d\mu}{dr} \right) \frac{dV_z}{dr} - \frac{\mu^2 \omega^2 \sigma}{r^2} V_z = \frac{\partial p}{\partial z} \tag{2.19}
\]
Now \( H_z \) and \( V_z \) are function of \( r \) only. It follows from equation (2.9) and (2.10) that \( \frac{\partial p}{\partial z} \) is independent of \( z \) and \( r \). Due to axis symmetric flow it is also independent of \( \phi \). Thus we take \( \frac{\partial p}{\partial z} \) is constant. We take it \((-P)\). Equation (2.19) then becomes
\[
\mu \frac{d^2 V_z}{dr^2} + \left( \frac{\mu}{r} + \frac{d\mu}{dr} \right) \frac{dV_z}{dr} - \frac{\mu^2 \omega^2 \sigma}{r^2} V_z = -P \tag{2.20}
\]
Now it will be convenient to express the equation in non-dimensional form by introducing the following transformation

\[ r = r_1 \tilde{r}, \quad V_z = \frac{Pr_1^2}{\mu_0} \tilde{V} \quad \text{and} \quad \mu = \mu_0 \tilde{\mu}(\tilde{r}). \]

Here characteristic velocity is taken as \( \frac{Pr_1^2}{\mu_0} \) and \( \mu_0 \) is characteristic viscosity; we may take it as average viscosity. Equations (2.20) in non-dimensional variables \( \tilde{r}, \tilde{V} \) and \( \tilde{\mu} \) can be written as

\[ \mu \frac{d^2V}{dr^2} + \left( \frac{\mu}{r} + \frac{d\mu}{dr} \right) \frac{dV}{dr} - \frac{m^2}{r^2} V = -1 \quad (2.21) \]

Where \( m^2 = \frac{\mu_0^2 \omega^2 \sigma}{\mu_0} \) is a dimensionless magnetic parameter. Here we drop the bars over the dimensionless variables for the convenience.

### 3. Solution of the Problem

Equation (2.21) is a differential equation of order two, thus two boundary conditions are required for the solution. These are no slip conditions (2.12) that in non-dimensional variables are given by

\[ \tilde{V} (1) = 0 \quad \text{and} \quad \tilde{V} (\theta) = 0 \quad (3.1) \]

Where \( \theta = \frac{r_2}{r_1} \) is the gap parameter. The analytic solution of the equation (2.21) for the general variation of viscosity is difficult to deal, hence here we consider two special cases. We will drop the bars over the non-dimensional variables for the convenience.

#### 3.1. Case -I

When \( \mu = (1 - \epsilon r) \), where \( \epsilon \) is a non-dimensional viscosity variation parameter such that \( |\epsilon\theta| < 1 \). With this viscosity distribution governing equation of motion (2.21) becomes

\[ (1 - \epsilon r) r^2 \frac{d^2V}{dr^2} + (1 - 2\epsilon r) r \frac{dV}{dr} - m^2 V = -r^2 \quad (3.2) \]

This is Hypergeometric equation. Its general solution subject to the boundary condition (3.1) may be expressed in terms of hypergeometric functions [Ref. Abramowitz and Stegun] as given below

\[ V = C_1 r^m F_1(m, 1 + m; 1 + 2m; \epsilon r) + C_2 r^{-m} F_1(-m, 1 - m; 1 - 2m; \epsilon r) \]

\[ - \frac{r^2}{(4 - m^2)} F_2(1, 2, 3; 3 - m, 3 + m; \epsilon r). \quad (3.3) \]
Where $C_1$ and $C_2$ are evaluated by applying boundary conditions (3.1)

\[
C_1 = \frac{\left( \theta^{2-m} F_1(-m, 1 - m; 1 - 2m; \epsilon) \right)}{(m^2 - 4) \left( \frac{2 F_1(-m, 1 - m; 1 - 2m; \epsilon \theta)}{2 F_1(m, 1 + m; 1 + 2m; \epsilon) - \theta^{2m} F_1(-m, 1 - m; 1 - 2m; \epsilon \theta)} \right)}, \quad (3.4)
\]

\[
C_2 = \frac{\left( \theta^{2-m} F_1(m, 1 + m; 1 + 2m; \epsilon) \right)}{(m^2 - 4) \left( \frac{2 F_1(-m, 1 - m; 1 - 2m; \epsilon \theta)}{2 F_1(m, 1 + m; 1 + 2m; \epsilon \theta) - \theta^{2m} F_1(-m, 1 - m; 1 - 2m; \epsilon \theta)} \right)}, \quad (3.5)
\]

Solution in particular cases, when $\epsilon = 0$, $m \neq 0$ and when $m = 0$, $\epsilon \neq 0$ can be obtain by solving the reduced equation of motion (2.21) corresponding to these cases separately with the use of boundary condition (3.1). That are

\[
V = \frac{1}{(m^2 - 4)} \left[ r^2 - \frac{\theta^2 \sinh(m \log r) - \sinh(m \log \theta)}{\sinh(m \log \theta)} \right]; \quad \epsilon = 0 \quad (3.6)
\]

and

\[
V = C_3 + C_4 [\log r - \log(1 - \epsilon r)] + \frac{[r \epsilon + \log(1 - \epsilon r)]}{2 \epsilon^2}; \quad m = 0 \quad (3.7)
\]

Where $C_3$ and $C_4$ are,

\[
C_3 = \frac{\epsilon [\log(1 - \epsilon \theta) - \theta \log(1 - \epsilon)] - \log \theta [\epsilon + \log(1 - \epsilon)]}{2 \epsilon^2 [\log(1 - \epsilon) + \log \theta - \log(1 - \epsilon \theta)]}, \quad (3.8)
\]

\[
C_4 = \frac{\log(1 - \epsilon) - \log(1 - \epsilon \theta) - \epsilon(\theta - 1)}{2 \epsilon^2 [\log(1 - \epsilon) + \log \theta - \log(1 - \epsilon \theta)]} \quad (3.9)
\]

In limiting case when $m \rightarrow 0$ in (3.6) and when $\epsilon \rightarrow 0$ in (3.9) we get

\[
V = \frac{(\theta^2 - 1) \log r - (r^2 - 1) \log \theta}{4 \log \theta} \quad (3.10)
\]
Figure 2: Velocity profiles of the flow when viscosity variation is \( \mu = 1 - \epsilon r \) and \( m = 2, 6 \) for different \( \epsilon \).

Figure 3: Velocity profiles of the flow for different \( m \) when viscosity variation is \( \mu = (1 - 0.2r) \).

Which is the velocity profile of the classical Poiseuille flow between two co-axial cylinders.

Figure (2) and (3) shows the effect of viscosity variation parameter \( \epsilon \) and magnetic parameter \( m \) on the velocity profile given by eq. (3.3). These figures reveal

- For fixed \( m \), increase in \( \epsilon \) causes increase in velocity. This is because increase in \( \epsilon \) results in decrease in the average viscosity according to the choice \( \mu = (1 - \epsilon r) \) (c.f. Fig. 2).

- Velocity profiles are almost symmetrical when viscosity is constant (\( \epsilon = 0 \)) and profiles get more and more asymmetric with position of maximum velocity shifting towards outer cylinder as \( \epsilon \) increases (c.f. Fig. 2).

- Effect of parameter \( \epsilon \) on the flow is strong for large values of \( m \) (c.f. Fig. 3).

- The effect of increase in magnetic parameter \( m \) is to flatten the velocity profile so that the core gets formed (c.f. Fig.3). This is because increase in the magnetic field leads to an increase in the Lorentz force in opposing the flow.

Thus we see that both the parameters \( m \) and \( \epsilon \) effect the flow considerably.
3.1.1 Rate of volume flow:

Non-dimensional volume flow rate is given by

$$Q = 2\pi \int_{\theta_1}^{\theta} V(r) r \, dr.$$  \hspace{1cm} (3.11)

Substituting $V$ from (3.3) in above equation, we get

$$Q = 2\pi C_1 \int_{\theta_1}^{\theta} r^{m+1} F_1(m, 1 + m; 1 + 2m; \epsilon r) dr$$

$$+ 2\pi C_2 \int_{\theta_1}^{\theta} r^{-m+1} F_1(-m, 1 - m; 1 - 2m; \epsilon r) dr$$

$$- \frac{2\pi}{4 - m^2} \int_{\theta_1}^{\theta} r^3 \, F_2(1, 2, 3; 3 - m, 3 + m; \epsilon r) dr$$  \hspace{1cm} (3.12)

where $C_1$ and $C_2$ are given by equations (3.4) and (3.5) and integration is performed with the use of following identity, which can be derived by direct integration after putting the Hypergeometric function as a sum of infinite series; thus, obtaining

$$\int x^k pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; x) dx = \frac{x^{k+1}}{k+1} p+1 F_{q+1}(a_1, \ldots, a_p, k+1; b_1, \ldots, b_q, k+2; x)$$  \hspace{1cm} (3.13)

Rate of volume flow in the absence of magnetic field (when $m = 0$) and when viscosity is constant ($\epsilon = 0$) can be obtained by evaluating integral (3.11) by substituting corresponding velocities given by (3.7) and (3.6). We get rate of volume flow in the absence of magnetic field as

$$Q = \pi C_3 (\theta^2 - 1) + \frac{\pi C_4}{\epsilon^2} \left( e^2 \theta^2 \log \theta - \left( 1 - e^2 \theta^2 \right) \log(1 - \epsilon \theta) \right)$$

$$+ \frac{\pi}{12e^4} \left( -6\epsilon(\theta - 1) - 3e^2(\theta^2 - 1) + 4e^3(\theta^3 - 1) + 6(1 - \epsilon^2) \log(1 - \epsilon) - 6(1 - \epsilon^2 \theta^2) \log(1 - \epsilon \theta) \right)$$  \hspace{1cm} (3.14)
Where $C_1$ and $C_4$ are given by eqs (3.8) and (3.9). Rate of volume flow for constant viscosity case is

$$Q = \frac{\pi}{2} \left( \frac{(4 + m^2)(\theta^4 - 1)\sinh (m \log \theta)}{-4m(\theta^4 + 1) \cosh (m \log \theta) + 8m \theta^2} \right)$$

(3.15)

In the limiting case when $\epsilon$ and $m$ tend to zero equations (3.14) and (3.15) reduce to

$$Q = \frac{\pi}{8} \left[ \frac{(\theta^4 - 1) \log \theta - (\theta^2 - 1)^2}{\log \theta} \right]$$

(3.16)

Which is rate of volume flow for the classical Poiseuille flow between two co-axial cylinders.

### 3.1.2 Stress on the two cylinders:

Non-dimensional shear stress at any point with in annular channel is given by

$$\tau_{r z}(r) = \mu \frac{dV}{dr}$$

(3.17)

Substituting $V$ from equation (3.3), we get after differentiation

$$\tau_{r z}(r) = (1 - \epsilon r) \left[ \begin{array}{c} C_1 m r^{m-1} \binom{1 + m}{1 + m; 1 + 2m; \epsilon r} \\ -C_2 m r^{-m-1} \binom{1 - m}{1 - m; 1 - 2m; \epsilon r} \\ -2r \\ (4 - m^2) \binom{3}{3} \binom{3 - m}{3 - m; 3 + m; \epsilon r} \end{array} \right]$$

(3.18)

Where $C_1$ and $C_2$ are given by eq.(3.4) and (3.5) and differentiation is performed with the use of following identity

$$\frac{d}{dz} \{z^{a_1} \binom{p}{a_1, \ldots, a_p; b_1, \ldots, b_q; z}\} = a_1 z^{a_1 - 1} \binom{p}{a_1 + 1, \ldots, a_p; b_1, \ldots, b_q; z}$$

(3.19)

Stress on the surface of inner and outer cylinder may be obtained from eq. (3.18) by putting $r = 1$ and $r = \theta$ with negative sign in case of outer cylinder.

$$\tau_{r z}(1) = (1 - \epsilon) \left[ \begin{array}{c} C_1 m \binom{2}{1 + m; 1 + m; 1 + 2m; \epsilon} \\ -C_2 m \binom{2}{1 - m; 1 - m; 1 - 2m; \epsilon} \\ -2 \\ (4 - m^2) \binom{3}{3} \binom{3 - m}{3 - m; 3 + m; \epsilon} \end{array} \right]$$

(3.20)
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\[ \tau_{rz}(\theta) = -(1 - \epsilon \theta) \left[ \begin{array}{c} C_1 \frac{m \theta^{m-1}}{2} \frac{1}{m \theta^{m}} \frac{1}{2} F_1(1 + m, 1 + m; 1 + 2m; \epsilon \theta) \\ -C_2 \frac{m \theta^{-m-1}}{2} \frac{1}{m \theta^{-m}} \frac{1}{2} F_1(1 - m, 1 - m; 1 - 2m; \epsilon \theta) \\ -\frac{2 \theta}{(4 - m^2)} F_2(1, 3, 3; 3 - m, 3 + m; \epsilon \theta) \end{array} \right] (3.21) \]

Stress at any point in the channel when viscosity is constant, i.e. when \( \epsilon = 0 \), is obtained by putting \( V \) from (3.6) in (3.17). We, thus, get

\[ \tau_{rz}(r) = \frac{1}{(m^2 - 4)} \left[ 2r - \frac{m \left[ \theta^2 \cosh(m \log r) - \cosh(m \log \theta) \right]}{r \sinh(m \log \theta)} \right]; \quad \epsilon = 0 \quad (3.22) \]

Stress on inner and outer cylinder when \( \epsilon = 0 \) are

\[ \tau_{rz}(1) = \frac{1}{(m^2 - 4)} \left[ 2 - \frac{m \left[ \theta^2 - \cosh(m \log \theta) \right]}{\sinh(m \log \theta)} \right] \quad (3.23) \]

\[ \tau_{rz}(\theta) = -\frac{1}{(m^2 - 4)} \left[ 2\theta - \frac{m \left[ \theta^2 \cosh(m \log \theta) - 1 \right]}{\theta \sinh(m \log \theta)} \right] \quad (3.24) \]

Stress at any point in the channel when applied magnetic field is zero, i.e. when \( m = 0 \), is obtained by putting \( V \) from (3.7) in (3.17). We get

\[ \tau_{rz}(r) = C_4 \frac{r}{2} - \frac{r}{2}; \quad m = 0 \quad (3.25) \]

where \( C_4 \) is given by eq.(3.32). In limiting case when \( m \rightarrow 0 \) in eq. (3.22) and \( \epsilon \rightarrow 0 \) in eq.(3.25). We get

\[ \tau_{rz}(r) = \frac{1}{4r} \left( -2r^2 + \frac{\theta^2 - 1}{\log \theta} \right) \quad (3.26) \]

which is the shear stress at any point in the annular channel for classical Poiseuille flow between two co-axial cylinders. Stress on inner and outer cylinder in this case are

\[ \tau_{rz}(1) = \frac{1}{4} \left( -2 + \frac{\theta^2 - 1}{\log \theta} \right) \quad (3.27) \]

\[ \tau_{rz}(\theta) = -\frac{1}{4\theta} \left( -2\theta^2 + \frac{\theta^2 - 1}{\log \theta} \right) \quad (3.28) \]

Figure (4) and (5) represents effect of parameters \( \epsilon \) and \( m \) on the shear stress at the surface of inner and outer cylinder governed by equations (3.20) and (3.21) when \( \theta = 2 \). We have following observation from these figures

- Effect of magnetic field (i.e. \( m \)) is to decrease the stress on the surface of cylinder. (c.f. Fig. 4 and 5). Thus, magnetic field can be used to control the stress on the cylinders.
Figure 4: Variation of shear stress on the inner cylinder with $m$ for $\epsilon = 0, 0.25$ and $0.4$, when viscosity variation is $\mu = (1 - \epsilon r)$.

Figure 5: Variation of shear stress on the outer cylinder with $m$ for $\epsilon = 0, 0.25$ and $0.4$, when viscosity variation is $\mu = (1 - \epsilon r)$.

- Stress on the outer cylinder decreases as $\epsilon$ increases (c.f. Fig.5). This is because increase in $\epsilon$ caused decrease in viscosity according to the law $\mu = (1 - \epsilon r)$.

- Stress on the inner cylinder increases with $\epsilon$ for small $m$ due to increase in viscosity and decreases as $\epsilon$ increases for large $m$ because increase in $m$ caused decrease in velocity gradient (c.f. Fig.4).

3.2. Case-II:

When $\mu = (1 - \epsilon r^2)$, where $\epsilon$ is a non-dimensional viscosity variation parameter such that $|\epsilon \theta^2| < 1$. In this case governing equation of motion (2.21) becomes

$$r^2 \left(1 - \epsilon r^2\right) \frac{d^2 V}{dr^2} + r \left(1 - 3\epsilon r^2\right) \frac{dV}{dr} - m^2 V = -r^2$$

(3.29)
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This is hypergeometric equation its solution subject to the boundary condition (3.1) may be expressed in terms of hypergeometric functions as follows

\[
V = D_1 r^m \, _2F_1 \left( \frac{m}{2}, 1 + \frac{m}{2}; 1 + m; \epsilon r^2 \right) \\
+ D_2 r^{-m} \, _2F_1 \left( -\frac{m}{2}, 1 - \frac{m}{2}; 1 - m; \epsilon r^2 \right) \\
+ \frac{r^2}{(m^2 - 4)} \, _3F_2 \left( 1, 1, 2; 2 - \frac{m}{2}, 2 + \frac{m}{2}; \epsilon r^2 \right)
\]

(3.30)

Where, \( D_1 \) and \( D_2 \), as obtained from boundary conditions (3.1), are

\[
D_1 = \frac{\theta^{2+m} \, _2F_1 \left( \frac{-m}{2}, 1 - \frac{m}{2}; 1 - m; \epsilon \theta^2 \right) \, _3F_2 \left( 1, 1, 2; 2 - \frac{m}{2}, 2 + \frac{m}{2}; \epsilon \theta^2 \right)}{(m^2 - 4) \left( _2F_1 \left( \frac{-m}{2}, 1 - \frac{m}{2}; 1 - m; \epsilon \theta^2 \right) - 2 \, _2F_1 \left( \frac{m}{2}, 1 + \frac{m}{2}; 1 + m; \epsilon \theta^2 \right) \right)}
\]

(3.31)

\[
D_2 = \frac{\theta^{2m} \, _2F_1 \left( \frac{m}{2}, 1 + \frac{m}{2}; 1 + m; \epsilon \theta^2 \right) \, _3F_2 \left( 1, 1, 2; 2 - \frac{m}{2}, 2 + \frac{m}{2}; \epsilon \theta^2 \right)}{(m^2 - 4) \left( _2F_1 \left( \frac{-m}{2}, 1 - \frac{m}{2}; 1 - m; \epsilon \theta^2 \right) - \theta^{2m} \, _2F_1 \left( \frac{m}{2}, 1 + \frac{m}{2}; 1 + m; \epsilon \theta^2 \right) \right)}
\]

(3.32)

Solutions, when \( \epsilon = 0 \) but \( m \neq 0 \) is same as obtained in case I and is given by eq.(3.7) and when \( m = 0 \) but \( \epsilon \neq 0 \) can be obtained, more easily, by solving the reduced equation.
of motion (3.29) for $m = 0$. Thus, we have

$$V = \frac{\log r[\log(1 - \epsilon) - \log(1 - \epsilon\theta^2)] - \log \theta[\log(1 - \epsilon) - \log(1 - \epsilon r^2)]}{2\epsilon \left[\log(1 - \epsilon) + 2\log \theta - \log(1 - \epsilon \theta^2)\right]}$$

(3.33)

In limiting case when $m \to 0$ in (3.33) above solution reduces to the classical case of Poiseuille flow between two co-axial cylinders and is given by (3.10). Velocity profiles of the flow for the viscosity variation $\mu = 1 - \epsilon r^2$, when $\theta = 2$, are shown in figures (6) and (7). The findings are similar to that corresponding to case I.

Figure 6: Velocity profiles of the flow when viscosity variation is $\mu = (1 - \epsilon r^2)$ and $m = 2, 6$ for different $\epsilon$.

Figure 7: Velocity profiles of the flow for different $m$ when viscosity variation is $\mu = (1 - 0.2r^2)$. 
3.2.1 Rate of volume flow:

Non-dimensional volume flow rate when viscosity varies according to the assumption \( \mu = (1 - \epsilon r^2) \) is obtained similarly as in case I on using the result (3.13). We thus have

\[
Q = 2\pi D_1 \left[ \frac{\epsilon^2 \left\{ \log(1 - \epsilon r^2) - \log(1 - \epsilon) \right\}}{\epsilon^2 \left[ \log(1 - \epsilon) + 2 \log \theta - \log(1 - \epsilon \theta^2) \right]} \right] \tag{3.35}
\]

In limiting case when \( \epsilon \) tend to zero above equation reduces to (3.16), which is the rate of volume flow of the classical Poiseuille flow between two co-axial cylinders.

3.2.2 Stress on the surface of cylinder:

Non-dimensional shear stress at any point within annular channel when viscosity of fluid varies according to the assumption \( \mu = (1 - \epsilon r^2) \) is obtained by substituting \( V \) from eq. (3.30) in eq. (3.17). We thus obtain

\[
\tau_{rz}(r) = \frac{\epsilon^2 \left\{ \log(1 - \epsilon r^2) - \log(1 - \epsilon) \right\}}{\epsilon^2 \left[ \log(1 - \epsilon) + 2 \log \theta - \log(1 - \epsilon \theta^2) \right]} \tag{3.36}
\]
Figure 8: Variation of shear stress on the inner cylinder with $m$ for $\epsilon = 0, 0.1$ and 0.2, when viscosity variation is $\mu = (1 - \epsilon r^2)$.

![Figure 8](image)

Figure 9: Variation of shear stress on the outer cylinder with $m$ for $\epsilon = 0, 0.1$ and 0.2, when viscosity variation is $\mu = (1 - \epsilon r^2)$.

![Figure 9](image)

Where $D_1$ and $D_2$ are given by eq. (3.31) and (3.32) and differentiation is performed with the use of following identity (3.19). Stress on the surface of inner and outer cylinder may be obtained from eq.(3.36) by putting $r = 1$ and $r = \theta$.

\[
\tau_{rz}(1) = (1 - \epsilon) \left[ \frac{D_1 m}{2} F_1 \left( 1 + \frac{m}{2}; 1 + \frac{m}{2}; 1 + m; \epsilon \right) \right. \\
- \left. D_2 \frac{m}{2} F_1 \left( 1 - \frac{m}{2}; 1 - \frac{m}{2}; 1 - m; \epsilon \right) \right] \\
+ \frac{2}{(m^2 - 4)} 3 F_2 \left( 1, 2, 2; 2 - \frac{m}{2}, 2 + \frac{m}{2}; \epsilon \right) 
\] (3.37)

\[
\tau_{rz}(\theta) = -(1 - \epsilon \theta^2) \left[ \frac{D_1 m \theta^{m-1}}{2} F_1 \left( 1 + \frac{m}{2}; 1 + \frac{m}{2}; 1 + m; \epsilon \theta^2 \right) \right. \\
- \left. D_2 \frac{\theta^{m-1}}{2} F_1 \left( 1 - \frac{m}{2}; 1 - \frac{m}{2}; 1 - m; \epsilon \theta^2 \right) \right] \\
+ \frac{2 \theta}{(m^2 - 4)} 3 F_2 \left( 1, 2, 2; 2 - \frac{m}{2}, 2 + \frac{m}{2}; \epsilon \theta^2 \right) 
\] (3.38)

Stress at any point in the channel when viscosity is constant, i.e. when $\epsilon = 0$, is same as obtained in case I and is given by eq.(3.22). When applied magnetic field is zero, i.e. when $m = 0$, stress at any point in the channel is obtained by putting $V$ from
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We get after simple differentiation

\[
\tau_{rz}(r) = \frac{\left[2r^2 \epsilon \log r - (1 - r^2 \epsilon)\right]\left[\log(1 - \epsilon \theta^2) - \log(1 - \epsilon)\right]}{2r^2 \epsilon \left[1 - \log(1 - \epsilon) + \log \left(1 - r^2 \epsilon\right)\right]\log \theta} \quad ; \quad m = 0
\]  

(3.39)

Stress on inner and outer cylinder when \(m = 0\), are

\[
\tau_{rz}(1) = \frac{(1 - \epsilon)\left[\log(1 - \epsilon) - \log(1 - \epsilon \theta^2)\right] - 2\epsilon \log \theta}{2\epsilon \left[\log(1 - \epsilon) + 2\log \theta - \log(1 - \epsilon \theta^2)\right]} \quad ; \quad \epsilon \rightarrow 0
\]  

(3.40)

\[
\tau_{rz}(\theta) = -\frac{\left[2\theta^2 \epsilon \log \theta - (1 - \theta^2 \epsilon)\right]\left[\log(1 - \epsilon \theta^2) - \log(1 - \epsilon)\right]}{2\epsilon \theta \left[1 - \log(1 - \epsilon) + \log \left(1 - \theta^2 \epsilon\right)\right]\log \theta} \quad ; \quad \epsilon \rightarrow 0
\]  

(3.41)

In limiting case when \(\epsilon \rightarrow 0\) expression (3.39) for stress reduces to the classical case of Poiseuille flow between two co-axial cylinders and is given by (3.26). Figure (8) and (9) represent effect of parameters \(\epsilon\) and \(m\) on the shear stress on inner and outer cylinder when viscosity variation is \(\mu = (1 - \epsilon r^2)\). The findings are similar to that corresponding to case I.

References


