Some Approximation Properties of q- MBS Operators

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ABSTRACT

This paper deals with the new type q-analogue of the modified Beta operators. We apply q-derivatives to obtain the central moments of the discrete q-Beta operators. We establish a direct result in terms of modulus of continuity for the q-operators. We also obtain some approximation properties and asymptotic formula for these operators.

KEY WORDS: Linear positive operators, Voronovskaja type asymptotic formula, Modulus of continuity, q-integers, q-Beta functions, q-derivatives, Weighted-approximation.

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1. INTRODUCTION

In the recent years, the q-calculus has a great deal of interest. The applications of q-calculus in the approximation theory are one of the main areas of research. After the q-analogue of Bernstein polynomials obtained by Phillips [23], Gupta and Heping [9] introduced q-Durrmeyer operators. Several other researchers have studied in this direction and obtained different approximation properties of many other operators. We mention some of them as [1], [4], [12], [21], [22] etc. In the present article, we propose the q-analogue of the modified Beta operators and study their convergence behavior. To approximate Lebesgue integrable function on the interval [0,∞), modified Beta operators [8] are defined as
\[ B_n(f, x) = \frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0, \infty) \]  
(1.1)

where
\[ b_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!} \frac{x^k}{(1+x)^{n+k+1}} \]
and
\[ p_{n,k}(t) = \frac{(n+k-1)!}{k!(n-1)!} \frac{t^k}{(1+t)^{n+k}}. \]

For every \( q \in (0, 1) \) we propose the q-analogue of these operators as
\[ B_q^n(f(t), x) = \frac{[n-1]_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k}^q(x) \int_0^1 q^k p_{n,k}^q(t) f(t) d_q t, \]
(1.2)

Where
\[ b_{n,k}^q(x) = \frac{[n+k]_q}{[k]_q [n-1]_q} q^{q(k-1)/2} \frac{x^k}{(1+x)^{n+k+1}} \]
and
\[ p_{n,k}^q(t) = \frac{[n+k-1]_q}{[k]_q [n-1]_q} q^{q(k-1)/2} \frac{t^k}{(1+t)^{n+k}}. \]

We can easily define
\[ \sum_{k=0}^{\infty} b_{n,k}^q(x) = [n]_q \]
and
\[ \int_0^1 q^k p_{n,k}^q(t) d_q t = \frac{1}{[n-1]_q}. \]

These operators reproduce constant function. Also \( B_q^n(f(t), x) = B_n(f(t), x) \). We consider the discrete q-Beta operators defined as
\[ V_q^n(f, x) = \frac{1}{[n]_q} \sum_{k=0}^{\infty} b_{n,k}^q(x) f \left( \frac{[k]_q}{[n]_q} q^{k-1} \right) \]
(1.3)

Maheshwari and Sharma [19] introduced the q-analogue of the Baskakov Beta Stancu operators and studied the rate of approximation and weighted approximation of these operators. Other approximation properties of these operators were studied by [17], [18], [20]. In this direction, we mention contribution of some other authors such as [6], [7]. Motivated by the [24], we suggest for \( 0 \leq \alpha \leq \beta \), the q-analogue of the operators \( B_{n,\alpha,\beta}^q \)
\[ B_{n,\alpha,\beta}^q(f, x) = \frac{[n-1]_q}{[n]_q} \sum_{k=0}^{\infty} b_{n,k}^q(x) \int_0^1 q^k p_{n,k}^q(t) f \left( \frac{[n]_q t + \alpha}{[n]_q + \beta} \right) d_q t. \]
(1.4)

For the notation and definition of q-calculus, we can refer the book written by Aral, Gupta and Agarwal [11]. De Sole and Kac [3] gave the q-analogue of Beta function of
second kind.

\[ B(r, s) = \int_{0}^{\infty} \frac{x^{r-1}}{(1+x)^{r+s}} \, dx \]

As follows

\[ B_{q}(r, s) = K(A, r) \int_{0}^{\infty} \frac{x^{r-1}}{(1+x)_{q}^{r+s}} \, dx \]

(1.5)

where

\[ K(x, r) = \frac{1}{(1+x)^{x}} x^{(1+1/2)} (1+x)^{1/2}. \]

That function is q-constant in x, that is \( K(qx, r) = K(x, r) \). It was observed in [3] that \( B_{q}(r, s) \) is independent of A. In particular for any positive integer n, we have \( K(x, n) = q^{n(n-1)/2} \), \( K(x, 0) = 1 \).

and we have

\[ B_{q}(r, s) = \frac{\Gamma_{q}(r)\Gamma_{q}(s)}{\Gamma_{q}(r+s)}. \quad (1.6) \]

For q-analogue of linear positive operators many researches such as [14], [15], [16] and [13] have done a lot of work in this direction.

2. MOMENT ESTIMATION AND AUXILIARY RESULTS

In this section, we establish certain lemmas which will be useful for the proof of our main theorems.

**Lemma 1:** [10] For \( V_{n}^{q}(t^{m}, x) \), \( m = 0, 1, 2, \ldots \)

\[ V_{n}^{q}(1, x) = \int_{x} \]

\[ V_{n}^{q}(t, x) = x \]

\[ V_{n}^{q}(t^{2}, x) = \frac{[n+2]_{q}x^{2} + x [n+1]_{q}}{q[n+1]_{q}}. \]

**Lemma 2:** [10]: The following equalities hold

\[ B_{n}^{q}(1, x) = 1 \]

\[ B_{n}^{q}(t, x) = \frac{[n+1]_{q}x + 1}{q[n-2]_{q}}, \quad n > 2 \]

\[ B_{n}^{q}(t^{2}, x) = \frac{[n+1]_{q}[n+2]_{q}x^{2} + [n+1]_{q}[2]_{q}^{2}x + [2]_{q}}{q[n-2]_{q}[n-3]_{q}}, \quad n > 3. \]

**Lemma 3:** The following equalities hold
\[ B_n^q(1, x) = 1 \]
\[ B_n^q(t, x) = \frac{[n+1]_q [n]_q}{q^2[n-2]_q ([n]_q + \beta)} x + \left( \frac{[n]_q}{q[n-2]_q} + \alpha \right) \frac{1}{([n]_q + \beta)}, \ n > 2 \]
\[ B_n^q(t^2, x) = \frac{[n+1]_q [n+2]_q [n]_q}{q^3[n-2]_q [n-3]_q ([n]_q + \beta)^2} x^2 + \left( \frac{[n]_q [2]_q^2}{q^3[n-3]_q} + 2\alpha \right) \frac{[n+1]_q [n]_q}{q^2[n-2]_q ([n]_q + \beta)^2} x \]
\[ + \frac{[2]_q [n]_q^2}{q^3[n-2]_q [n-3]_q ([n]_q + \beta)^2} + \frac{2\alpha [n]_q}{q[n-2]_q ([n]_q + \beta)^2} + \frac{\alpha}{([n]_q + \beta)^2}, \ n > 3. \]

**Proof:** The operators \( B_n^q \) are well defined on function \( 1, t, t^2 \). By Lemma 2, for every \( n > 3 \) and \( x \in [0, \infty) \), we have
\[ B_n^q(1, x) = B_n^q(1, x) = 1. \]
\[ B_n^q(t, x) = \frac{[n]_q}{([n]_q + \beta)} B_n^q(t, x) + \left( \frac{[n]_q}{[n]_q + \beta} \right) B_n^q(1, x) \]
\[ = \frac{[n+1]_q [n]_q}{q^2[n-2]_q ([n]_q + \beta)} x + \left( \frac{[n]_q}{q[n-2]_q} + \alpha \right) \frac{1}{([n]_q + \beta)}. \]

Similarly
\[ B_n^q(t^2, x) = \frac{[n+1]_q [n+2]_q [n]_q}{q^3[n-2]_q [n-3]_q ([n]_q + \beta)^2} x^2 + \left( \frac{[n]_q [2]_q^2}{q^3[n-3]_q} + 2\alpha \right) \frac{[n+1]_q [n]_q}{q^2[n-2]_q ([n]_q + \beta)^2} x \]
\[ + \frac{[2]_q [n]_q^2}{q^3[n-2]_q [n-3]_q ([n]_q + \beta)^2} + \frac{2\alpha [n]_q}{q[n-2]_q ([n]_q + \beta)^2} + \frac{\alpha}{([n]_q + \beta)^2}. \]

**Remark 1.** If we put \( q = 1 \) and \( \alpha = \beta = 0 \), we get the moments of the modified Beta operators [8] as
\[ B_n^1(t, x) = \frac{(n+1)x+1}{n-2}, \ n > 2 \]
\[ B_n^1(t^2, x) = \frac{(n+1)(n+2)x^2 + 4(n+1)x + 2}{n(n-2)(n-3)}, \ n > 3 \]
Remark 2. From Lemma 3, we have
\[ E_{n,\alpha,\beta}(x) = B_{n,\alpha,\beta}^q ((t-x), x) \]
\[ = \left( \frac{[n+1]_q[n]_q}{q^2 [n-2]_q ([n]_q + \beta)} \right) x + \left( \frac{[n]_q}{q [n-2]_q} + \alpha \right) \frac{1}{([n]_q + \beta), \quad n > 2} \]
\[ F_{n,\alpha,\beta}(x) = B_{n,\alpha,\beta}^q ((t-x)^2, x) \]
\[ = B_{n,\alpha,\beta}^q (t^2, x) - 2 x B_{n,\alpha,\beta}^q (t, x) + x^2 \]
\[ = \left( \frac{[n+1]_q[n+2]_q[n]_q^2}{q^3 [n-2]_q [n-3]_q ([n]_q + \beta)^2} - \frac{[n+1]_q[n]_q}{q^2 [n-2]_q ([n]_q + \beta)} + 1 \right) x^2 \]
\[ + \left( \frac{[n+1]_q[n]_q^2 [2]_q}{q^2 [n-2]_q [n-3]_q ([n]_q + \beta)^2} + \frac{[n+1]_q[n]_q \alpha}{q^2 [n-2]_q ([n]_q + \beta)} - 2 \right) \frac{[n]_q}{q [n-2]_q} + \alpha \right) \frac{1}{([n]_q + \beta)} x \]
\[ + \left( \frac{2 \alpha [n]_q}{q^2 [n-2]_q [n-3]_q ([n]_q + \beta)^2} + \frac{\alpha}{([n]_q + \beta)^2} + \frac{\alpha}{([n]_q + \beta)} \right) x, \quad n > 3 \]

Remark 3. [10] We define the central moment as
\[ T_{n,m}(x) = B_{n,\alpha,\beta}^q (t^n, x) = \frac{[n-1]_q [n]_q \sum_{k=0}^\infty b^\gamma_{\alpha,\beta}^k (x)}{[n]_q} \int_0^1 q^k p^\gamma_{\alpha,\beta}^k (t) t^m d_q t, \]
then for \( n > m + 2 \), we have the following recurrence relation
\[ ([n]_q - [m+2]_q) T_{n,m+1}(q x) = q x (1 + x) D_q T_{n,m}(x) + q ([m+1]_q + [n+1]_q x) T_{n,m}(q x). \]

3. DIRECT ESTIMATES
Definition: Let the space \( C_q[0,\infty) \) of all real valued continuous bounded functions \( f \) on \([0,\infty)\) endowed with the norm \( ||f|| = \sup \{|f(t) : x \in [0,\infty)\}| \). Further, let us consider the following K functional
\[ K_2(f, \delta) = \inf \{ ||f - g|| + \delta ||g'|| \}, \]
where \( \delta > 0 \) and \( W^2 = \{ g \in C_q[0,\infty) : g', g'' \in C_q[0,\infty) \} \), according to [2], there exist an absolute constant \( C > 0 \) such that
\[ K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \quad (3.1) \]
where
\[ \omega_2(f, \sqrt{\delta}) = \sup_{0 < h < \sqrt{\delta}} \sup_{q \in [0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)| \]
is the second order modulus of smoothness of \( f \in C_q[0,\infty) \) and
\[ \omega(f, \sqrt{\delta}) = \sup_{0 < h < \sqrt{\delta}} \sup_{q \in [0,\infty)} |f(x + h) - f(x)| \]
Theorem 1. Let \( f \in C_q(0,\infty), \) with \( q \in (0,1), \) then for every \( x \in [0,\infty) \) and \( n \geq 3, \) we
have

$$
|B_{n,a,b}^q(f,x) - f(x)| \leq C \omega_2(f,\delta_n(x)) + \alpha \left( f, \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} - 1 \right) x + \left( \frac{[n]_q}{q[n-2]_q + \alpha} \right) \frac{1}{([n]_q + \beta)}.
$$

where C is an absolute constant and

$$
\delta_n(x) = F_{n,a,b}(x) + \left( \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} - 1 \right) x + \left( \frac{[n]_q}{q[n-2]_q + \alpha} \right) \frac{1}{([n]_q + \beta)} \right)^{1/2}.
$$

**Proof:** Introducing the auxiliary operators as follows

$$
\overline{B}_{n,a,b}^q = B_{n,a,b}^q + f(x) - f \left( \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} \right) x + \left( \frac{q^2 \alpha[n-2]_q + q}{[n]_q [n+1]_q} \right) (3.2)
$$

From (3.2) and Lemma 3, we have

$$
\overline{B}_{n,a,b}^q((t-x),x) = 0. \tag{3.3}
$$

Let \( x \in [0, \infty) \) and \( g \in \mathcal{W}^2[0, \infty) \). Using Taylor's formula

$$
g(t) = g(x) + (t-x)g'(x) + \int_0^t (t-u)g''(u)du, \quad t \in [0, \infty)
$$

applying \( B_{n,a,b}^q \) and by (3.3), we get

$$
\overline{B}_{n,a,b}^q(g,x) = g(x) + \overline{B}_{n,a,b}^q \left( \int_0^t (t-u)g''(u)du, x \right)
$$

Hence, by (3.2), we have

$$
|\overline{B}_{n,a,b}^q(g,x) - g(x)| \leq |B_{n,a,b}^q| \left( \int_0^t (t-u)g''(u)du, x \right) + \left| \int_0^t \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} \right| \frac{1}{([n]_q + \beta)}
$$

$$
\leq |B_{n,a,b}^q| \left( \int_0^t (t-u)g''(u)du, x \right) + \left( \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} \right) \frac{1}{([n]_q + \beta)}
$$

$$
\leq |B_{n,a,b}^q| \left( (t-x)^2, x \right) + \left( \frac{[n+1]_q[n]_q}{q^2[n-2]_q ([n]_q + \beta)} - 1 \right) x + \left( \frac{[n]_q}{q[n-2]_q + \alpha} \right) \frac{1}{([n]_q + \beta)} \right)^2 ||g''|| \tag{3.4}
$$

From (3.2), we can write
Now from (3.2), (3.4) and (3.5), we get

\[ |B_{n,a,b}^q(f,x) - f(x)| \leq B_{n,a,b}^q(f-g,x) - (f-g)(x) + B_{n,a,b}^q(g,x) - g(x) \]

\[
\frac{f(x) - f\left(\frac{[n+1]_q [n]_q}{[n-2]_q ([n]_q + \beta)} x + \frac{q\alpha [n-2]_q}{[n]_q [n+1]_q} x + \frac{q}{[n+1]_q}\right)}{[n]_q [n+1]_q} \leq 4\|f-g\| + \delta_n^2(x)\|g'\|.
\]

Now taking infimum on the right hand side over all \( g \in W^2[0, \infty) \), we get

\[ B_{n,a,b}^q(f,x) - f(x) \leq CK_2(f,\delta_n^2(x)) + \omega\left(f,\left(\frac{[n+1]_q [n]_q}{[n-2]_q ([n]_q + \beta)} x + \frac{q\alpha [n-2]_q}{[n]_q [n+1]_q} x + \frac{q}{[n+1]_q}\right)\right) \]

From definition (3.1), we get

\[ B_{n,a,b}^q(f,x) - f(x) \leq C\omega_2(f,\delta_n(x)) + \omega\left(f,\left(\frac{[n+1]_q [n]_q}{[q-2]_q ([n]_q + \beta)} x + \frac{[n]_q}{[q-2]_q} + \alpha\right)\right) \]

Hence proves the theorem.

**Definition:** Let \( H^*_{\omega}[0, \infty) \) be the set of all functions \( f \) defined on \([0, \infty)\), satisfying the condition \( |f(x)| \leq M_f (1 + x^2) \), where \( M_f \) is a constant depending only on \( f \). By \( C^*_{\omega}[0, \infty) \), we denote the subspace of all continuous functions belonging to \( H^*_{\omega}[0, \infty) \). Also let \( C^*_{\omega}[0, \infty) \) be the subspace of all functions \( f \in C^*_{\omega}[0, \infty) \), for which \( \lim_{x \to \infty} \frac{f(x)}{1 + x^2} \) is finite. The norm on \( C^*_{\omega}[0, \infty) \) is \( \|f\|_{C^*} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2} \). We denote the modulus of continuity on \( f \) closed interval \([0, a]\), \( a > 0 \) as by \( \omega_2(f, \delta) = \sup_{\delta < \delta_0 \leq 0} \sup_{x \in [0, \infty)} |f(x) - f(x)| \)

We observe that for the function \( f \in C^*_{\omega}[0, \infty) \), the modulus of continuity \( \omega_2(f, \delta) \) tends to zero.

**Theorem 2.** Let \( q = q_n \) satisfies \( 0 < q_n < 1 \) and let \( q_n \to 1 \) as \( n \to \infty \). For each \( f \in C^*_{\omega}[0, \infty) \), we have

\[ \lim_{n \to \infty} \|B_{n,a,b}^q(f) - f\|_{C^*} = 0 \]

**Proof:** By using the Korovkin theorem [5], we see that it is sufficient to verify the
following three conditions
\[ \lim_{n \to \infty} \| B_{n,\alpha,\beta}^{(v)}(t^v, x) - x^t \|_\varepsilon = 0, \quad v = 0, 1, 2 \] (3.6)

Since \( B_{n,\alpha,\beta}^{(1)}(1, x) = 1 \), therefore (3.6) holds true for \( v = 0 \).

By Lemma 3, we have for \( n > 2 \)
\[
\left\| B_{n,\alpha,\beta}^{(v)}(t, x) - x \right\|_\varepsilon \leq \left( \frac{[n + 1]_q [n]_q}{q^2 [n - 2]_q ([n]_q + \beta)} \right)^{-1} \sup_{\alpha \in (0, \infty)} \frac{x^2}{1 + x^2} + \left( \frac{[n]_q + \alpha}{q[n - 2]_q ([n]_q + \beta)} \right)^{-1} \sup_{\alpha \in (0, \infty)} \frac{1}{1 + x^2}
\]

Thus
\[ \lim_{n \to \infty} \left\| B_{n,\alpha,\beta}^{(v)}(t, x) - x \right\|_\varepsilon = 0. \]

Similarly we can write for \( n > 3 \)
\[
\left\| B_{n,\alpha,\beta}^{(v)}(t^2, x) - x^2 \right\|_\varepsilon \leq \left( \frac{[n + 1]_q [n + 2]_q [n]_q^2}{q^6 [n - 2]_q [n - 3]_q ([n]_q + \beta)^2} \right)^{-1} \sup_{\alpha \in (0, \infty)} \frac{x^2}{1 + x^2}
\]
\[ + \left[ \frac{[n]_q [2]_q^2}{q^3 [n - 3]_q ([n]_q + \beta)^2} \right] \sup_{\alpha \in (0, \infty)} \frac{x}{1 + x^2}
\]
\[ + \left[ \frac{[2]_q [n]_q^2}{q^4 [n - 2]_q [n - 3]_q ([n]_q + \beta)^2} + \frac{2\alpha [n]_q}{q[n - 2]_q ([n]_q + \beta)^2} + \left( \frac{\alpha}{([n]_q + \beta)} \right)^2 \right] \sup_{\alpha \in (0, \infty)} \frac{1}{1 + x^2}
\]

Which implies that
\[ \lim_{n \to \infty} \left\| B_{n,\alpha,\beta}^{(v)}(t^2, x) - x^2 \right\|_\varepsilon = 0. \]

Thus the proof is completed.

REFERENCES

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