On combinatorial extensions of some mock theta functions using signed partitions

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Abstract

Recently Agarwal and Sood, in their paper entitled “Split $n+t$-color partitions and Gordon-McIntosh eight order mock theta functions, The Electronic Journal of Combinatorics 21(2), (2014), #P2.46”, have defined and used the ‘split $(n+t)$-color partitions’ to obtain the combinatorial interpretations of two mock theta functions. The purpose of this paper is to extend their results using ‘signed partitions’. We further interpret them combinatorially using classical partitions and convolution properties which give rise to three way combinatorial identities.

AMS Subject Classification: 11P81, 83, 84, 05A17, 19.
Keywords: Mock theta functions, split $(n+t)$-color partitions, signed partitions.

1. Introduction

In literature we find the combinatorial interpretations to the following five Ramanujan’s mock theta functions:

\[ \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}, \quad \text{(1.1)} \]

\[ F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}, \quad \text{(1.2)} \]
\[
\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2} (-q; q^2)_n, \quad (1.3)
\]
\[
\phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} (-q; q^2)_n, \quad (1.4)
\]
\[
F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}}. \quad (1.5)
\]

Fine [9] has used the classical partitions as a combinatorial tool to obtain the following interpretation of \( \chi(q) \).

**Theorem 1.1.** \( \chi(q) \) generate the number of partitions into odd parts without gap.

Using the same combinatorial tool Andrews [8] obtained the following interpretation of \( F_0(q) \).

**Theorem 1.2.** \( F_0(q) \) generate the number of partitions into odd parts without gap and repeating atleast twice.

Agarwal [1, 2] used the \( n \)-color partitions and lattice paths to obtain the combinatorial meanings of \( (1.1) - (1.4) \). Agarwal and Narang [5] extended their results and interpreted combinatorially \( (1.1) - (1.4) \) using Frobenius partitions. Agarwal and Rana [4, 12] used the \( n \)-color partitions, lattice paths and Frobenius partitions to provide the combinatorial interpretations of \( (1.5) \).

Recently the two Gordon–McIntosh mock theta functions [10]
\[
V_0(q) = -1 + 2 \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q; q^2)_n} \quad \text{and} \quad (1.6)
\]
\[
V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} (-q; q^2)_n}{(q; q^2)_{n+1}} \quad (1.7)
\]
were interpreted combinatorially by Agarwal and Sood [6] using extension of \( (n + t) \)-color partitions given as ‘split \( (n + t) \)-color partitions’.

Before proceeding further, we recall some definitions and notations. Informally, a partition [7] of an integer \( n \) is a representation of \( n \) as a sum of positive integers where the order of the summands is considered irrelevant. Thus the five partitions of 4 are 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. The summands are called the ‘parts’ of the partition and since the order of the parts is irrelevant, 2 + 1 + 1, 1 + 2 + 1, and 1 + 1 + 2 are all considered to be the same partition of 4. The number of parts in a partition \( \pi \) is called the length of the partition \( \pi \) and is denoted by \( l(\pi) \). The sum of all parts of a partition \( \pi \) is called the weight of the partition \( \pi \) and is denoted by \( |\pi| \).

Note that an exponent can be used to denote that the part is repeating a certain number of times in a partition and in such notation we consider \( \lambda \) as the number of distinct parts appearing in a particular partition.
For example, the five partitions of 4 can be written as \((4), (31), (2^2), (21^2)\) and \((1^4)\).
In particular for the partition \((21^2)\) of 4, the value of \(\lambda\) is ‘2’.

**Definition 1.3.** \([3]\) A partition with \(\text{"}(n + t)\text{— copies of } n\text{"}, t \geq 0\) is a partition in which a part of size \(n\), \((n \geq 0)\), can come in \((n + t)\text{— different colors denoted by the subscripts},
\(n_1, n_2, n_3, \cdots, n_{n+t}\).

For example, the relevant partitions of 3 with \(\text{"} n + 1 \text{— copies of } n\text{"} \) are,
\[
3_1, \ 3_10_1, \ 1_11_11_1, \ 1_11_10_11_0, \ 2_11_1, \ 2_11_01_1, \ 2_11_2, \ 2_11_01_0,
3_2, \ 3_20_1, \ 1_2_11, \ 1_2_11_01, \ 2_2_1, \ 2_2_10_1, \ 2_2_12, \ 2_2_1_01,
3_3, \ 3_30_1, \ 1_2_2_1, \ 1_2_2_1_01, \ 2_3_1, \ 2_3_10_1, \ 2_3_1_2, \ 2_3_1_2_01,
3_4, \ 3_40_1, \ 1_2_1_2_1, \ 1_2_1_2_1_01.
\]
Note that zeros are permitted if and only if \(t \geq 1\). Also in no partition zeros are permitted to repeat.

**Definition 1.4.** \([3]\) The weighted difference of two parts \(m_i, n_j, \ m \geq n\) is defined by \(m - n - i - j\) and denoted by \((m_i - n_j))\).

**Definition 1.5.** \([11]\) A signed partition \(\sigma\) of an integer \(\nu\) is a partition pair \((\pi^+, \pi^-)\) where
\[
\nu = |\pi^+| - |\pi^-|.
\]
\(\pi^+\) (resp. \(\pi^-\)) the positive (resp. negative) subpartition of \(\sigma\) and \(\pi_{1^+}, \pi_{2^+}, \ldots, \pi_{l(\pi^+)}\)
(resp. \(\pi_{1^-}, \pi_{2^-}, \ldots, \pi_{l(\pi^-)}\)) the positive (resp. negative) parts of \(\sigma\).

For example, \(((8^221), (321^2))\), which represents \(8 + 2 + 2 + 1 - 1 - 1 - 2 - 3,\) is a signed partition of 6.

**Remark 1.6.** It is obvious that there are infinitely many unrestricted signed partitions of any integer, but when we place restrictions on how parts may appear, signed partitions arise naturally in the study of certain q-series.

**Definition 1.7.** \([6]\) Let \(m_i\) be a part in an \((n + t)\text{— color partition of a non negative}\) integer \(\nu\). Split the color ‘\(i\)’ into two parts - ‘the green part’ and ‘the red part’ and denote them by ‘\(g\)’ and ‘\(r\)’, respectively, such that \(1 \leq g \leq i, 0 \leq r \leq i - 1\) and \(i = g + r\). An \((n + t)\text{— color partition in which each part splits in this manner is called a split \((n + t)\text{— color partition.}\)

For example, In \(7_{3+2}\), the green part is 3 and the red part is 2.

**Remark 1.8.** If the red part is 0, it will not be written. Thus for example, \(7_{5+0}\) is written as \(7_5\).
Using these ‘split \((n+1)\)– color partitions’, Agarwal and Sood [6] obtained the following interpretations of (1.6) and (1.7).

**Theorem 1.9.** For \(\nu \geq 1\), let \(A_1(\nu)\) denote the number of ‘split \(n\)– color partitions’ of \(\nu\) such that

(i) the parts and their subscripts have the same parity,

(ii) the red part of the subscripts cannot exceed 1,

(iii) the least part is either \(k_k (k \geq 1)\) or \(k_{(k-1)+1} (k \geq 2)\), and

(iv) the weighted difference of any two consecutive parts is 0.

Then

\[
V_0(q) = 1 + 2 \sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu}.
\]

**Remark 1.10.** In conditions (i) and (iv) the whole subscript \(i\) is considered, not its parts \(g\) and \(r\), separately.

**Theorem 1.11.** For \(\nu \geq 1\), let \(A_2(\nu)\) denote the number of ‘split \(n\)– color partitions’ of \(\nu\) such that

(i) the parts and their subscripts have the same parity,

(ii) the red part of the subscripts cannot exceed 1,

(iii) the least part is \(k_k (k \geq 1)\), and

(iv) the weighted difference of any two consecutive parts is 0.

Then

\[
V_1(q) = \sum_{\nu=1}^{\infty} A_2(\nu)q^{\nu}.
\]

**Remark 1.12.** As in Theorem 1.11, here also, in conditions (i) and (iv) the whole subscript \(i\) is considered, not its parts \(g\) and \(r\), separately.

In this paper we use the signed partitions and the classical partitions with convolution properties separately in order to provide the combinatorial meanings to (1.6) and (1.7) which further extend the results of [6] and give rise to infinite three way combinatorial identities.
2. Combinatorial interpretations of (1.6) and (1.7) by using ‘signed partitions’

Theorem 2.1. For \( v \geq 0 \), let \( B_1(v) \) denote the number of signed partitions \( \sigma = (\pi^+, \pi^-) \) of \( v \), where

(i) \( \pi^+ \) contains the odd parts without gaps appearing at least twice, and

(ii) \( \pi^- \) contains the odd, distinct parts \( < 2\lambda \).

where \( \lambda \) is the number of distinct parts in \( \pi^+ \). Then,

\[
-1 + 2 \sum_{v=0}^{\infty} B_1(v) q^v = V_0(q). \tag{2.1}
\]

For example, consider \( v = 6 \), then \( B_1(6) = 4 \), and the relevant signed partitions are;

\[ 1^6, \ 1^7 - 1, \ 1^4 + 3^2 - 1 - 3, \ 1^3 + 3^2 - 3. \]

Proof. We have,

\[
\sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)^n}{(q; q^2)^n} = \sum_{n=0}^{\infty} q^{n^2} \prod_{k=1}^{n} (1 + q^{(2k-1)}) \tag{2.2}
\]

\[
= \sum_{n=0}^{\infty} q^{2n^2} \prod_{k=1}^{n} (1 + q^{-(2k-1)}) \tag{2.3}
\]

Now, \( \frac{q^{2n^2}}{(q; q^2)_n} \) generate the partitions into odd parts without gap appearing at least twice.

Also, \( \prod_{k=1}^{n} (1 + q^{-(2k-1)}) \) generate the partitions into distinct odd parts less than \( 2\lambda \). Taking summation over all \( n \), we obtain that (2.2) generate the signed partitions enumerated by \( B_1(v) \). Hence,

\[
-1 + 2 \sum_{v=0}^{\infty} B_1(v) q^v = V_0(q). \tag{2.4}
\]

Theorem 2.2. For \( v \geq 0 \), let \( B_2(v) \) denote the number of signed partitions \( \sigma = (\pi^+, \pi^-) \) of \( v \), where
(i) \( \pi^+ \) contains an odd part \( r \), atleast once and \( \frac{(r - 1)}{2} \) parts which are odd and appear atleast twice without gaps, and

(ii) \( \pi^- \) contain odd, distinct parts \( \leq r - 2 \).

Then,

\[
\sum_{\nu=0}^{\infty} B_2(\nu)q^\nu = V_1(q).
\] (2.5)

For example, consider \( \nu = 6 \), then \( B_2(\nu) = 3 \), and the relevant signed partitions are;

\[
1^6, \ 1^7 - 1, \ 1^4 + 3 - 1.
\]

We have,

\[
\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{(n+1)^2} \prod_{k=1}^{n} (1 + q^{(2k-1)}) \] (2.6)

\[
= \sum_{n=0}^{\infty} q^{2n^2 + 2n + 1} \prod_{k=1}^{n} (1 + q^{-(2k-1)}) \] (2.7)

\[
= \sum_{n=0}^{\infty} \frac{q^{2n^2} q^{2n+1}}{(q;q^2)_n (1 - q^{2n+1})} \prod_{k=1}^{n} (1 + q^{-(2k-1)}). \] (2.8)

Now, \( \frac{q^{2n^2}}{(q;q^2)_n} \) generate the partitions into odd parts without gap appearing atleast twice.

The factor \( \frac{q^{2n+1}}{(1 - q^{2n+1})} \) generate the partitions consisting of the part \( 2n + 1 \) atleast once.

Also, \( \prod_{k=1}^{n} (1 + q^{-(2k-1)}) \) generate the partitions into distinct odd parts less than or equals to \( 2n - 1 \). Taking summation over all \( n \), we obtain that (2.6) generate the signed partitions enumerated by \( B_2(\nu) \). Hence,

\[
\sum_{\nu=0}^{\infty} B_2(\nu)q^\nu = V_1(q).
\]
3. Combinatorial interpretations of (1.6) and (1.7) by using ‘classical partitions with convolution properties’

Theorem 3.1. For $m, ν ≥ 0$, let $D_1(m, ν)$ enumerate the number of partitions of $ν$ into exactly $m$ odd parts without gap and $E_1(m, ν)$ denote the number of partitions of $ν$ into atmost $m$ distinct odd parts less than $2m$. Let

$$C_1(ν) = \sum_{m=0}^{∞} \sum_{k=0}^{ν} D_1(m, k) E_1(m, ν − k)$$

Then,

$$V_0(q) = -1 + 2 \sum_{ν=0}^{∞} C_1(ν) q^ν.$$ 

For example, if we consider $ν = 6$, then $C_1(ν) = 4$ and the relevant partitions for $D_1(m, k)$ and $E_1(m, k)$, $(0 ≤ k ≤ ν)$ respectively are given in the table below:

Table 1: Partitions enumerated by $D_1(m, k)$

<table>
<thead>
<tr>
<th>$m \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>0</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

d|d|  

Table 2: Partitions enumerated by $E_1(m, k)$

<table>
<thead>
<tr>
<th>$m \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>
\[ C_1(6) = [D_1(0, 0)E_1(0, 6) + D_1(0, 1)E_1(0, 5) + \cdots + D_1(6, 5)E_1(6, 1) + D_1(6, 6)E_1(6, 0)] \]
\[ = D_1(1, 5)E_1(1, 1) + D_1(1, 6)E_1(1, 0) + D_1(2, 5)E_1(2, 1) + D_1(2, 6)E_1(2, 0) \]
\[ = 1 + 1 + 1 + 1 \]
\[ = 4. \]

**Proof.** We shall show that,
\[ \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q; q^2)_n} = \sum_{v=0}^{\infty} C_1(v)q^v. \]

The theorem will be proved in the following three steps:

**Step 1.** By Theorem 1.1, we have,
\[ \frac{q^{m^2}}{(q; q^2)_m} \]
generate the partitions into exactly \( m \) odd parts without gap such as:
\[ a_1.1 + a_2.3 + a_3.5 + \cdots + a_m.(2m - 1) \]

. Therefore,
\[ \sum_{v=0}^{\infty} D_1(m, v)q^v = \frac{q^{m^2}}{(q; q^2)_m}. \] (3.1)

**Step 2.** Clearly,
\[ (-q; q^2)_m = \prod_{k=1}^{m} (1 + q^{2k-1}) \] (3.2)
generate the number of partition into distinct odd parts less than \( 2m \). Therefore,
\[ \sum_{v=0}^{\infty} E_1(m, v)q^v = (-q; q^2)_m \] (3.3)

**Step 3.** Since,
\[ \sum_{v=0}^{\infty} C_1(v)q^v = \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} C_1(m, v)q^v \] (3.4)
We have,
\[
\sum_{\nu=0}^{\infty} C_1(\nu) q^\nu = \sum_{\nu=0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\nu} D_1(m, k) E_1(m, \nu - k) \right) q^\nu
\]
\[
= \sum_{\nu=0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\nu} D_1(m, k) E_1(m, \nu) \right) q^{\nu+k}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\nu} D_1(m, k) q^k \right) \left( \sum_{\nu=0}^{\infty} E_1(m, \nu) q^\nu \right)
\]
\[
= \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-q; q^2)_m (q; q^2)_m}. \tag{3.5}
\]
Hence the result.

\[\square\]

**Theorem 3.2.** For \( m, \nu \geq 0 \), let \( D_2(m, \nu) \) enumerate the number of partitions of \( \nu \) into exactly \( m + 1 \) odd parts without gap and \( E_1(m, \nu) \) denote the number of partitions of \( \nu \) into distinct odd parts less than \( 2m \). Let,

\[
C_2(\nu) = \sum_{m=0}^{\infty} \sum_{k=0}^{\nu} D_2(m, k) E_2(m, \nu - k)
\]

Then,

\[
\sum_{\nu=0}^{\infty} C_2(\nu) q^\nu = V_1(q).
\]

**Proof.** From Theorem 1.1, we have, \( \frac{q^{(m+1)^2}}{(q; q^2)_{m+1}} \) generate the partitions into exactly \((m + 1)\) odd parts such as:

\[
a_{1.1} + a_{2.3} + a_{3.5} + \cdots + a_{m+1}.(2m + 1)
\]

Therefore,

\[
\sum_{\nu=0}^{\infty} D_2(\nu, m) q^\nu = \frac{q^{(m+1)^2}}{(q; q^2)_{m+1}}. \tag{3.5}
\]

Now proceeding same as in Theorem 3.1, we get,

\[
\sum_{\nu=0}^{\infty} C_2(\nu) q^\nu = \sum_{m=0}^{\infty} \frac{q^{(m+1)^2} (-q; q^2)_m}{(q; q^2)_{m+1}}
\]
\[
= V_1(q).
\]

Hence the result.  \[\square\]
By proving the above results we have established the three way combinatorial identities given as:

For \( \nu \geq 1 \) and \( i = 1, 2 \)
\[
A_i(\nu) = B_i(\nu) = C_i(\nu). \tag{3.6}
\]

4. Conclusion

With the combinatorial extension of mock theta functions given by (1.6) and (1.7) using signed partitions and classical partitions with convolution properties, we have provided the three way combinatorial results. The obvious question arises from this work is: Is it possible to provide the direct bijection between ‘split \( n+t \) – color partitions’ and signed partitions?

Acknowledgement

The first author is fully supported by University Grant Commission through Grant No. F. 2-16/2011(SA-1).

References


