On Blowup Phenomenon of Solutions to the Euler Equations for Generalized Chaplygin Gas

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Abstract
We obtain some blowup results of the Euler equations for Generalized Chaplygin Gas (GCG). In particular, we show that the solutions with velocity of the form

\[ u(t, x) = \frac{\dot{a}(t)}{a(t)} x \]

blow up on finite time if the parameter of the ordinary differential equation related to \( a(t) \) is negative. Moreover, by the substitution and perturbation methods, we construct a family of non-spherical symmetric blowup solutions for the one-dimensional GCG system.

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1. Introduction and Main Results

In this paper, we consider the \( N \)-dimensional Euler equations for Generalized Chaplygin Gas:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho [u_t + (u \cdot \nabla)u] + \nabla p &= 0, \\
p &= -A\rho^{-\gamma}, \quad A > 0, \quad 0 < \gamma < 1,
\end{align*}
\]

(1.1)
where $\rho(t, x) : (\mathbb{R}_{\geq 0}, \mathbb{R}^N) \to \mathbb{R}_{\geq 0}$, $u(t, x) : (\mathbb{R}_{\geq 0}, \mathbb{R}^N) \to \mathbb{R}^N$ represent the density and velocity of the substance considered respectively. $p$ is the pressure function which is governed by the state equation (1.1)_3. (1.1)_1 is derived from the mass conservation law while (1.1)_2 is a result of the momentum conservation law.

Most real-world physical systems are modeled by non-linear partial differential equations. Examples include gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics and so on. However, in some cases, the equations reach a point at which they fail to continue and describe the real world. This phenomenon is called “blowup”.

The Euler equations are one of the most fundamental equations in fluid dynamics. Many interesting fluid dynamic phenomena can be described by the Euler equations [3, 4]. Due to its physical significance and mathematical challenge, the singularity formation in fluid mechanics has been attracting the attention of a number of researchers [9, 5, 6, 7, 2, 8, 1].

In particular, in [1], the authors considered the gas expansion problem for system (1.1) with $N = 2$. More precisely, the authors proved the global existence of solution to the expansion problem of a wedge of gas into vacuum with the half angel $\theta \in (0, \pi/2)$ for the generalized Chaplygin gas after obtaining some priori estimates. On the other hand, the authors in [2] investigated the corresponding system for compressible fluid in the spherical symmetry case and obtained some blowup results. To be specific, they considered solutions with velocity of the form

$$u(t, x) = c(t)x$$

and show that, by using the argument of phase diagram, the solutions will be expanding if the initial data $c(0)$ and $\dot{c}(0)$ satisfy some inequalities. It is natural to further consider the finite time singularities problem for system (1.1) by the approach in [2]. We start with a more general form:

$$u(t, x) = c(t)|x|^{\alpha-1}x.$$  \hspace{1cm} (1.3)

It turns out that we show in Theorem 3.1 that there are only trivial solutions if $\alpha \neq 1$. The mathematics developed in Theorem 3.1 are then used to prove that the solutions will be expanding and blowing up when the initial data $c(0)$ and $\dot{c}(0)$ satisfy some simple inequalities in the case of $\alpha = 1$. By analyzing ordinary differential equations with different natures and properties when compared with those in [2], we show in Corollary 3.3 that the solutions with velocity of the form

$$u(t, x) = \frac{\dot{a}(t)}{a(t)}x$$

(1.4)

blow up in finite time if the parameter of the ordinary differential equation that $a(t)$ satisfies is negative. Moreover, by the substitution and perturbation methods as in [10], we construct a family of non-spherical symmetric blowup solutions for the system (1.1) with $N = 1$. 
2. Lemmas

For solutions in spherical symmetry, namely, \( \rho(t, x) = \rho(t, r) \) and \( u(t, x) = u(t, r) \frac{x}{r} \), system (1.1) is transformed to

\[
\begin{cases}
\rho_t + u \rho_r + \rho u_r + \frac{N-1}{r} \rho u = 0, \\
\rho(u_t + uu_r) + p_r = 0, \\
p = -A \rho^{-\gamma}, \ 0 < \gamma < 1,
\end{cases}
\]

(2.1)

where \( r = |x| \) is the length of the spatial variable \( x \).

First, we give a relation that holds for (2.1) in a general setting.

**Lemma 2.1.** For system (2.1), we have the following relation.

\[
\frac{d}{dr} \left( \frac{1}{\rho^{\gamma+1}(t, 0)} \right) + \frac{\gamma+1}{A \rho} \int_0^r [u_t + uu_r]_r ds + \frac{\gamma+1}{A \rho} u[u_t + uu_r] = 0. 
\]

(2.2)

\[
-(\gamma+1) \left[ u_r + \frac{N-1}{r} u \right] \left[ \frac{1}{\rho^{\gamma+1}(t, 0)} \right] + \frac{\gamma+1}{A \rho} \int_0^r [u_t + uu_r] ds = 0.
\]

(2.3)

**Proof.** From (2.1), one has

\[
\rho_t + u \rho_r + \rho \left[ u_r + \frac{N-1}{r} u \right] = 0
\]

(2.4)

\[
\rho^{-\gamma-2} \left[ \rho_t + u \rho_r + \rho \left[ u_r + \frac{N-1}{r} u \right] \right] = 0
\]

(2.5)

\[
\frac{1}{-\gamma-1} \left[ \frac{1}{\rho^{\gamma+1}} \right]_r + \frac{1}{-\gamma-1} u \left[ \frac{1}{\rho^{\gamma+1}} \right]_r + \frac{1}{\rho^{\gamma+1}} \left[ u_r + \frac{N-1}{r} u \right] = 0
\]

(2.6)

\[
\left[ \frac{1}{\rho^{\gamma+1}} \right]_r + u \left[ \frac{1}{\rho^{\gamma+1}} \right]_r - \frac{\gamma+1}{\rho^{\gamma+1}} \left[ u_r + \frac{N-1}{r} u \right] = 0.
\]

(2.7)

On the other hand, from (2.1)\(_2\) and (2.1)\(_3\), one has

\[
(u_t + uu_r) + \frac{d}{\rho} \frac{d}{dr} \left[ -A \rho^{-\gamma} \right] = 0
\]

(2.8)

\[
(u_t + uu_r) + A \gamma \rho^{-\gamma-2} \rho_r = 0
\]

(2.9)

\[
(u_t + uu_r) - \frac{A \gamma}{\gamma+1} \left[ \frac{1}{\rho^{\gamma+1}} \right]_r = 0
\]

(2.10)

\[
\left[ \frac{1}{\rho^{\gamma+1}} \right]_r = \frac{\gamma+1}{A \gamma} [u_t + uu_r].
\]

(2.11)

It follows from (2.11) that one has the following two equalities.

\[
\frac{1}{\rho^{\gamma+1}} = \frac{1}{\rho^{\gamma+1}(t, 0)} + \frac{\gamma+1}{A \gamma} \int_0^r [u_t + uu_r] ds
\]

(2.12)
and
\[
\left[ \frac{1}{\rho^{r+1}} \right]_t = \frac{d}{dt} \left( \frac{1}{\rho^{r+1}(t, 0)} \right) + \frac{\gamma + 1}{A^\gamma} \int_0^x [u_t + uu_x]_y dy. \tag{2.13}
\]
Substituting (2.13), (2.11) and (2.12) into (2.7), the result follows. ■

With similar techniques, one can obtain a corresponding relation for system (1.1) in the non-spherical symmetric case.

**Lemma 2.2.** For the one-dimensional system (1.1), we have the following relation.

\[
\frac{d}{dt} \left( \frac{1}{\rho^{r+1}(t, 0)} \right) + \frac{\gamma + 1}{A^\gamma} \int_0^x [u_t + uu_x]_y dy + \frac{\gamma + 1}{A^\gamma} u[u_t + uu_x] = 0. \tag{2.14}
\]

\[
-(\gamma + 1)u_x \left[ \frac{1}{\rho^{r+1}(t, 0)} \right] + \frac{\gamma + 1}{A^\gamma} \int_0^x [u_t + uu_x]_y dy = 0. \tag{2.15}
\]

**Proof.** The one-dimensional system (1.1) is expressed as

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
\rho[u_t + uu_x] + p_x &= 0, \\
p &= -A\rho^{-\gamma}.
\end{align*}
\tag{2.16}
\]

From (2.16)\textsubscript{2} and (2.16)\textsubscript{3}, one has

\[
u_t + uu_x + \gamma A\rho^{-\gamma-2}\rho_x = 0 \tag{2.17}
\]

\[
A\gamma\rho^{-\gamma-2}\rho_x = -[u_t + uu_x] \tag{2.18}
\]

\[
\frac{A\gamma}{\gamma + 1} \left[ \frac{1}{\rho^{r+1}} \right]_x = -[u_t + uu_x] \tag{2.19}
\]

\[
\frac{1}{\rho^{r+1}} = \frac{1}{\rho^{r+1}(t, 0)} + \int_0^x [u_t + uu_x]_y dy. \tag{2.20}
\]

It follows that one has following identities

\[
\left[ \frac{1}{\rho^{r+1}} \right]_t = \frac{d}{dt} \left( \frac{1}{\rho^{r+1}(t, 0)} \right) + \frac{\gamma + 1}{A^\gamma} \int_0^x [u_t + uu_x]_y dy. \tag{2.21}
\]

On the other hand, from (2.16)\textsubscript{1}, one has

\[
(-\gamma - 1)\rho^{-\gamma-2}\rho_t + u \left[ (-\gamma - 1)\rho^{-\gamma-2}\rho_x \right] + (-\gamma - 1)\rho^{-\gamma-2}\rho u_x = 0 \tag{2.22}
\]

\[
\left[ \frac{1}{\rho^{r+1}} \right] + u \left[ \frac{1}{\rho^{r+1}} \right]_x - (\gamma + 1)u_x \frac{1}{\rho^{r+1}} = 0. \tag{2.23}
\]
Substituting (2.22), (2.20) and (2.21) into (2.25), the results follows.

Third, the analysis of the following ordinary differential equation is crucial to prove the blowup properties of system (2.1).

**Lemma 2.3.** For the differential equation

\[
\ddot{a}(t) = \xi a^\beta(t), \quad a(t) \geq 0
\]  

with \(\beta > 1\) and initial data

\[
\begin{align*}
    a(0) &= a_0 > 0, \\
    \dot{a}(0) &= a_1 \in \mathbb{R},
\end{align*}
\]  

one has

(i) If \(\xi > 0\) and \(|a_1| < \sqrt{\frac{2\xi}{\beta + 1} a_0^{\frac{\beta + 1}{2}}}\), then

\[
\lim_{t \to +\infty} a(t) = \lim_{t \to +\infty} \dot{a}(t) = +\infty.
\]

(ii) If \(\xi > 0\) and \(|a_1| \geq \sqrt{\frac{2\xi}{\beta + 1} a_0^{\frac{\beta + 1}{2}}}\), then

\[
\lim_{t \to +\infty} a(t) = \lim_{t \to +\infty} \dot{a}(t) = 0.
\]

(iii) If \(\xi < 0\), then there exists a finite \(T_1 > 0\) such that \(a(T_1) = 0\) and \(\dot{a}(T_1) < 0\).

**Proof.** From (2.26), one always has

\[
\frac{1}{2} \dot{a}^2 - \frac{\xi}{\beta + 1} a^{\beta + 1} = h
\]  

and

\[
\dot{a} = a_1 + \xi \int_0^t a^\beta(s)ds,
\]  

where

\[
h := \frac{1}{2} a_1^2 - \frac{\xi}{\beta + 1} a_0^{\beta + 1}
\]  

is a constant. Moreover, from (2.28), one obtains that

\[
\xi a^{\beta + 1} \geq -h(\beta + 1).
\]
Suppose $\xi > 0$, then $a$ is strictly convex. If the conditions in (i) hold, then $h < 0$ and from (2.31),

$$a \geq \left( \frac{-h(\beta + 1)}{\xi} \right)^{\frac{1}{\beta + 1}}. \quad (2.32)$$

It follows from (2.29) that

$$\dot{a}(t) \geq a_1 + \xi \left[ \frac{-h(\beta + 1)}{\xi} \right]^\beta t \quad (2.33)$$

for all $t$. Thus, $\dot{a}(t) > 0$ for all sufficiently large $t$. As $a$ is strictly convex, one has reached the conclusion in part (i).

If the conditions in (ii) hold, then $h \geq 0$. We will show that $\dot{a}(t)$ is always negative. For, suppose on the contrary, $\dot{a}(T) \geq 0$ for some $T$. Then, for all $t > T$, $\dot{a}(t) > 0$. From (2.28) and the fact that $h \geq 0$,

$$\dot{a} \geq \sqrt{\frac{2\xi}{\beta + 1}} a^{\frac{\beta + 1}{2}} \quad (2.34)$$

for all $t > T$. It follows that

$$\frac{1}{a^{\frac{\beta + 1}{2}}} \leq \frac{1}{a_0^{\frac{\beta + 1}{2}}} - \frac{\beta - 1}{2} \sqrt{\frac{2\xi}{\beta + 1}} t. \quad (2.35)$$

Thus, $\lim_{t \to +\infty} a(t) = 0$, which is impossible as $a$ is strictly convex and is strictly increasing after time $T$. Thus, $\dot{a}(t)$ is always negative and (2.34) becomes

$$\dot{a} \leq -\sqrt{\frac{2\xi}{\beta + 1}} a^{\frac{\beta + 1}{2}}. \quad (2.36)$$

It follows that

$$a^{\frac{\beta - 1}{2}} \leq \frac{1}{\frac{\beta - 1}{2} \sqrt{\frac{2\xi}{\beta + 1}}} t + \frac{1}{a_0^{\frac{\beta - 1}{2}}}. \quad (2.37)$$

Thus, $\lim_{t \to +\infty} a(t) = \lim_{t \to +\infty} \dot{a}(t) = 0$ and part (ii) is proved.

Third, suppose $\xi < 0$, then $a$ is strictly concave. Note that $h > 0$. From (2.31),

$$a \leq \left[ \frac{h(\beta + 1)}{-\xi} \right]^\frac{1}{\beta + 1}. \quad (2.38)$$
Thus, \( a \) is bounded above. We will show that \( \dot{a}(T) \leq 0 \) for some \( T \). For, suppose on the contrary, \( \dot{a}(t) > 0 \) for all \( t \). Then, \( \lim_{t \to +\infty} \dot{a}(t) = 0 \) and \( a(t) \) tends to some positive constant as \( t \) tends to \( +\infty \). Thus,

\[
\int_0^{+\infty} a^\beta(s)ds = +\infty.
\]

(2.39)

However, from (2.29) we know that

\[
\int_0^{+\infty} a^\beta(s) = -\frac{a_1}{\xi} < +\infty,
\]

(2.40)

which is a contradiction. Thus, \( \dot{a}(T) \leq 0 \) for some \( T \). It follows that \( \dot{a}(t) < 0 \) for all \( t > T \) and there is a finite \( T_1 \) such that \( a(T_1) = 0 \). The proof is complete. \( \blacksquare \)

**Remark 2.4.** If \( \xi = 0 \), then \( a(t) \) is linear and we have

(iii) If \( a_1 > 0 \), then \( \lim_{t \to +\infty} a(t) = +\infty \) and \( \dot{a}(t) \equiv a_1 \).

(iv) If \( a_1 = 0 \), then \( a(t) \equiv a_0 \).

(vi) If \( a_1 < 0 \), then \( a(t) = 0 \) for \( t = -a_0/a_1 \).

Last, the following lemma will be used to prove the family of one-dimensional solutions that we construct in Theorem 3.4 contains blowup cases.

**Lemma 2.5.** For the differential equation

\[
\ddot{a}(t) = \frac{\xi}{a^\gamma(t)}
\]

(2.41)

with \( 0 < \gamma < 1 \) and initial data

\[
\begin{cases}
a(0) =: a_0 > 0 \\
\dot{a}(0) =: a_1 \in \mathbb{R},
\end{cases}
\]

(2.42)

we have

(i) If \( \xi < 0 \), then there exists a finite time \( T \) such that

\[
\lim_{t \to T^-} a(t) = 0.
\]

(2.43)

(ii) If \( \xi > 0 \), then the solution \( a(t) \) exists globally.

(iii) If \( \xi = 0 \) and \( a_1 < 0 \), then the solution \( a(t) \) blows up in the finite time \( T = -a_0/a_1 \).

(iv) If \( \xi = 0 \) and \( a_1 \geq 0 \), then the solution \( a(t) \) exists globally.

**Remark 2.6.** For a proof of Lemma 2.5, reader may refer to Lemma 3 of [9].
3. Proofs of Main Results

We start with a general case and it turns out that only the case \( u(t, x) = c(t)x \) is meaningful.

**Theorem 3.1.** There are only trivial solutions to (2.1) of the form \( u(t, r) = c(t)r^\alpha \) with \( \alpha \neq 1 \).

**Proof.** Set

\[
 u = cr^\alpha. \tag{3.1}
\]

First, note that if \( \alpha < 0 \), then \( u \) is undefined at \( r = 0 \) unless \( c = 0 \). It follows that one must have \( u = 0 \). Then, from (2.1), \( \rho \) is independent of \( t \). From (2.11), \( \rho \) is independent of \( r \). Thus, one has only trivial solutions if \( \alpha < 0 \).

For \( \alpha \geq 0 \), one has

\[
 u_r + uu_r = \dot{c}r^\alpha + \alpha c^2 r^{2\alpha - 1} \tag{3.2}
\]

\[
 [u_t + uu_r]_r = \ddot{c}r^\alpha + 2\alpha c\dot{c}r^{2\alpha - 1} \tag{3.3}
\]

\[
 u_r + \frac{N-1}{r}u = c(\alpha + N - 1)r^{\alpha - 1}. \tag{3.4}
\]

Substituting (3.2), (3.3) and (3.4) into (2.2) and (2.3), one has, for \( \alpha \neq -1 \) and 0,

\[
 F_1r^{3\alpha - 1} + F_2r^{2\alpha} + F_3r^{\alpha + 1} + F_4r^{\alpha - 1} + F_5r^0 = 0, \tag{3.5}
\]

where

\[
 F_1 = \frac{(\gamma + 1)[2\alpha - (\gamma + 1)(\alpha + N - 1)]c^3}{2A\gamma}, \tag{3.6}
\]

\[
 F_2 = \left[ 2 - \frac{(\gamma + 1)(\alpha + N - 1)}{\alpha + 1} \right] \frac{\gamma + 1}{A\gamma} c\dot{c}, \tag{3.7}
\]

\[
 F_3 = \frac{\gamma + 1}{A\gamma(\alpha + 1)}, \tag{3.8}
\]

\[
 F_4 = -\frac{(\gamma + 1)(\alpha + N - 1)}{\rho^{\gamma+1}(t, 0)} c, \tag{3.9}
\]

\[
 F_5 = \frac{1}{\rho^{\gamma+1}(t, 0)} \frac{d}{dr} \left( \frac{c}{\rho^{\gamma+1}(t, 0)} \right). \tag{3.10}
\]

For \( \alpha \neq 1, 0 \) and \( 1/3, 3\alpha - 1 \) and \( \alpha - 1 \) are different and unique among the powers of \( r \) in (3.5). Thus, one has

\[
 \begin{cases}
 F_1 = 0 \\
 F_4 = 0.
 \end{cases} \tag{3.11}
\]
If $\alpha + N - 1 \neq 0$, then $c = 0$ from $F_4 = 0$. If $\alpha + N - 1 = 0$, then $c = 0$ from $F_1 = 0$. Hence, one has only trivial solutions for $\alpha \neq 1, 0$ and $1/3$.

For $\alpha = 1/3$, $\alpha - 1$ is unique among the powers of $r$ in (3.5). Thus,

$$F_4 = 0.$$  \hfill (3.12)

As $\alpha + N - 1 \neq 0$ for $\alpha = 1/3$, one has $c = 0$ and the solutions are trivial.

For $\alpha = 0$, the corresponding equation of (3.5) is

$$G_1 r^1 + G_2 r^0 + G_3 r^{-1} = 0,$$  \hfill (3.13)

where

$$G_1 = \frac{\gamma + 1}{A\gamma} \dot{c},$$  \hfill (3.14)

$$G_2 = \frac{\gamma + 1}{A\gamma} \left[ 1 - (\gamma + 1)(N - 1) \right] \dot{c} + \frac{d}{dt} \left( \frac{1}{\rho^{\gamma+1}(t,0)} \right),$$  \hfill (3.15)

$$G_3 = -\frac{(\gamma + 1)(N - 1)}{\rho^{\gamma+1}(t,0)} c.$$  \hfill (3.16)

From $G_3 = 0$, one has $c = 0$ if $N > 1$. Thus, one has only trivial solutions if $\alpha = 0$ and $N > 1$.

For $\alpha = 0$ and $N = 1$, one has, from $G_1 = 0$ that

$$\ddot{c} = 0$$

$$c = c_1 t + c_0,$$  \hfill (3.17)

where $c_1 := \dot{c}(0)$ and $c_0 := c(0)$.

On the other hand, from $G_2 = 0$,

$$\frac{\gamma + 1}{A\gamma} \ddot{c} + \frac{d}{dt} \left( \frac{1}{\rho^{\gamma+1}(t,0)} \right) = 0$$

$$\frac{1}{\rho^{\gamma+1}(t,0)} = \frac{1}{\rho^{\gamma+1}(0,0)} - \frac{\gamma + 1}{A\gamma} \int_0^t c \dot{c} ds$$  \hfill (3.20)

$$\frac{1}{\rho^{\gamma+1}(t,0)} = \frac{1}{\rho^{\gamma+1}(0,0)} - \frac{\gamma + 1}{A\gamma} \left( \frac{c_1^2}{2} t^2 + c_1 c_0 t \right).$$  \hfill (3.21)

As it is impossible that the coefficient of $t^2$ is negative while $\rho$ remains positive for all $t$, one concludes that $c_1 = 0$. It follows that $\rho(t,0)$ is a constant and $u = c = c_0$. Moreover, we see that $\rho$ is a constant from (2.12) and $u = 0$ from (2.1)_1. The proof is complete.

With the steps developed in the previous theorem, one obtains the following result.

**Theorem 3.2.** There exists a family of solutions $(\rho, u)(t, r)$ with $u(t, r) = c(t)r$ for (2.1) and $N > 1$ satisfying the following properties.

(i) If \( \dot{c}(0) + c^2(0) < 0 \), then \( \dot{c}(t) + c^2(t) < 0 \) and \( (\rho, u)(t, r) \to (\infty, -\infty) \) as \( t \to \infty \), for all \( r \geq 0 \).

(ii) If \( \dot{c}(0) + c^2(0) > 0 \), \( c(0) < 0 \) and \( \dot{c}(0) + \frac{2 - (\gamma + 1)N}{2} c^2(0) < 0 \), then \( \dot{c}(t) + c^2(t) > 0 \), \( c(t) < 0 \), \( \dot{c}(t) + \frac{2 - (\gamma + 1)N}{2} c^2(t) < 0 \) and \( (\rho, u)(t, r) \to (\infty, -\infty) \) as \( t \to \infty \), for all \( r \geq 0 \).

(iii) If \( \dot{c}(0) + \frac{2 - (\gamma + 1)N}{2} c^2(0) > 0 \), then \( \dot{c}(t) + \frac{2 - (\gamma + 1)N}{2} c^2(t) > 0 \) and \( (\rho, u)(t, r) \to (0, \infty) \) as \( t \to \infty \), for all \( r \geq 0 \).

(iv) If \( \dot{c}(0) + c^2(0) > 0 \), \( c(0) > 0 \) and \( \dot{c}(0) + \frac{2 - (\gamma + 1)N}{2} c^2(0) < 0 \), then \( \dot{c}(t) + c^2(t) > 0 \), \( c(t) > 0 \), \( \dot{c}(t) + \frac{2 - (\gamma + 1)N}{2} c^2(t) < 0 \) and \( (\rho, u)(t, r) \to (0, \infty) \) as \( t \to \infty \), for all \( r \geq 0 \).

**Proof.** For \( \alpha = 1 \), (3.5) becomes

\[
H_1 r^2 + H_2 r^0 = 0,
\]

where

\[
H_1 = \frac{\gamma + 1}{2A\gamma} \left[ [2 - (\gamma + 1)N] c^3 + [4 - (\gamma + 1)N] c\dot{c} + \ddot{c} \right],
\]

\[
H_2 = -\frac{(\gamma + 1)N}{\rho^{\gamma+1}(t, 0)} c + \frac{d}{dt} \left( \frac{1}{\rho^{\gamma+1}(t, 0)} \right).
\]

From \( H_2 = 0 \), one has, by solving the first order O.D.E. (3.24),

\[
\frac{1}{\rho^{\gamma+1}(t, 0)} = \frac{1}{\rho^{\gamma+1}(0, 0)} e^{(\gamma+1)N \int_0^t c(s) ds}.
\]

It follows from (2.12) that

\[
\frac{1}{\rho^{\gamma+1}} = \frac{1}{\rho^{\gamma+1}(t, 0)} + \frac{\gamma + 1}{A\gamma} (\dot{c} + c^2) \frac{r^2}{2}
\]

\[
\rho^{\gamma+1} = \frac{1}{\rho^{\gamma+1}(0, 0)} e^{(\gamma+1)N \int_0^t c(s) ds} + \frac{\gamma + 1}{2A\gamma} (\dot{c} + c^2) r^2.
\]

On the other hand, from \( H_1 = 0 \), one has

\[
[2 - (\gamma + 1)N] c^3 + [4 - (\gamma + 1)N] c\dot{c} + \ddot{c} = 0.
\]
Note that $c$ is a solution of (3.28) if $\dot{c} + c^2 = 0$ as (3.28) is equivalent to

$$\frac{d}{dt}(\dot{c} + c^2) + [2 - (\gamma + 1)N](\dot{c} + c^2) c = 0. \quad (3.29)$$

Similarly, one can easily obtain that $c$ is a solution of (3.28) if

$$\dot{c} + \frac{2 - (\gamma + 1)N}{2}c^2 = 0. \quad (3.30)$$

Consider the dynamical system obtained from (3.28):

$$\begin{cases}
\dot{V} = \left[ (\gamma + 1)N - 4 \right] V + [(\gamma + 1)N - 2]c^2, \\
\dot{c} = V.
\end{cases} \quad (3.31)$$

The direction fields and phase portrait for $V$ and $c$ are shown in Fig. 1, where $N$ and $\gamma$ are set to be 3 and $2/3$ respectively.

It can be seen that there are four invariant regions separated by the curves $V + c^2 = 0$ and $V + \frac{2 - (\gamma + 1)N}{2}c^2 = 0$. Moreover, it can be observed that

a) Solutions curves originating from regions I and II will approach to $(-\infty, -\infty)$.

b) Solutions curves originating from regions III and IV will approach to $(+\infty, +\infty)$.

However, we will show that

a’) Solution curves originating from regions I and II will approach to the curve $V + c^2 = 0$ asymptotically.

b’) Solution curves originating from regions III and IV will approach to the curve $V + \frac{2 - (\gamma + 1)N}{2}c^2 = 0$ asymptotically.

Combining with (3.27), the proof of Theorem 3.2 will be completed.

First, consider

$$W_1 := (V + c^2)^2. \quad (3.32)$$

By (3.31),

$$\frac{dW_1}{dt} = 2(V + c^2)(\dot{V} + 2c\dot{c}) = 2\left[ (\gamma + 1)N - 2 \right] (V + c^2)c. \quad (3.33)$$

Note that $c(t) < 0$ in region II and $c(t) < 0$ as $t \to \infty$ in region I. Thus, $\frac{dW_1}{dt} < 0$ and hence

$$V + c^2 \to 0 \text{ as } t \to \infty \quad (3.35)$$
Figure 1: The direction fields and phase portrait for $V$ and $c$.

in regions I and II.

Second, consider

$$W_2 := \left[ V + \frac{2 - (\gamma + 1)N}{2} c^2 \right]^2. \quad (3.36)$$

By (3.31),

$$\frac{dW_2}{dt} = 2 \left[ V + \frac{2 - (\gamma + 1)N}{2} c^2 \right] \left[ \dot{V} + [2 - (\gamma + 1)N] \dot{c} \dot{c} \right]$$

$$= -4 \left[ V + \frac{2 - (\gamma + 1)N}{2} c^2 \right]^2 c. \quad (3.37)$$

Note that $c(t) > 0$ in region IV and $c(t) > 0$ as $t \to \infty$ in region III. Thus, $\frac{dW_2}{dt} < 0$
and hence
\[ V + \frac{2 - (\gamma + 1)N}{2} c^2 \to 0 \text{ as } t \to \infty \] (3.39)
in regions III and IV. The proof is complete.  

Next, the finite time blowup result is given by the following corollary.

**Corollary 3.3.** Let \((\rho, u)\) be a solution for (2.1) with \(N > 1\), \(u(t, r) = \frac{\dot{a}(t)}{a(t)} r\) and \(a(t) > 0\). Then \(a(t)\) satisfies (2.26) with \(\beta = N(\gamma + 1) - 1 > 1\). Moreover,

(i) If \(\xi > 0\) and \(|a_1| < \sqrt{\frac{2\xi}{\beta + 1} a_0^{\frac{\beta+1}{2}}}\), then \((\rho, u)(t, r) \to (0, 0)\) as \(t \to +\infty\), for all \(r \geq 0\).

(ii) If \(\xi > 0\) and \(|a_1| \geq \sqrt{\frac{2\xi}{\beta + 1} a_0^{\frac{\beta+1}{2}}}\), then \((\rho, u)(t, r) \to (+\infty, 0)\) as \(t \to +\infty\), for all \(r \geq 0\).

(iii) If \(\xi < 0\), then the solution blows up on finite time.

**Proof.** Putting \(c(t) = \frac{\dot{a}(t)}{a(t)}\) in (3.29), one has
\[
\frac{\ddot{a}}{a} + \left[1 - (\gamma + 1)N\right] \frac{\dot{a} \ddot{a}}{a^2} = 0 \tag{3.40}
\]
\[
\frac{d}{dt} \left(a^{1-(\gamma+1)N} \dot{a}\right) = 0 \tag{3.41}
\]
\[
\dot{a} = \xi a^{N(\gamma+1)-1} \tag{3.42}
\]
for some constant \(\xi\). Thus, \(a(t)\) satisfies (2.26) with \(\beta = N(\gamma + 1) - 1 > 1\).

If the conditions of (i) hold, then from part (i) of Lemma 2.3 and (3.27), \(\rho \to 0\) as \(t \to +\infty\). Moreover,
\[
\lim_{t \to +\infty} \frac{\dot{a}(t)}{a(t)} = \lim_{t \to +\infty} \frac{\ddot{a}(t)}{\dot{a}(t)} = \lim_{t \to +\infty} \frac{\xi a^\beta(t)}{\ddot{a}(t)} = \lim_{t \to +\infty} \frac{\xi \beta a^{\beta-1} \dot{a}(t)}{\ddot{a}(t)} = \lim_{t \to +\infty} \beta \frac{\dot{a}(t)}{a(t)}. \tag{3.43}
\]
As \(\beta > 1\), one concludes that \(\lim_{t \to +\infty} \frac{\dot{a}(t)}{a(t)} = 0\).

The result of part (ii) can be shown in a similar way.

Lastly, part (iii) is followed from part (iii) of Lemma 2.3. The proof is complete.  

Last but not the least, we construct a family of blowup solutions for system (1.1) with \(N = 1\).
Theorem 3.4. The one-dimensional Euler equations for generalized Chaplygin gas, that is, system \((1.1)\) with \(N = 1\), have a family of solutions \((\rho, u)\) in the following form.

\[
\begin{align*}
\rho^{-\gamma - 1}(t, x) &= \rho^{-\gamma - 1}(t, 0) + \frac{\gamma + 1}{\rho^{\gamma + 1}(t, 0)} \left[ \frac{\xi}{2\alpha^{\gamma + 1}} x^2 + \left( \dot{b} + \frac{\dot{\alpha}}{\alpha} \right) x \right], \\
u(t, x) &= \frac{\dot{\alpha}}{\alpha(t)} x + b(t),
\end{align*}
\]

(3.44)

where \(\rho(t, 0)\) satisfies

\[
\frac{d}{dt} \left( \frac{1}{\rho^{\gamma + 1}(t, 0)} \right) - \left[ (\gamma + 1) \frac{\dot{\alpha}}{\alpha} \right] \frac{1}{\rho^{\gamma + 1}(t, 0)} + \left[ \frac{\gamma + 1}{\alpha} \left( \dot{b} + \frac{\dot{\alpha}}{\alpha} \right) \right] b = 0.
\]

(3.45)

\(a(t)\) and \(b(t)\) are governed by

\[
\ddot{a} = \frac{\xi}{a^{\gamma}}
\]

(3.46)

and

\[
\ddot{b} + \left[ (1 - \gamma) \frac{\dot{a}}{a} \right] \dot{b} + \left[ \frac{2\xi}{a^{\gamma + 1}} - (\gamma + 1) \frac{\dot{a}^2}{a^2} \right] b = 0.
\]

(3.47)

respectively.

Proof. Set

\[
u(t, x) = c(t)x + b(t).
\]

(3.48)

Then,

\[
\begin{align*}
u_t + uu_x &= (\dot{c} + c^2)x + \dot{b} + bc, \\
[u_t + uu_x]_t &= \frac{d}{dt} \left( \frac{1}{\rho^{\gamma + 1}(t, 0)} \right) - (\gamma + 1) \frac{1}{\rho^{\gamma + 1}(t, 0)} + \frac{\gamma + 1}{\alpha} \left( \dot{b} + \frac{\dot{\alpha}}{\alpha} \right) b.
\end{align*}
\]

(3.50)

Substituting (3.49) and (3.50) into (2.14) and (2.15), one has

\[
J_1 x^2 + J_2 x^1 + J_3 x^0 = 0,
\]

(3.51)

where

\[
\begin{align*}
J_1 &= \frac{\gamma + 1}{2A^{\gamma}} \left[ \frac{d}{dt} \left( \frac{1}{\rho^{\gamma + 1}(t, 0)} \right) \right] \left( \dot{c} + c^2 \right) + (1 - \gamma)(\dot{c} + c^2)c, \\
J_2 &= \frac{\gamma + 1}{A^{\gamma}} \left[ \frac{d}{dt} \left( \dot{b} + bc \right) \right] - \gamma \left( \dot{b} + bc \right)c + (\dot{c} + c^2)b, \\
J_3 &= \frac{d}{dt} \left( \frac{1}{\rho^{\gamma + 1}(t, 0)} \right) - (\gamma + 1)c \frac{1}{\rho^{\gamma + 1}(t, 0)} + \frac{\gamma + 1}{A^{\gamma}} \left( \dot{b} + bc \right)b.
\end{align*}
\]

(3.52) (3.53) (3.54)
With
\[
c(t) = \frac{\dot{a}(t)}{a(t)}, \quad (3.55)
\]
one has, from \(J_1 = 0\),
\[
\ddot{a} = \frac{\xi}{a^\gamma}, \quad (3.56)
\]
for some constant \(\xi\). From \(J_2 = 0\), one sees that \(b(t)\) satisfies (3.47); from \(J_3 = 0\), one
has that \(\rho(t, 0)\) satisfies (3.45). Lastly, from (2.21), one has the form (3.44) of \(\rho\). The
proof is completed.

**Corollary 3.5.** For the family of solutions (3.44),

(i) If \(\xi < 0\), then the solutions blow up on finite time.

(ii) If \(\xi > 0\), then the solutions exist globally in time.

(iii) If \(\xi = 0\) and \(a_1 < 0\), then the solutions blow up on finite time \(T = -a_0/a_1\).

(iv) If \(\xi = 0\) and \(a_1 \geq 0\), then the solutions exist globally in time.

**Proof.** If \(\xi < 0\), then by lemma 2.5, there is a finite \(T\) such that
\[
\lim_{t \to T^-} a(t) = 0 \quad (3.57)
\]
and
\[
\lim_{t \to T^-} \dot{a}(t) = 0, \quad (3.58)
\]
then
\[
\lim_{t \to T^-} \frac{\dot{a}(t)}{a(t)} = \lim_{t \to T^-} \frac{\ddot{a}(t)}{\dot{a}(t)} = \lim_{t \to T^-} \frac{\ddot{a}(t)}{a'(t)\dot{a}(t)} = \infty. \quad (3.59)
\]
Thus, (i) is proved. (ii), (iii) and (iv) follow from (ii), (iii) and (iv) of lemma 2.5 respectively. The proof is completed.

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