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Logarithmic Class of Estimators of the Population Mean under Stratified Ranked Set Sampling

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Abstract

We study the logarithmic class of estimators of a finite population mean under stratified ranked set sampling (SRSS). Combined and separate versions are defined, the sampling design is specified fully, and closed-form rank-selection probabilities are given. These probabilities make every within-rank moment exactly computable for a finite population. The biases and mean squared errors (MSEs) themselves are first-order approximations in terms of these exact moments. We give optimum and estimated constants, and an allocation of the sample that minimises the within-rank residual variance. Three results guide practice. (i) Among first-order optimal members, the logarithmic form has a smaller bias than the power-ratio and exponential forms if and only if the optimum constant lies in $(0, 2/3)$; for product-type estimators it is never the less biased form. Applied stratum by stratum, this rule defines hybrid separate and combined estimators. (ii) Under proportional allocation the bias of a separate estimator is $O(H/n)$ while its MSE is $O(1/n)$, so the ratio of squared bias to first-order MSE equals (H^2/n) times a population constant. This ratio vanishes when the number of strata H is fixed, but it dominates in finite samples with many small strata, or asymptotically when H grows faster than $\sqrt{n}$. (iii) The first-order MSE plus the squared first-order bias is a useful heuristic approximation to the finite-sample MSE. A simulation study in R uses one synthetic population and three real ones has been conducted to verify the theoratical findings

Keywords: Stratified ranked set sampling; Logarithmic estimator; Ratio and product estimators; Separate and combined estimators; Bias accumulation; Optimum allocation.
MSC (2020): 62D05.

1. INTRODUCTION

When the population mean $\bar{X}$ of an auxiliary variable is known, the ratio estimator $\bar{y}\bar{X}/\bar{x}$ (for positive correlation), the product estimator $\bar{y}\bar{x}/\bar{X}$ (for negative correlation) [1] and the exponential estimators of Bahl and Tuteja [2] are the standard choices. Brar, Bajwa and Kaur [3] introduced logarithmic functional forms of the ratio and product estimators. These forms are consistent, and their first-order MSEs coincide with those of

the Bahl–Tuteja estimators while their biases differ. After a change of origin [4, 5], their minimum MSE equals that of the regression estimator. A class that contains logarithmic forms of both types is

$$t_L(\alpha) = \bar{y}\Big\{1 + \alpha \log \frac{\bar{X}}{\bar{x}}\Big\}, \qquad \alpha \in \mathbb{R}. \tag{1}$$

For $\alpha > 0$ it is ratio-type. For $\alpha < 0$ it is the product-type estimator $t_{LP}(|\alpha|) = \bar{y}\{1 + |\alpha| \log(\bar{x}/\bar{X})\}$. The member $\alpha = 1/2$ has, to first order, the same MSE as the Bahl–Tuteja exponential ratio estimator but a different bias. The member $\alpha = 1$ has the first-order MSE of the classical ratio estimator, and $\alpha = -1$ that of the classical product estimator.

Ranked set sampling (RSS) [6–8] with ranking on a concomitant variabl, Stokes [9] improves the precision of ratio-type estimators [10, 11]. In stratified populations, Samawi [12] is carried out RSS independently within strata, giving stratified ranked set sampling (SRSS) . Ratio estimation under SRSS was developed by Samawi and Siam [13], and modified ratio estimators that use known population parameters by Mandowara and Mehta [14]. The logarithmic class (1) has not been studied under SRSS. Stratification raises three questions that do not arise under simple random sampling or unstratified RSS: the choice between separate and combined estimators; the accumulation of bias over strata; the allocation of the sample, which under SRSS depends on within-rank rather than ordinary stratum variances.

The contributions of this paper are as follows. Section 2 specifies the design and notation, gives the first-order theory of combined and separate estimators of both types, and proves a bias comparison of the logarithmic, power-ratio and exponential forms (Proposition 2.1). It then defines hybrid estimators that apply this comparison stratum by stratum (Corollary 2.3), states the rate of bias accumulation precisely (Proposition 2.2), and derives the optimum allocation (Theorem 2.4). Section 3 reports a simulation study with Monte Carlo standard errors. Sections 4 and 5 discuss the results and give practical rules.

2. MATERIALS AND METHODS

2.1. Population, design and notation

The finite population U of N units is divided into H non-overlapping strata of sizes N_h, with weights $W_h = N_h/N$, $h = 1, \ldots, H$. For unit i of stratum h we observe the study variable y_{hi} and the auxiliary variable $x_{hi} > 0$. The stratum means are $\bar{Y}_h, \bar{X}_h$, and the population means are $\bar{Y} = \sum_h W_h \bar{Y}_h$ and $\bar{X} = \sum_h W_h \bar{X}_h$. Variances and covariances use divisor N_h. The coefficients of variation are $C_y = \sigma_y/\bar{Y}$ and $C_x = \sigma_x/\bar{X}$, and ρ is the correlation coefficient.

Assumption 2.1 (Auxiliary information)**.** *The value x_{hi} is available for every unit on the sampling frame. The combined estimators require only $\bar{X}$. The separate estimators require the stratum means $\bar{X}_h$ for every h. If only $\bar{X}$ is known, only the combined estimators are feasible.*

In all populations used below x is a register variable, so both $\bar{X}$ and every $\bar{X}_h$ are known.

The SRSS design. Fix a set size m and, for each stratum, a number of cycles $r_h \geq 2$. Independently in each stratum and in each cycle $l = 1, \ldots, r_h$:

1. draw m sets of m units from stratum h by simple random sampling *with replacement*;

2. rank the units of each set by their x-values (perfect ranking on x, which is an induced or concomitant ranking for y);

3. from the jth set measure the unit of rank j, giving the pair $(y_{h[j]l},\, x_{h(j)l})$, $j = 1, \ldots, m$.

Stratum h thus yields $n_h = mr_h$ measured units, and the total sample size is $n = \sum_h n_h$. The stratum and stratified sample means are

$$\bar{y}_{h(rss)} = \frac{1}{n_h} \sum_{j=1}^{m} \sum_{l=1}^{r_h} y_{h[j]l}, \qquad \bar{x}_{h(rss)} = \frac{1}{n_h} \sum_{j=1}^{m} \sum_{l=1}^{r_h} x_{h(j)l},$$

$$\bar{y}_{st(rss)} = \sum_{h=1}^{H} W_h\, \bar{y}_{h(rss)}, \qquad \bar{x}_{st(rss)} = \sum_{h=1}^{H} W_h\, \bar{x}_{h(rss)}.$$

The rank-stratum sample means are $\bar{y}_{h[j]} = r_h^{-1} \sum_l y_{h[j]l}$ and $\bar{x}_{h(j)} = r_h^{-1} \sum_l x_{h(j)l}$. The comparison design is stratified simple random sampling with replacement with the same n_h, whose mean we denote $\bar{y}_{st(srs)}$.

Lemma 2.1 (Exact rank-selection probabilities)**.** *Order the units of stratum h by x, with ties broken by a fixed random permutation, and let $u_{(k)}$ be the unit in position k. Under step 1 of the design, the unit selected as the jth order statistic of a set is $u_{(k)}$ with probability*

$$w_{jk}^{(h)} = I_{k/N_h}(j, m - j + 1) - I_{(k-1)/N_h}(j, m - j + 1), \qquad k = 1, \ldots, N_h,$$

where $I_p(a, b)$ is the regularised incomplete beta function. Moreover $\sum_{j=1}^{m} w_{jk}^{(h)} = m/N_h$.

Proof. The jth order statistic lies in positions $1, \ldots, k$ if and only if at least j of the m independent draws do, an event of probability $P\{\mathrm{Bin}(m, k/N_h) \geq j\} = I_{k/N_h}(j, m - j + 1)$; differencing in k gives $w_{jk}^{(h)}$. The column sum is the expected number of draws that select position k, namely m/N_h. □

Within-rank moments. Let $\mu_{yh[j]} = \sum_k w_{jk}^{(h)} y_{h(k)}$ be the mean of y in rank stratum j of stratum h, and let $\sigma^2_{yh[j]}$, $\sigma^2_{xh(j)}$ and $\sigma_{yxh[j]}$ be the corresponding variances and covariance, all computed with the probabilities $w_{jk}^{(h)}$. The *within-rank* stratum moments are

$$\Sigma_{yh} = \frac{1}{m}\sum_{j=1}^{m} \sigma^2_{yh[j]}, \qquad \Sigma_{xh} = \frac{1}{m}\sum_{j=1}^{m} \sigma^2_{xh(j)}, \qquad \Sigma_{yxh} = \frac{1}{m}\sum_{j=1}^{m} \sigma_{yxh[j]}.$$

Lemma 2.2 (Moments of the SRSS means)**.** $\mathrm{E}(\bar{y}_{h(rss)}) = \bar{Y}_h$ *and* $\mathrm{E}(\bar{x}_{h(rss)}) = \bar{X}_h$*, and, exactly,*

$$\mathrm{Var}(\bar{y}_{h(rss)}) = \frac{\Sigma_{yh}}{n_h}, \qquad \mathrm{Var}(\bar{x}_{h(rss)}) = \frac{\Sigma_{xh}}{n_h}, \qquad \mathrm{Cov}(\bar{y}_{h(rss)}, \bar{x}_{h(rss)}) = \frac{\Sigma_{yxh}}{n_h},$$

with independence between strata. Moreover $\Sigma_{yh} = \sigma^2_{yh} - m^{-1}\sum_j (\mu_{yh[j]} - \bar{Y}_h)^2 \le \sigma^2_{yh}$.

Proof. By Lemma 2.1, $m^{-1}\sum_j \mu_{yh[j]} = \bar{Y}_h$. The n_h measured units are independent, with r_h of them drawn from each rank stratum, so $\mathrm{Var}(\bar{y}_{h(rss)}) = n_h^{-2} r_h \sum_j \sigma^2_{yh[j]} = \Sigma_{yh}/n_h$. The decomposition is the within/between identity for the equal mixture of rank strata. □

The moments in Lemma 2.2 are exact. The biases and MSEs of the non-linear estimators below are first-order approximations in terms of these exact moments, valid as $n_h \to \infty$.

Relative error moments. For the combined estimators we write $\bar{y}_{st(rss)} = \bar{Y}(1 + e_0)$ and $\bar{x}_{st(rss)} = \bar{X}(1 + e_1)$. Then

$$V_y = \mathrm{E}(e_0^2) = \frac{1}{\bar{Y}^2}\sum_h \frac{W_h^2 \Sigma_{yh}}{n_h}, \qquad V_x = \mathrm{E}(e_1^2) = \frac{1}{\bar{X}^2}\sum_h \frac{W_h^2 \Sigma_{xh}}{n_h},$$
$$V_{yx} = \mathrm{E}(e_0 e_1) = \frac{1}{\bar{Y}\bar{X}}\sum_h \frac{W_h^2 \Sigma_{yxh}}{n_h}.$$

For the separate estimators the corresponding stratum quantities are

$$v_{yh} = \frac{\Sigma_{yh}}{n_h \bar{Y}_h^2}, \qquad v_{xh} = \frac{\Sigma_{xh}}{n_h \bar{X}_h^2}, \qquad v_{yxh} = \frac{\Sigma_{yxh}}{n_h \bar{Y}_h \bar{X}_h}.$$

All symbols are collected in the List of Symbols and Abbreviations.

2.2. Estimators

Table 1 lists every estimator compared in this paper.

Combined estimators. For $\alpha \in \mathbb{R}$ (ratio-type for $\alpha > 0$, product-type for $\alpha < 0$),

$$t_{LC}(\alpha) = \bar{y}_{st(rss)}\Big\{1 + \alpha \log \frac{\bar{X}}{\bar{x}_{st(rss)}}\Big\},$$

and the classical combined estimators are

$$t_{CR} = \bar{y}_{st(rss)}\,\frac{\bar{X}}{\bar{x}_{st(rss)}}, \qquad t_{CP} = \bar{y}_{st(rss)}\,\frac{\bar{x}_{st(rss)}}{\bar{X}}.$$

Separate estimators. For $\alpha = (\alpha_1, \ldots, \alpha_H)$,

$$t_{LS}(\alpha) = \sum_{h=1}^{H} W_h\,\bar{y}_{h(rss)}\Big\{1 + \alpha_h \log \frac{\bar{X}_h}{\bar{x}_{h(rss)}}\Big\},$$

and the classical separate estimators are

$$t_{SR} = \sum_h W_h\,\bar{y}_{h(rss)}\,\frac{\bar{X}_h}{\bar{x}_{h(rss)}}, \qquad t_{SP} = \sum_h W_h\,\bar{y}_{h(rss)}\,\frac{\bar{x}_{h(rss)}}{\bar{X}_h}.$$

2.3. First-order approximate results

Every estimator considered is of the form $\bar{y}\,g(\bar{x}/\bar{X})$ (combined, or within a stratum) with $g(1) = 1$, $g'(1) = -k$ and $g''(1)/2 = c$. Since $\log(\bar{X}/\bar{x}) = -\log(1 + e_1) = -e_1 + e_1^2/2 + O_p(n^{-3/2})$, the logarithmic member with constant α has $k = \alpha$ and $c = \alpha/2$. The power form $\bar{y}(\bar{X}/\bar{x})^k$ has $c = k(k+1)/2$; this includes the classical ratio estimator ($k = 1$, $c = 1$) and the classical product estimator ($k = -1$, $c = 0$). The exponential form $\bar{y}\exp\{2k(\bar{X} - \bar{x})/(\bar{X} + \bar{x})\}$ also has $c = k(k+1)/2$.

Theorem 2.1 (Combined estimators)**.** *For a combined estimator with constants (k, c), to the first order of approximation,*

$$\mathrm{B} = \bar{Y}\,(cV_x - kV_{yx}), \qquad \mathrm{MSE} = \bar{Y}^2\,(V_y - 2kV_{yx} + k^2V_x).$$

For $t_{LC}(\alpha)$ this gives $\mathrm{B} = \bar{Y}(\frac{\alpha}{2}V_x - \alpha V_{yx})$*. The MSE is minimised at $\alpha_c^* = V_{yx}/V_x = (\bar{X}/\bar{Y})B_c$, where*

$$B_c = \frac{\sum_h W_h^2 \Sigma_{yxh}/n_h}{\sum_h W_h^2 \Sigma_{xh}/n_h},$$

and the minimum is $\bar{Y}^2(V_y - V_{yx}^2/V_x)$. The optimum is ratio-type ($\alpha_c^* > 0$) for a positive and product-type ($\alpha_c^* < 0$) for a negative within-rank covariance.

Proof. $t/\bar{Y} - 1 = e_0 - ke_1 - ke_0e_1 + ce_1^2 + O_p(n^{-3/2})$. Taking expectations with Lemma 2.2 gives the bias, and squaring the first-order part $e_0 - ke_1$ gives the MSE; the minimisation is elementary. □

Theorem 2.2 (Separate estimators)**.** *For a separate estimator that uses constants (k_h, c_h) in stratum h, to the first order,*

$$\mathrm{B} = \sum_h W_h \bar{Y}_h \,(c_h v_{xh} - k_h v_{yxh}), \qquad \mathrm{MSE} = \sum_h W_h^2 \bar{Y}_h^2 \,(v_{yh} - 2k_h v_{yxh} + k_h^2 v_{xh}).$$

The MSE is minimised at $\alpha_h^* = v_{yxh}/v_{xh} = (\bar{X}_h/\bar{Y}_h)B_h$, with $B_h = \Sigma_{yxh}/\Sigma_{xh}$, and the minimum is $\sum_h W_h^2 \phi_h / n_h$, where $\phi_h = \Sigma_{yh} - \Sigma_{yxh}^2/\Sigma_{xh}$.

Proof. Apply the expansion of Theorem 2.1 within each stratum. The strata are sampled independently, so the biases add over strata, the MSEs add with weights W_h^2, and the cross terms vanish. □

Estimated constants. With $r_h \geq 2$, unbiased estimators of the within-rank moments are

$$\hat{\Sigma}_{yxh} = \frac{1}{m(r_h - 1)} \sum_{j=1}^{m} \sum_{l=1}^{r_h} \big(y_{h[j]l} - \bar{y}_{h[j]}\big)\big(x_{h(j)l} - \bar{x}_{h(j)}\big),$$

$$\hat{\Sigma}_{xh} = \frac{1}{m(r_h - 1)} \sum_{j=1}^{m} \sum_{l=1}^{r_h} \big(x_{h(j)l} - \bar{x}_{h(j)}\big)^2.$$

The estimated constants are

$$\hat{\alpha}_c = \hat{B}_c \frac{\bar{X}}{\bar{y}_{st(rss)}}, \qquad \hat{B}_c = \frac{\sum_h W_h^2 \hat{\Sigma}_{yxh}/n_h}{\sum_h W_h^2 \hat{\Sigma}_{xh}/n_h};$$

$$\hat{\alpha}_h = \hat{B}_h \frac{\bar{X}_h}{\bar{y}_{h(rss)}}, \qquad \hat{B}_h = \frac{\hat{\Sigma}_{yxh}}{\hat{\Sigma}_{xh}},$$

with $\hat{\alpha}_h = 0$ if $\hat{\Sigma}_{xh} = 0$. By the argument of Srivastava and Jhajj [15], $t_{LC}(\hat{\alpha}_c)$ and $t_{LS}(\hat{\alpha})$ attain the minimum MSE to first order as $r_h \to \infty$. Each $\hat{\Sigma}_{xh}$ has only $m(r_h - 1)$ degrees of freedom, however, so with $r_h = 2$ the first-order statement is not a reliable guide for the separate version (Section 4).

2.4. Logarithmic versus power and exponential forms

Proposition 2.1 (Combined estimators)**.** *Let the logarithmic, power-ratio and exponential combined estimators all use the optimum constant $k^* = \alpha_c^*$, so that they have the same minimum first-order MSE. Their first-order biases are*

$$\mathrm{B}_{\log} = \bar{Y} V_x \, k^* \big(\tfrac{1}{2} - k^*\big), \qquad \mathrm{B}_{\mathrm{pow}} = \mathrm{B}_{\exp} = \bar{Y} V_x \, \tfrac{1}{2} k^* \big(1 - k^*\big).$$

Hence $|\mathrm{B}_{\log}| < |\mathrm{B}_{\mathrm{pow}}|$ if and only if $0 < k^* < 2/3$. In particular, for product-type estimators ($k^* < 0$) the logarithmic form is always the more biased one.

Proof. Substitute $c = k^*/2$ and $c = k^*(k^*+1)/2$ into Theorem 2.1, using $V_{yx} = k^* V_x$. The inequality $|\frac{1}{2} - k^*| < \frac{1}{2}|1 - k^*|$ holds for $0 < k^* \leq \frac{1}{2}$, holds for $\frac{1}{2} < k^* \leq 1$ exactly when $k^* < \frac{2}{3}$, and fails for $k^* > 1$ and for $k^* \leq 0$. □

Proposition 2.1 concerns a single constant and therefore applies directly to the combined estimators only. For a separate estimator each stratum has its own α_h^*, and the total bias is a sum of stratum biases that may differ in sign. The comparison can, however, be made stratum by stratum.

Corollary 2.3 (Hybrid estimators)**.** *Define the hybrid separate estimator*

$$t_{HS} = \sum_h W_h\, \bar{y}_{h(rss)}\, g_h\big(\bar{x}_{h(rss)}/\bar{X}_h\big), \qquad g_h(u) = \begin{cases} 1 - \alpha_h^* \log u, & 0 < \alpha_h^* < 2/3, \\ u^{-\alpha_h^*}, & \textit{otherwise,} \end{cases}$$

and the hybrid combined estimator t_{HC} in the same way with the single constant α_c^*. Then t_{HS} has the minimum first-order MSE of Theorem 2.2, and it minimises every stratum bias $|\mathrm{B}_h|$ among the three forms. Hence it minimises the bound $\sum_h |\mathrm{B}_h|$ on the absolute total bias. The total bias itself can be smaller for another choice only through cancellation of stratum biases of opposite signs.

Proof. Within stratum h both forms have constant $k_h = \alpha_h^*$, so the MSE is unchanged. The stratum bias comparison is Proposition 2.1 applied to $(v_{yh}, v_{xh}, v_{yxh})$. The triangle inequality gives the last statement. □

2.5. Accumulation of bias in separate estimators

Proposition 2.2. *Let $n_h = nW_h$ (proportional allocation) and let the separate estimator use constants (k_h, c_h). Define*

$$a_h = \bar{Y}_h\Big(c_h \frac{\Sigma_{xh}}{\bar{X}_h^2} - k_h \frac{\Sigma_{yxh}}{\bar{Y}_h \bar{X}_h}\Big), \qquad \bar{a}_H = \frac{1}{H}\sum_{h=1}^{H} a_h, \qquad \bar{\phi}_W = \sum_{h=1}^{H} W_h \phi_h(k_h),$$

where $\phi_h(k) = \Sigma_{yh} - 2kR_h\Sigma_{yxh} + k^2 R_h^2 \Sigma_{xh}$ and $R_h = \bar{Y}_h/\bar{X}_h$. Then, exactly in terms of the first-order quantities,

$$\mathrm{B} = \frac{H}{n}\,\bar{a}_H, \qquad \mathrm{MSE}_1 = \frac{\bar{\phi}_W}{n}, \qquad \frac{\mathrm{B}^2}{\mathrm{MSE}_1} = \frac{H^2}{n} \cdot \frac{\bar{a}_H^2}{\bar{\phi}_W}.$$

Consequently:

- *(a) if H is fixed and $n \to \infty$, then $\mathrm{B}^2/\mathrm{MSE}_1 = O(n^{-1}) \to 0$, and the first-order MSE is asymptotically adequate;*
- *(b) if $H = H_n \to \infty$ with $n \geq 2mH_n$, $\bar{a}_H \to \bar{a} \neq 0$ and $\bar{\phi}_W \to \bar{\phi} \in (0, \infty)$, then $\mathrm{B}^2/\mathrm{MSE}_1 \to 0$ if and only if $H_n = o(\sqrt{n})$. The ratio tends to a positive constant if $H_n \asymp \sqrt{n}$, and diverges if $H_n/\sqrt{n} \to \infty$;*

(c) in a finite sample the squared bias dominates the first-order MSE whenever

$$\frac{H^2}{n} \cdot \frac{\bar{a}_H^2}{\bar{\phi}_W} > 1.$$

This happens with many small strata, strong correlation (small $\bar{\phi}_W$) and stratum biases of a common sign (no cancellation in $\bar{a}_H$).

Proof. By Theorem 2.2 with $v_{xh} = \Sigma_{xh}/(n_h \bar{X}_h^2)$ and $n_h = nW_h$, each stratum contributes $W_h \bar{Y}_h (c_h v_{xh} - k_h v_{yxh}) = a_h/n$, so $\mathrm{B} = n^{-1} \sum_h a_h = (H/n)\bar{a}_H$. Similarly $\mathrm{MSE}_1 = \sum_h W_h^2 \phi_h / n_h = n^{-1} \sum_h W_h \phi_h$. Statements (a)–(c) follow directly. □

With rounding and the requirement $r_h \geq 2$, the allocation is proportional only approximately, and the expressions then hold approximately. Proposition 2.2 also suggests a heuristic finite-sample approximation, $\mathrm{MSE} \approx \mathrm{MSE}_1 + \mathrm{B}^2$, with both terms taken from Theorem 2.2. It is not exact: it omits second-order variance terms, which are not negligible for the strongly curved logarithmic transformation in heavy-tailed populations. Its accuracy is examined in Section 4.

2.6. Optimum allocation

Theorem 2.4. *For the separate estimator with optimum constants, minimising the minimum MSE of Theorem 2.2 subject to the linear cost $\sum_h c_h n_h = C_0$ gives*

$$n_h \propto \frac{W_h \sqrt{\phi_h}}{\sqrt{c_h}}, \qquad \phi_h = \Sigma_{yh} - \frac{\Sigma_{yxh}^2}{\Sigma_{xh}} = \frac{1}{m} \sum_{j=1}^{m} \mathrm{Var}_{[j]}\big(y - B_h x\big).$$

Proof. The objective is $\sum_h W_h^2 \phi_h / n_h$, and the Lagrangian argument of Neyman allocation applies unchanged. □

We call this the SRSS-Neyman allocation. In the simulations $c_h = 1$ for all h, so $\sum_h n_h = n$. The n_h are rounded to multiples of m by the largest-remainder method subject to $r_h \geq 2$, so that the totals equal n exactly. The stratified SRS Neyman allocation uses ordinary residual variances $\sigma_{yh}^2(1 - \rho_h^2)$ in place of ϕ_h. The two coincide when the within-stratum regressions are linear and homoscedastic, because ranking on x does not change residuals that are independent of x.

3. SIMULATION DESIGN

Four populations were used (Table 2):

- P1 is synthetic, with three strata of sizes 2000, 1500 and 1000, $x \sim \mathrm{Gamma}(4, 5h)$ in stratum h, $y = b_{0h} + b_{1h}x +$ heteroscedastic normal noise, and slopes $b_{1h} = 0.6, 1.0, 1.6$.

- P2 is the MU281 population of Särndal, Swensson and Wretman [16] (the MU284 data with the three largest municipalities removed), stratified by its eight regions, with $y =$ RMT85 (municipal taxation revenue, 1985) and $x =$ P75 (population in 1975).

- P3 and P4 are the California Academic Performance Index population of 6194 schools [17], stratified by school type (elementary, high, middle), with $y =$ api00. In P3 the auxiliary is $x =$ api99 (positive correlation, ratio-type). In P4 it is $x =$ meals, the percentage of students eligible for subsidised meals (negative correlation, product-type).

The total sample sizes were $n = 60$ (P1, P3, P4) and $n = 96$ (P2), with $m \in \{2, 3, 4\}$ and either proportional or SRSS-Neyman allocation. For MU281 we also used $m = 2$ with $n \in \{32, 48, 96, 192\}$ under proportional allocation, to examine Proposition 2.2 at a fixed $H = 8$. Each configuration used $20{,}000$ independent SRSS samples. The efficiency measure is

$$\mathrm{PRE}(t) = 100 \times \frac{\mathrm{MSE}(\bar{y}_{st(srs)})}{\mathrm{MSE}(t)},$$

where $\bar{y}_{st(srs)}$ is the stratified SRS (with replacement) mean with the same allocation. Theoretical values ("Th.") use the exact moments of Lemmas 2.1–2.2 in the first-order formulas. Monte Carlo values ("MC") are reported with standard errors: the SE of a Monte Carlo MSE is the standard deviation of the squared errors divided by $\sqrt{20000}$. All computations used R [18]; the complete code is in Appendix A.

4. RESULTS

For P1, P3 and P4 the combined estimators with known constants agree with the theory to within about 3% (Tables 4–5), well within the relative Monte Carlo SE of about 1% per cell plus the omitted second-order terms. The separate logarithmic product-type estimators in P4 fall up to 13% below their first-order PRE, a bias effect that Table 9 quantifies.

In P2, with eight strata and n_h between 6 and 18, the separate logarithmic estimators fail. With $m = 2$, the PRE of $t_{LS}(1)$ is 756 against a first-order value of 4069, and that of $t_{LS}(\alpha^*)$ is 652 against 5699. Their biases are about -3.1% and -3.4% of $\bar{Y}$, and the first-order theory predicts both almost exactly (Table 9). The cause is identified by Proposition 2.1: all eight α_h^* exceed $2/3$ (Table 3), so the logarithmic form is the more biased one in every stratum, and the stratum biases share a sign. The hybrid separate estimator t_{HS}, which here uses the power form in every stratum, has a bias of about -0.3%. Its PRE (5533–5950) is the highest of all estimators. The same holds for the combined estimators: t_{HC} attains its first-order PRE (4783 against 4787), whereas $t_{LC}(\alpha_c^*)$ falls to 3843.

Table 10 shows that, with $H = 8$ fixed, $\mathrm{B}^2/\mathrm{MSE}_1$ for $t_{LS}(1)$ falls from 11.8 at $n = 32$ to 7.4, 3.7 and 1.9 at $n = 48$, 96 and 192. It roughly halves each time n doubles, as Proposition 2.2(a) states. The squared bias still dominates at $n = 192$, as in regime (c),

whereas for t_{HS} the ratio is 0.13 at $n = 32$ and 0.02 at $n = 192$. The heuristic $\mathrm{MSE}_1 + \mathrm{B}^2$ reduces the error of the first-order approximation from a factor of up to 24 to between -29% and $+9\%$. For the logarithmic estimators it undershoots the Monte Carlo MSE, by 5–17% for the separate and 3–29% for the combined versions.

In P4 ($\alpha_c^* \approx -0.27$) the classical product estimators are far worse than the stratified mean (PRE 25–41), because their implicit constant -1 is far from the optimum. The logarithmic member $t_{LC}(-\frac{1}{2})$ is better (Monte Carlo PRE 166–220), and the optimum members reach about 440. As Proposition 2.1 predicts for $k^* < 0$, the hybrid forms (power-product) are slightly less biased than the logarithmic optimum, for example -0.079% against -0.095% for the combined estimators with $m = 2$.

When the stratum slopes differ (P1), the optimum separate estimators are much better than the combined ones (PRE about 1100 against 730–856). When the slopes are similar (P3, P4), the two are equivalent.

The SRSS-Neyman allocation raises the efficiency of the optimum separate estimator by 31–44% in P1 and by 11–12% in MU281 relative to proportional allocation (Table 11). In API the gain is at most 2%. All allocations sum to the stated n. After rounding, the SRSS-Neyman and stratified-SRS Neyman allocations coincided in every configuration.

Table 8 reports Monte Carlo results only, because the first-order theory does not describe the estimated separate constants when r_h is small. The combined estimator with $\hat{\alpha}_c$ loses at most 7% relative to the known optimum. The separate estimator with $\hat{\alpha}$ loses 6–20% in P1, P3 and P4. In P3 and P4 the largest losses occur in configurations where some stratum has $r_h = 2$ (marked †). In MU281 both separate logarithmic versions are dominated by bias, so neither the known-constant nor the estimated-constant result is meaningful there.

5. CONCLUSION

The logarithmic ratio- and product-type estimators extend naturally to stratified ranked set sampling, with first-order theory expressed through exactly computable within-rank moments. Choosing between the logarithmic and the power or exponential form by the size of the optimum constant (Proposition 2.1), stratum by stratum, removes most of the bias without any cost in first-order MSE. The resulting hybrid estimators were the most efficient estimators in every population studied. The bias of separate estimators accumulates at rate H/n, and the first-order MSE should be supplemented by B^2 whenever the diagnostic $\hat{D}$ is not small. Finally, allocating the sample in proportion to W_h times the within-rank residual standard deviation gives gains of up to 44% when the strata differ in their relationships.

TABLES

Table 1: Estimators compared. $s = +1$ for the ratio-type populations (P1–P3) and $s = -1$ for the product-type population (P4).

Symbol	Definition	First-order (k, c)
$\bar{y}_{st(rss)}$	$\sum_h W_h \bar{y}_{h(rss)}$	$(0, 0)$
t_{CR}, t_{CP}	$\bar{y}_{st(rss)}(\bar{X}/\bar{x}_{st(rss)})^s$ (classical ratio / product)	$(s,\ s(s+1)/2)$
$t_{LC}(\alpha)$	$\bar{y}_{st(rss)}\{1 + \alpha \log(\bar{X}/\bar{x}_{st(rss)})\}$; $\alpha = \pm\frac{1}{2}, \pm 1, \alpha_c^*, \hat{\alpha}_c$	$(\alpha,\ \alpha/2)$
t_{HC}	$t_{LC}(\alpha_c^*)$ if $0 < \alpha_c^* < 2/3$, else $\bar{y}_{st(rss)}(\bar{X}/\bar{x}_{st(rss)})^{\alpha_c^*}$	$(\alpha_c^*,\ \cdot)$
t_{SR}, t_{SP}	$\sum_h W_h \bar{y}_{h(rss)}(\bar{X}_h/\bar{x}_{h(rss)})^s$	$(s,\ s(s+1)/2)$ per stratum
$t_{LS}(\alpha)$	$\sum_h W_h \bar{y}_{h(rss)}\{1 + \alpha_h \log(\bar{X}_h/\bar{x}_{h(rss)})\}$; $\alpha_h \equiv \pm\frac{1}{2}, \pm 1, \alpha_h^*, \hat{\alpha}_h$	$(\alpha_h,\ \alpha_h/2)$ per stratum
t_{HS}	Corollary 2.3	$(\alpha_h^*,\ \cdot)$ per stratum
$\bar{y}_{st(srs)}$	stratified SRS mean, same n_h (benchmark)	–

Table 2: Discription of Populations.

Pop.	Description	H	N	$\bar{Y}$	$\bar{X}$	C_y	C_x	ρ (range of ρ_h)
P1	Synthetic, 3 strata, unequal slopes	3	4500	57.36	35.30	0.827	0.697	0.911 (0.865, 0.959)
P2	MU281 by region, y = RMT85, x = P75	8	281	189.15	24.26	1.056	0.956	0.987 (0.968, 0.997)
P3	API by school type, y = api00, x = api99	3	6194	664.71	631.91	0.193	0.210	0.975 (0.977, 0.983)
P4	API by school type, y = api00, x = meals	3	6194	664.71	48.04	0.193	0.635	-0.828 (-0.890, -0.762)

Table 3: Optimum constants α_c^* and α_h^* under proportional allocation.

Pop.	m	α_c^*	α_h^*, $h = 1, \ldots, H$
P1	2	0.772	0.55, 0.66, 0.75
	3	0.792	0.55, 0.66, 0.75
	4	0.789	0.55, 0.66, 0.75
P2	2	1.086	0.94, 1.10, 1.13, 1.10, 1.00, 1.13, 1.11, 1.17
	3	1.091	0.94, 1.10, 1.12, 1.10, 0.99, 1.14, 1.12, 1.18
	4	1.095	0.93, 1.10, 1.12, 1.11, 0.98, 1.14, 1.12, 1.18
P3	2	0.896	0.88, 0.95, 0.94
	3	0.896	0.88, 0.95, 0.94
	4	0.897	0.88, 0.95, 0.94
P4	2	-0.272	-0.29, -0.16, -0.27
	3	-0.272	-0.29, -0.16, -0.27
	4	-0.271	-0.29, -0.16, -0.27

Table 4: PRE (relative to $\bar{y}_{st(srs)}$ with the same allocation) of the estimators with known constants under proportional allocation

		$m=2$		$m=3$		$m=4$		
Pop.	Estimator	Th.	MC	Th.	MC	Th.	MC	SE (%)
P1	$\bar{y}_{st(rss)}$	138.6	139.3	173.1	171.1	204.3	208.3	1.01
	t_{CR}	525.4	536.8	642.3	644.7	694.5	694.2	1.01
	$t_{LC}(\frac{1}{2})$	472.3	477.0	538.4	536.9	597.4	606.3	1.00
	$t_{LC}(1)$	525.4	536.0	642.3	643.6	694.5	696.0	1.01
	$t_{LC}(\alpha_c^*)$	716.7	730.4	805.0	809.2	851.6	856.1	1.01
	t_{HC}	716.7	731.8	805.0	809.5	851.6	855.2	1.01
	t_{SR}	532.5	521.9	618.5	601.4	669.9	661.5	1.05
	$t_{LS}(\frac{1}{2})$	652.9	659.0	723.5	722.7	775.5	788.1	1.01
	$t_{LS}(1)$	532.5	528.5	618.5	603.9	669.9	665.8	1.03
	$t_{LS}(\alpha^*)$	1104.6	1102.8	1123.1	1112.6	1123.6	1126.6	1.01
	t_{HS}	1104.6	1118.3	1123.1	1121.3	1123.6	1133.9	1.01
P2	$\bar{y}_{st(rss)}$	125.3	124.0	148.2	151.6	170.3	168.7	1.00
	t_{CR}	3878.4	3871.4	4098.3	4223.8	4135.6	4111.6	1.00
	$t_{LC}(\frac{1}{2})$	404.6	401.9	471.5	484.9	533.4	530.1	0.98
	$t_{LC}(1)$	3878.4	3376.6	4098.3	3784.8	4135.6	3812.1	1.08
	$t_{LC}(\alpha_c^*)$	4786.9	3843.1	5039.4	4323.2	5027.1	4409.2	1.03
	t_{HC}	4786.9	4783.4	5039.4	5195.3	5027.1	5022.5	1.02
	t_{SR}	4068.5	3610.1	4304.0	3928.4	4311.6	3908.0	0.98
	$t_{LS}(\frac{1}{2})$	406.3	377.2	473.4	456.5	535.9	505.0	1.00
	$t_{LS}(1)$	4068.5	755.6	4304.0	995.0	4311.6	1197.0	0.88
	$t_{LS}(\alpha^*)$	5699.3	652.2	5963.2	867.0	5844.6	1055.0	0.86
	t_{HS}	5699.3	5533.4	5963.2	5950.4	5844.6	5717.2	0.98

Table 5: As Table 4, populations P3–P4 (P4: product-type; $t_{LC}(-\alpha) \equiv t_{LP}(\alpha)$).

		$m=2$		$m=3$		$m=4$		
Pop.	Estimator	Th.	MC	Th.	MC	Th.	MC	SE (%)
P3	$\bar{y}_{st(rss)}$	146.2	144.0	190.2	186.2	232.2	235.0	1.00
	t_{CR}	1917.8	1907.0	2012.5	2002.6	2085.6	2048.5	1.01
	$t_{LC}(\frac{1}{2})$	593.7	581.4	727.8	713.9	841.8	852.5	1.01
	$t_{LC}(1)$	1917.8	1895.1	2012.5	1996.0	2085.6	2049.2	1.01
	$t_{LC}(\alpha_c^*)$	2301.6	2255.2	2316.6	2297.7	2335.1	2309.1	1.01
	t_{HC}	2301.6	2272.3	2316.6	2304.5	2335.1	2309.6	1.01
	t_{SR}	1887.7	1870.3	1987.8	1972.2	2067.3	2025.2	1.02
	$t_{LS}(\frac{1}{2})$	595.6	582.0	729.8	714.8	842.6	852.4	1.01
	$t_{LS}(1)$	1887.7	1838.9	1987.8	1952.0	2067.3	2018.6	1.01
	$t_{LS}(\alpha^*)$	2312.2	2232.8	2324.9	2287.6	2342.5	2303.5	1.00
	t_{HS}	2312.2	2279.4	2324.9	2312.0	2342.5	2313.4	1.01
P4	$\bar{y}_{st(rss)}$	135.1	134.0	164.0	162.6	187.7	185.6	1.00
	t_{CP}	25.5	25.6	33.3	33.4	40.8	40.7	1.01
	$t_{LC}(-\frac{1}{2})$	169.6	165.8	201.2	197.6	224.8	220.1	1.08
	$t_{LC}(-1)$	25.5	25.1	33.3	32.8	40.8	39.9	1.08
	$t_{LC}(\alpha_c^*)$	442.0	433.2	446.7	441.9	443.6	438.9	1.01
	t_{HC}	442.0	435.6	446.7	443.0	443.6	439.7	1.01
	t_{SP}	24.8	24.8	32.6	32.4	39.0	38.6	1.01
	$t_{LS}(-\frac{1}{2})$	159.7	138.7	192.1	173.1	210.5	190.9	1.09
	$t_{LS}(-1)$	24.8	22.4	32.6	29.9	39.0	35.7	1.09
	$t_{LS}(\alpha^*)$	443.7	414.4	448.3	430.1	445.2	430.6	1.02
	t_{HS}	443.7	423.5	448.3	435.3	445.2	434.6	1.01

Table 6: PRE of the estimators with known constants under SRSS-Neyman allocation.

		$m=2$		$m=3$		$m=4$		
Pop.	Estimator	Th.	MC	Th.	MC	Th.	MC	SE (%)
P1	$\bar{y}_{st(rss)}$	137.8	138.4	170.9	172.8	199.8	197.7	1.01
	t_{CR}	323.4	323.8	381.5	388.6	389.9	397.6	1.02
	$t_{LC}(\frac{1}{2})$	486.6	493.3	559.4	570.7	592.4	598.1	1.00
	$t_{LC}(1)$	323.4	324.5	381.5	389.2	389.9	398.4	1.02
	$t_{LC}(\alpha_c^*)$	585.6	593.9	657.3	673.2	662.1	675.6	1.00
	t_{HC}	585.6	593.9	657.3	673.2	662.1	675.6	1.00
	t_{SR}	444.5	429.1	503.4	503.8	531.5	532.2	1.03
	$t_{LS}(\frac{1}{2})$	633.6	638.1	693.5	708.1	716.6	724.4	1.00
	$t_{LS}(1)$	444.5	437.5	503.4	510.9	531.5	536.8	1.01
	$t_{LS}(\alpha^*)$	975.4	965.7	978.5	1004.4	945.5	965.7	1.01
	t_{HS}	975.4	972.2	978.5	1006.6	945.5	968.1	1.01
P2	$\bar{y}_{st(rss)}$	125.3	124.0	148.3	146.1	170.4	176.4	1.00
	t_{CR}	4370.4	4339.5	4556.1	4429.5	4634.4	4760.9	1.01
	$t_{LC}(\frac{1}{2})$	409.7	405.8	480.1	474.1	545.0	566.3	0.98
	$t_{LC}(1)$	4370.4	3655.5	4556.1	3868.1	4634.4	4319.9	1.14
	$t_{LC}(\alpha_c^*)$	5443.5	4169.0	5537.8	4341.0	5534.7	4850.4	1.08
	t_{HC}	5443.5	5484.1	5537.8	5390.8	5534.7	5662.7	1.00
	t_{SR}	4575.9	3899.3	4823.0	4127.5	4872.6	4483.2	1.00
	$t_{LS}(\frac{1}{2})$	408.3	377.7	479.3	444.4	543.5	534.1	1.00
	$t_{LS}(1)$	4575.9	713.3	4823.0	953.8	4872.6	1234.2	0.88
	$t_{LS}(\alpha^*)$	6773.8	609.6	6950.0	831.7	6868.1	1083.0	0.85
	t_{HS}	6773.8	6330.1	6950.0	6405.2	6868.1	6756.2	0.98

Table 7: As Table 6, populations P3–P4.

		$m=2$		$m=3$		$m=4$		
Pop.	Estimator	Th.	MC	Th.	MC	Th.	MC	SE (%)
P3	$\bar{y}_{st(rss)}$	146.2	144.1	190.1	191.5	232.2	228.3	1.00
	t_{CR}	1950.7	1932.7	2021.2	2008.8	2085.6	2073.3	1.01
	$t_{LC}(\frac{1}{2})$	592.0	584.0	725.7	729.9	841.8	826.1	1.01
	$t_{LC}(1)$	1950.7	1931.2	2021.2	2005.0	2085.6	2072.1	1.01
	$t_{LC}(\alpha_c^*)$	2325.9	2303.8	2318.5	2301.1	2335.1	2310.1	1.01
	t_{HC}	2325.9	2308.5	2318.5	2304.8	2335.1	2313.3	1.01
	t_{SR}	1929.0	1900.5	2001.7	1983.5	2067.3	2053.4	1.01
	$t_{LS}(\frac{1}{2})$	591.5	581.8	725.8	728.9	842.6	826.1	1.01
	$t_{LS}(1)$	1929.0	1876.7	2001.7	1968.2	2067.3	2040.9	1.01
	$t_{LS}(\alpha^*)$	2338.6	2278.8	2327.6	2290.5	2342.5	2294.8	1.01
	t_{HS}	2338.6	2318.9	2327.6	2309.1	2342.5	2311.8	1.01
P4	$\bar{y}_{st(rss)}$	135.1	136.1	164.0	164.1	187.7	187.3	1.00
	t_{CP}	25.5	25.5	33.3	33.6	40.8	39.8	1.01
	$t_{LC}(-\frac{1}{2})$	169.6	162.7	201.2	196.7	224.8	216.2	1.06
	$t_{LC}(-1)$	25.5	24.8	33.3	32.9	40.8	39.2	1.08
	$t_{LC}(\alpha_c^*)$	442.0	435.1	446.7	439.3	443.6	440.0	1.00
	t_{HC}	442.0	437.3	446.7	440.7	443.6	440.9	1.00
	t_{SP}	24.8	24.6	32.6	32.8	39.0	37.9	1.01
	$t_{LS}(-\frac{1}{2})$	159.7	134.9	192.1	172.6	210.5	187.3	1.08
	$t_{LS}(-1)$	24.8	21.9	32.6	30.1	39.0	35.1	1.10
	$t_{LS}(\alpha^*)$	443.7	415.3	448.3	426.8	445.2	430.2	1.00
	t_{HS}	443.7	424.5	448.3	432.5	445.2	434.4	1.00

Table 8: Estimated versus known constants: Monte Carlo PRE only.

		Proportional			SRSS-Neyman		
Pop.	Estimator	$m=2$	$m=3$	$m=4$	$m=2$	$m=3$	$m=4$
P1	$t_{LC}(\alpha_c^*)$	730.4	809.2	856.1	593.9	673.2	675.6
	$t_{LC}(\hat{\alpha}_c)$	689.8	750.6	793.5	564.7	641.9	642.5
	$t_{LS}(\alpha^*)$	1102.8	1112.6	1126.6	965.7	1004.4	965.7
	$t_{LS}(\hat{\alpha})$	930.1	912.1	932.9	845.5	882.5	843.8
P2	$t_{LC}(\alpha_c^*)$	3843.1	4323.2	4409.2	4169.0	4341.0	4850.4
	$t_{LC}(\hat{\alpha}_c)$	3822.9	4296.4	4177.9	4289.6	4348.5	4659.3
	$t_{LS}(\alpha^*)$	652.2	867.0	1055.0	609.6	831.7	1083.0
	$t_{LS}(\hat{\alpha})$	1086.6	1317.5†	1326.2†	937.8†	1320.6†	1497.2†
P3	$t_{LC}(\alpha_c^*)$	2255.2	2297.7	2309.1	2303.8	2301.1	2310.1
	$t_{LC}(\hat{\alpha}_c)$	2232.7	2248.0	2240.5	2245.4	2250.8	2249.0
	$t_{LS}(\alpha^*)$	2232.8	2287.6	2303.5	2278.8	2290.5	2294.8
	$t_{LS}(\hat{\alpha})$	2107.4	2124.0	1998.8†	1998.2	1964.3†	1977.8†
P4	$t_{LC}(\alpha_c^*)$	433.2	441.9	438.9	435.1	439.3	440.0
	$t_{LC}(\hat{\alpha}_c)$	427.9	432.6	428.0	430.1	432.0	429.4
	$t_{LS}(\alpha^*)$	414.4	430.1	430.6	415.3	426.8	430.2
	$t_{LS}(\hat{\alpha})$	366.6	378.4	345.2†	367.2	383.8	344.4†

Table 9: Relative bias (% of $\bar{Y}$), proportional allocation: first-order theory and Monte Carlo ($\pm$ one SE).

		$m = 2$		$m = 4$	
Pop.	Estimator	Th.	MC ($\pm$SE)	Th.	MC ($\pm$SE)
P1	t_{CR}	0.077	0.034 ($\pm$0.018)	0.049	0.079 ($\pm$0.017)
	$t_{LC}(1)$	-0.092	-0.133 ($\pm$0.018)	-0.067	-0.035 ($\pm$0.017)
	$t_{LC}(\alpha_c^*)$	-0.071	-0.101 ($\pm$0.015)	-0.053	-0.026 ($\pm$0.015)
	t_{HC}	0.030	-0.001 ($\pm$0.015)	0.019	0.045 ($\pm$0.015)
	t_{SR}	0.301	0.281 ($\pm$0.018)	0.207	0.231 ($\pm$0.017)
	$t_{LS}(1)$	-0.209	-0.235 ($\pm$0.018)	-0.148	-0.125 ($\pm$0.017)
	$t_{LS}(\alpha^*)$	-0.152	-0.163 ($\pm$0.013)	-0.108	-0.091 ($\pm$0.013)
	t_{HS}	0.023	0.014 ($\pm$0.012)	0.021	0.037 ($\pm$0.013)
P2	t_{CR}	-0.062	-0.075 ($\pm$0.012)	-0.050	-0.039 ($\pm$0.012)
	$t_{LC}(1)$	-0.424	-0.436 ($\pm$0.013)	-0.314	-0.302 ($\pm$0.012)
	$t_{LC}(\alpha_c^*)$	-0.460	-0.470 ($\pm$0.012)	-0.344	-0.334 ($\pm$0.011)
	t_{HC}	-0.034	-0.044 ($\pm$0.011)	-0.027	-0.019 ($\pm$0.010)
	t_{SR}	-0.501	-0.539 ($\pm$0.012)	-0.405	-0.429 ($\pm$0.012)
	$t_{LS}(1)$	-3.140	-3.117 ($\pm$0.027)	-2.333	-2.335 ($\pm$0.021)
	$t_{LS}(\alpha^*)$	-3.458	-3.427 ($\pm$0.029)	-2.594	-2.602 ($\pm$0.023)
	t_{HS}	-0.284	-0.325 ($\pm$0.010)	-0.232	-0.264 ($\pm$0.010)
P3	t_{CR}	0.005	-0.001 ($\pm$0.004)	0.003	0.006 ($\pm$0.004)
	$t_{LC}(1)$	-0.019	-0.026 ($\pm$0.004)	-0.012	-0.009 ($\pm$0.004)
	$t_{LC}(\alpha_c^*)$	-0.017	-0.026 ($\pm$0.004)	-0.011	-0.007 ($\pm$0.004)
	t_{HC}	0.002	-0.006 ($\pm$0.004)	0.001	0.005 ($\pm$0.004)
	t_{SR}	0.010	0.004 ($\pm$0.004)	0.007	0.009 ($\pm$0.004)
	$t_{LS}(1)$	-0.053	-0.059 ($\pm$0.004)	-0.035	-0.032 ($\pm$0.004)
	$t_{LS}(\alpha^*)$	-0.049	-0.057 ($\pm$0.004)	-0.032	-0.028 ($\pm$0.004)
	t_{HS}	0.005	-0.004 ($\pm$0.004)	0.003	0.007 ($\pm$0.004)
P4	t_{CP}	-0.117	-0.069 ($\pm$0.035)	-0.070	-0.088 ($\pm$0.028)
	$t_{LC}(-1)$	-0.332	-0.286 ($\pm$0.035)	-0.200	-0.219 ($\pm$0.028)
	$t_{LC}(\alpha_c^*)$	-0.090	-0.095 ($\pm$0.008)	-0.054	-0.057 ($\pm$0.008)
	t_{HC}	-0.074	-0.079 ($\pm$0.008)	-0.045	-0.047 ($\pm$0.008)
	t_{SP}	-0.330	-0.285 ($\pm$0.035)	-0.213	-0.224 ($\pm$0.028)
	$t_{LS}(-1)$	-1.048	-1.030 ($\pm$0.037)	-0.679	-0.706 ($\pm$0.029)
	$t_{LS}(\alpha^*)$	-0.246	-0.255 ($\pm$0.009)	-0.158	-0.163 ($\pm$0.008)
	t_{HS}	-0.206	-0.213 ($\pm$0.009)	-0.132	-0.136 ($\pm$0.008)

Table 10: MU281 ($H = 8$, $m = 2$, proportional allocation) for increasing n.

n	Estimator	MSE_1	B^2	$\text{MSE}_1 + \text{B}^2$	MC MSE ($\pm$SE)	B^2/MSE_1	$\frac{\text{MSE}_1+\text{B}^2}{\text{MC}}$
32 ($r_h \geq 2$)	t_{CR}	31.79	0.15	31.94	31.53 ($\pm$0.31)	0.005	1.013
	$t_{LC}(1)$	31.79	7.73	39.52	47.47 ($\pm$0.56)	0.243	0.833
	$t_{LC}(\alpha_c^*)$	25.83	9.03	34.86	48.91 ($\pm$0.71)	0.350	0.713
	t_{HC}	25.83	0.04	25.87	25.51 ($\pm$0.27)	0.002	1.014
	t_{SR}	29.76	8.38	38.13	35.69 ($\pm$0.32)	0.282	1.069
	$t_{LS}(1)$	29.76	351.58	381.34	400.02 ($\pm$3.09)	11.815	0.953
	$t_{LS}(\alpha^*)$	20.86	422.95	443.81	491.64 ($\pm$4.37)	20.272	0.903
	t_{HS}	20.86	2.66	23.52	21.58 ($\pm$0.20)	0.128	1.090
48 ($r_h \geq 2$)	t_{CR}	20.89	0.06	20.95	20.94 ($\pm$0.21)	0.003	1.001
	$t_{LC}(1)$	20.89	2.65	23.55	27.01 ($\pm$0.30)	0.127	0.872
	$t_{LC}(\alpha_c^*)$	16.61	3.16	19.77	25.49 ($\pm$0.29)	0.190	0.775
	t_{HC}	16.61	0.02	16.63	16.80 ($\pm$0.18)	0.001	0.990
	t_{SR}	19.86	4.01	23.87	23.67 ($\pm$0.22)	0.202	1.008
	$t_{LS}(1)$	19.86	147.45	167.31	182.67 ($\pm$1.32)	7.426	0.916
	$t_{LS}(\alpha^*)$	13.98	180.04	194.02	222.72 ($\pm$1.81)	12.880	0.871
	t_{HS}	13.98	1.30	15.27	14.42 ($\pm$0.14)	0.093	1.059
96 ($r_h \geq 3$)	t_{CR}	10.13	0.01	10.15	10.20 ($\pm$0.10)	0.001	0.995
	$t_{LC}(1)$	10.13	0.64	10.77	11.60 ($\pm$0.13)	0.063	0.929
	$t_{LC}(\alpha_c^*)$	8.21	0.76	8.97	10.33 ($\pm$0.11)	0.092	0.868
	t_{HC}	8.21	0.00	8.21	8.32 ($\pm$0.09)	0.000	0.987
	t_{SR}	9.66	0.90	10.56	10.93 ($\pm$0.11)	0.093	0.966
	$t_{LS}(1)$	9.66	35.28	44.93	51.64 ($\pm$0.42)	3.653	0.870
	$t_{LS}(\alpha^*)$	6.89	42.77	49.67	59.75 ($\pm$0.49)	6.204	0.831
	t_{HS}	6.89	0.29	7.18	7.23 ($\pm$0.07)	0.042	0.993
192 ($r_h \geq 5$)	t_{CR}	5.15	0.00	5.15	5.06 ($\pm$0.05)	0.001	1.018
	$t_{LC}(1)$	5.15	0.16	5.31	5.45 ($\pm$0.06)	0.031	0.974
	$t_{LC}(\alpha_c^*)$	4.15	0.19	4.34	4.58 ($\pm$0.05)	0.046	0.949
	t_{HC}	4.15	0.00	4.15	4.08 ($\pm$0.04)	0.000	1.018
	t_{SR}	4.92	0.24	5.16	5.24 ($\pm$0.05)	0.048	0.985
	$t_{LS}(1)$	4.92	9.15	14.07	16.10 ($\pm$0.15)	1.859	0.874
	$t_{LS}(\alpha^*)$	3.52	11.12	14.63	17.52 ($\pm$0.15)	3.159	0.835
	t_{HS}	3.52	0.08	3.60	3.56 ($\pm$0.04)	0.022	1.009

Table 11: Allocations (n_h by stratum, with total) and the theoretical efficiencies.

Pop.	m	Allocation	n_h (total)	$t_{LC}(\alpha_c^*)$	$t_{LS}(\alpha^*)$
P1	2	proportional	26/20/14 (60)	64.9	100.0
		SRSS-Neyman	14/20/26 (60)	78.4	130.5
		StSRS-Neyman	14/20/26 (60)	78.4	130.5
P1	3	proportional	27/21/12 (60)	71.7	100.0
		SRSS-Neyman	15/18/27 (60)	96.0	142.9
		StSRS-Neyman	15/18/27 (60)	96.0	142.9
P1	4	proportional	28/20/12 (60)	75.8	100.0
		SRSS-Neyman	12/20/28 (60)	100.8	144.0
		StSRS-Neyman	12/20/28 (60)	100.8	144.0
P2	2	proportional	8/16/12/12/18/14/6/10 (96)	84.0	100.0
		SRSS-Neyman	12/16/20/12/14/8/4/10 (96)	90.1	112.1
		StSRS-Neyman	12/16/20/12/14/8/4/10 (96)	90.1	112.1
P2	3	proportional	9/15/12/12/18/15/6/9 (96)	84.5	100.0
		SRSS-Neyman	12/15/21/12/12/9/6/9 (96)	88.3	110.9
		StSRS-Neyman	12/15/21/12/12/9/6/9 (96)	88.3	110.9
P2	4	proportional	8/16/12/12/20/12/8/8 (96)	86.0	100.0
		SRSS-Neyman	12/16/20/12/12/8/8/8 (96)	89.8	111.4
		StSRS-Neyman	12/16/20/12/12/8/8/8 (96)	89.8	111.4
P3	2	proportional	42/8/10 (60)	99.5	100.0
		SRSS-Neyman	46/6/8 (60)	100.8	101.3
		StSRS-Neyman	46/6/8 (60)	100.8	101.3
P3	3	proportional	42/9/9 (60)	99.6	100.0
		SRSS-Neyman	45/6/9 (60)	101.3	101.7
		StSRS-Neyman	45/6/9 (60)	101.3	101.7
P3	4	proportional	44/8/8 (60)	99.7	100.0
		SRSS-Neyman	44/8/8 (60)	99.7	100.0
		StSRS-Neyman	44/8/8 (60)	99.7	100.0
P4	2	proportional	42/8/10 (60)	99.6	100.0
		SRSS-Neyman	42/8/10 (60)	99.6	100.0
		StSRS-Neyman	42/8/10 (60)	99.6	100.0
P4	3	proportional	42/9/9 (60)	99.7	100.0
		SRSS-Neyman	42/9/9 (60)	99.7	100.0
		StSRS-Neyman	42/9/9 (60)	99.7	100.0
P4	4	proportional	44/8/8 (60)	99.7	100.0
		SRSS-Neyman	44/8/8 (60)	99.7	100.0
		StSRS-Neyman	44/8/8 (60)	99.7	100.0

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