Advances in Theoretical and Applied Mathematics.
ISSN 0973-4554 Vol.1 No.1 (2006), pp. 49-62
© Research India Publications
http://www.ripublication.com/atam.htm

Some Manifold Maps

Nural Yüksel
Department of Mathematics, Erciyes University,
38039 Kayseri, Turkey
E-mail: yukseln@erciyes.edu.tr

Abstract

In this paper we will show that some properties are preserved under the Normal map, Stereographic projection, Inversion map, Conform map and Lambert projection.

Keywords: Map, manifold, projection.

1. Introduction

Let M be a topological n-manifold and S be an atlas of M. If the atlas S is differentiable of class C^k, than M is called C^k n-manifold [1]. Let M and N be C^∞ n-manifolds. If the map $f : M \longrightarrow N$ is differentiable, then the map f_* which is defined as $f_* : T_M(p) \longrightarrow T_N(f(p))$ for every $p \in M$ is called jacobian map or differentiation map of [1].

Let M be a hypersurface of E^n and $N = (a_1, a_2, \ldots, a_n)$ be a unit normal vector field of M. The transformation

$$S_p : T_M(p) \longrightarrow T_M(p)$$

$$X_p \longrightarrow S_p(X_p) = D_{X_p}N = (N[a_1], N[a_2], \ldots, N[a_n])$$

is called shape operator of M [1].

Definition 1.1: Let M be a hypersurface of E^n. Given a curve α on M. If tangent vector at every point of α, is a principal direction then curve α is called a line of curvature of M [2].

Theorem 1.2: Let M_1 and M_2 be two hypersurfaces in E^n. If the curve α of of intersection M_1 and M_2 is a line of curve on M_1 and M_2, then their angle is constant along the curve α.

If the angle of M_1 and M_2 surfaces is constant along their curve α of intersection and if the curve α is curvature line on one surface, then the curve α is a curvature line also on the other one.

Proof: Let N_1 and N_2 be normal vector fields of M_1 and M_2, respectively. If a curve α is a curvature line on M_1 and if $\dot{\alpha}$ is tangent vector field of α , then by Radrigues's formula, we have

$$S(\dot{\alpha}) = D_{\dot{\alpha}} N_1 = \frac{dN_1}{dt} = \lambda\dot{\alpha},$$

If a curve α is a curvature line on M_2 and if $\dot{\alpha}$ is tangent vector field of α, then

$$S(\dot{\alpha}) = D_{\dot{\alpha}} N_2 = \frac{dN_2}{dt} = \lambda^*\dot{\alpha} \tag{1.1}$$

It follows that

$$cos\theta =< N_1, N_2 >$$

and

$$\begin{aligned} \frac{d}{dt}(cos\theta) &= \frac{d}{dt} < N_1, N_2 > \\ &= < \frac{dN_1}{dt}, N_2 > + < N_1, \frac{dN_2}{dt} > \end{aligned}$$

where θ is the angle between M_1 and M_2 surfaces. By (2) and (1), we have

$$\frac{d}{dt}(cos\theta) = \lambda < \dot{\alpha}, N_2 > +\lambda^* < N_1, \dot{\alpha} >= 0$$

Because, the tangent $\dot{\alpha}$ of a curve α is orthogonal to normal vector fields N_1 and N_2 at every point. Hence we have

$$\frac{d}{dt}(cos\theta) = 0$$

that is

$$\theta = \text{constant}.$$

Let the curve α be curvature line of M_1. By Rodrigues's formula

$$\frac{dN_1}{dt} = \lambda\dot{\alpha}$$

Since the angle θ between M_1 and M_2 is constant, we have

$$\begin{aligned} \frac{d}{dt}(cos\theta) &= < \frac{dN_1}{dt}, N_2 > + < N_1, \frac{dN_2}{dt} > \\ 0 &= < \lambda\dot{\alpha}, N_2 > + < N_1, \frac{dN_2}{dt} > . \end{aligned}$$

Since $< \lambda\dot{\alpha}, N_2 >= 0$, $< N_1, (\frac{dN_2}{dt}) >= 0$, this means that at any point of α, $(\frac{dN_2}{dt}) \in T_{M_1}(\alpha)$. We also know $(\frac{dN_2}{dt}) \in T_{M_2}(\alpha)$. Therefore,

$$\frac{dN_2}{dt} \in T_{M_1}(\alpha) \bigcap T_{M_2}(\alpha),$$

that is $\frac{dN_2}{dt}$ is in the direction of tangent of α. The intersection of the tangent phanes $T_{M_1}(\alpha)$ and $T_{M_2}(\alpha)$is the line in the direction of the tangent $\dot{\alpha}$ of α, where $\theta \neq 0$. Thus, $\frac{dN_2}{dt} = \lambda^* \dot{\alpha}$ whenever $\theta \neq 0$. This proves that α is a curvature line on hypersurface M_2.

If $\theta = 0$, $N_1 = N_2$ along α and $\frac{dN_2}{dt} = \lambda \alpha$. ■

2. Preserved Properties under Manifold Maps

Definition 2.1: Let M and M_r be two hypersurface in E^n and let

$$f_* : \chi(M) \to \chi(M_r)$$

be an adjoint transformation of

$$f_* : \chi(M) \to \chi(M_r).$$

It is said that the curvature line α is preserved by f if $S(T) = \lambda T$ and $(S_r)(f_*(T)) = \mu(f_*(T))$ where α is the line of curvature and T is the tangent vector of α. Here S and Sr are shape operators of hypersurface M and M_r, respectively [2].

2.1. Normal Map

Definition 2.2: The hypersurface M_r, defined by,

$$M_r = (p_1 + ra_1(p)), (p_2 + ra_2(p)), \ldots, (p_n + ra_n(p))$$

is called a parallel hypersurface of M, where M is a hyper surface in E^n, and the unit normal vector field N of M is defined by

$$N = \sum a_i \frac{\partial}{\partial x_i}, a_i \in C^\infty(M, R).$$

The map

$$f : M \longrightarrow M_r$$

$$p \longrightarrow f(p) = (p_1 + ra_1(p)), (p_2 + ra_2(p)), \ldots, (p_n + ra_n(p))$$

is called a normal mapping from M into M_r [3].

Lemma 2.3: Let M be a hypersurface of E^n and $T_M(p)$ be a tangent space of M. Then for $p \epsilon M$, $X_p \in T_M(p)$

$$f_*(X_p) = \bar{X}_p|_{f(p)} + r.S\bar{X}_p|_{f(p)}.$$

Lemma 2.4: Let M_r be a parallel hypersurface of M and let $f : M \longrightarrow M_r$ be a normal map. Then

1) for every $X \in \chi(M)$

$$S_r(f_*(X)) = S\bar{X}$$

2) if $S(X) = kX$ for every $X \in \chi(M)$

$$S_r(f_*(X)) = \frac{k}{1+rk} f_*(X).$$

Proof: 1) Let us consider the values

$$\begin{aligned} N &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \\ N_r &= \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}, \, a_i(p) = c_i(f(p)) \\ X &= \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \end{aligned}$$

Then we get

$$\begin{aligned} S(x) &= D_X N \\ &= \sum_{i=1}^{n} X[a_i] \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^{n} X[c_i o f] \frac{\partial}{\partial x_i} \\ &= \sum_{i,j,k=1}^{n} b_j \frac{\partial c_i}{\partial f_k} . \frac{\partial f_k}{\partial x_j} . \frac{\partial}{\partial x_i} \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{n} (\sum_{k=1}^{n} \frac{\partial c_i}{\partial f_k} . \sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j} b_j) \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^{n} f_* X[c_i] \frac{\partial}{\partial x_i} \\ S(x) &= \sum_{i=1}^{n} f_* X[c_i] \frac{\partial}{\partial x_i} |_{f(p)} \\ \Rightarrow S(\bar{x}) &= D_{f_* X} N_r \\ S(\bar{x}) &= S_r(f_* X) \end{aligned}$$

2) Let $X \in \chi(M)$, for $S(X) = kX$. Since

$$f_*|_p(X) = \bar{X}|_{f(p)} + r.S(\bar{X})|_{f(p)}$$

by Lemma 2.1 and from the relation $S(X) = kX \Rightarrow S(\bar{X}) = k\bar{X}$, we get

$$\begin{aligned} f_*|_p(X) &= \bar{X}|_{f(p)} + rk\bar{X}|_{f(p)} \\ \bar{X}|_{f(p)} &= \frac{1}{1+rk}(f_*X). \end{aligned}$$

Since

$$S_r(f_*X) = \bar{S}X$$

by 1, we obtain

$$S_r(f_*X) = k\bar{X} = \frac{k}{1+rk}(f_*X).$$

Theorem 2.5: Let M_r be a parallel hypersurface of M and let $f : M \longrightarrow M_r$ be a normal map. Then f preserves curvature lines, umbilik points and third fundamental forms.

Proof: Assume that $\alpha : I \longrightarrow M$ is a curvature line on M. Then

$$S(T) = \lambda T$$

for each tangent vector field T. By Lemma (2.1)

$$\begin{aligned} S_r(f_*(T)) &= \frac{k}{1+rk} f_*(T), \quad \frac{k}{1+rk} = \lambda \\ S_r(f_*(T)) &= \lambda f_*(T) \end{aligned}$$

This shows that the line of curvature is preserved under the normal map.

Let p be an umbilik point of M. Then

$$S(X_p) = \lambda X_p$$

for every $X_P \in T_M(p)$ and $\lambda \in R$. Since

$$f_*(X_p) = \bar{X_p}|_{f(p)} + r.S\bar{X_p}|_{f(p)},$$

by Lemma (2.1)

$$f_*(X_p) = (1 + r\lambda)\bar{X}_p$$

So, we have, for every $f_*(X_p) \in T_{Mr}(f(p))$

$$\begin{aligned} S_r(f_*(X_p)) &= S(\bar{X_p}) = \bar{X}_p|_{f(p)} \\ &= \frac{\lambda}{1+r\lambda}|_f*(X_p) \end{aligned}$$

or

$$S_r = \frac{\lambda}{1+r\lambda} I|_{f(p)}$$

This means that $f(p)$ is an umbilic point of M_r.

Assume that III and III_r are third fundamental forms of M and M_r, respectively. Then, it is clear that, for every $X, Y \in \chi(M)$ and $p \in M$,

$$\begin{aligned} III_r(f_*(X_p), f_*(Y_p)) &= < S_r(f_*(X_p), S_r(f_*(Y_p) > \\ &= < S(\bar{X}), S(\bar{Y}) > |_{f(p)} \\ &= < S(X), S(Y) > \\ &= III(X_p, Y_p) \end{aligned}$$

3. Stereographic Projection

Definition 2.6: Given a hypersphere S_c^n and a hyperplane H_n in E^{n+1}. Let $S_c^n \bigcap H_n = 0$. A map

$$\sigma : S_c^n/b \to H_n$$

defined by $\sigma(p) = q$ for every $P \in S_c^n/b$ and $q \in H_n$ is called stereographic projection, where $b \in S_c^n$ is a symetric of $a \in E^{n+1}$ with respect to c [3].

The analytic statement of stereographic projection:

If S_c^n passes through origin and its center c is on $axis - x_{n+1}$,we have (Figure 2.1),

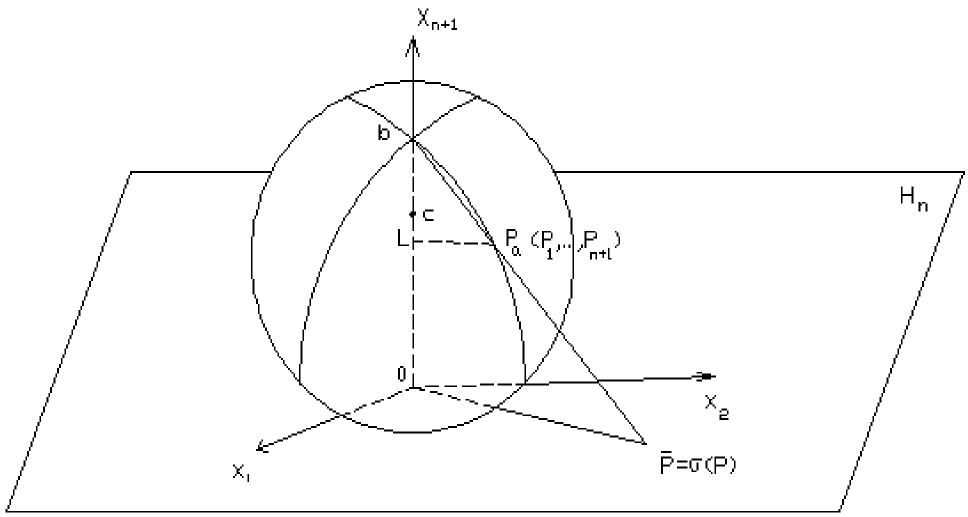
$$\sigma(P) = \frac{rP - bP_{n+1}}{r - P_{n+1}}) \tag{3.1}$$

The derivative map of σ: Consider

$$\sigma : S_c^n \to H_n$$

$$(x_1, x_2, \ldots, x_{n+1}) = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, 0)$$

$$\sigma(X) = \bar{x_i} = \frac{rx_i}{r - x_{n+1}} \tag{3.2}$$



By (4), we obtain the derivate operator σ_* as

$$\sigma_* \mid_p = \frac{x_1}{r - x_{n+1}} \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \frac{x_1}{r - x_{n+1}} \\ 0 & 1 & \ldots & 0 & 0 & \frac{x_2}{r - x_{n+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & \ldots & 0 & 1 & \frac{x_n}{r - x_{n+1}} \end{pmatrix}$$

Theorem 2.7: The stereographic projection preserves the line of curvature and umbilic point.

Proof: Let us consider stereographic projection

$$\sigma : {S_c}^n/b \longrightarrow H_n.$$

Let S_k be a shape operator for hypersphere and let S_d be a shape operator for hyperplane. We get

$$S_k(T) = \lambda T, \forall T \in \chi({S_c}^n/b)$$

when the curve $\alpha : I \longrightarrow {S_c}^n/b$ be a line of curvature on hypersphere and T is a tangent vector field of α. Since

$$S_d(\sigma_*(T)) = O.\sigma_*(T),$$

$\sigma_*(T)$, which is the image for the tangent vector fields under σ_*, is a principal direction. Hence every curve is a line of curvature on hyperplane. This completes the proof of l.

Let $p \in {S_c}^n/b$ be a umbilic point. Then

$$S_k(X_p) = \lambda X_p$$

for every $X_p \in T_{{S_c}^n/b}(p)$. On the other hand $\sigma(p)$ is an umbilic point since

$$S_d(\sigma_*(X_p)) = O.\sigma_*(X_p) \mid_f (p)$$

and

$$S_d = 0.I_n.$$

3.1. Inversion Map

Definition 2.8: M^n defined by

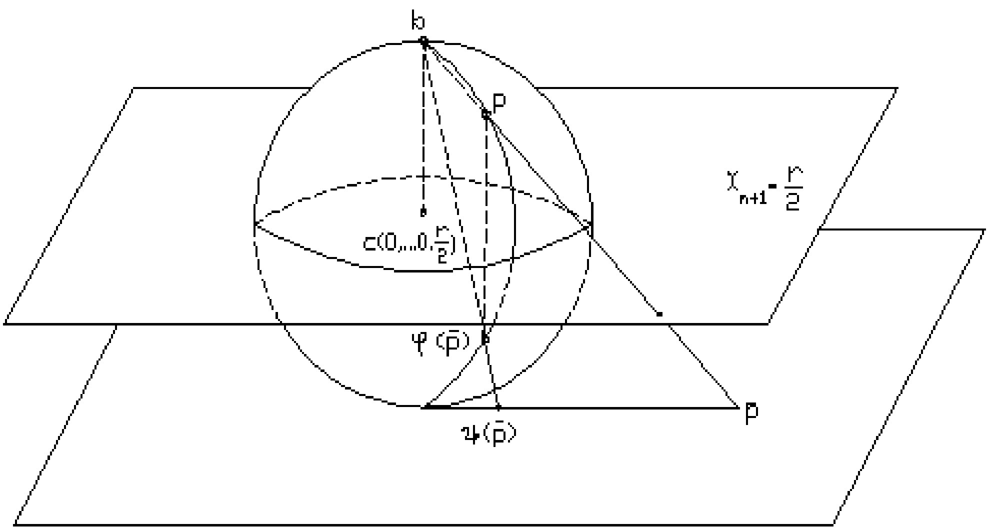
$$M^n = H_n \bigcup \{P_\infty\}$$

is called n- dimensional Mobius space under $\sigma : {S_c}^n \longrightarrow H_n$ stereographic projection, where $\sigma(p) = p_\infty$.

Definition 2.9: Let $\varphi : S^n \longrightarrow S^n$ be a symetric map with respect to hyperplane H_n, which passes from the center of $S^n$and is parallel of M^n. Then ψ by

$$\psi = \sigma_o \varphi_o \sigma^{-1} : M^n \longrightarrow M^n$$

is called inversion map (Figure 2.2).



The analytic statement of ψ : We obtain

$$\psi(p) = \frac{r^2 = p}{<= p, = p>}$$

since

$$\varphi(p_1, p_2, \ldots, p_{n+1}) = (p_1, p_2, \ldots, p_n, r - p_{n+1})$$

$$\sigma(p) = b + \frac{r}{r - p_{n+1}}(p - b) = \frac{rp - bp_{n+1}}{r - p_{n+1}}$$

$$\sigma^{-1}(=p) = \frac{b <=p, =p> + =pr^2}{r^2 + <=p, =p>}$$

where $=p \neq p, \quad =p \neq 0 \quad and \quad =p \in M^n.$

This map ψ doesn't change the points of hypersphere with center 0 and radius r.

The derivative map of a map ψ is obtained in the following way,

$$\psi_* \mid_X = \frac{r^2}{(\sum_{i=1}^n x_i{}^2)^2} \begin{pmatrix} \sum_{i=1}^n x_i{}^2 - 2x_1{}^2 & -2x_1x_2 & \ldots & -2x_1x_n \\ -2x_1x_2 & \sum_{i=1}^n x_i{}^2 - 2x_2{}^2 & \ldots & -2x_2x_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ -2x_1x_n & -2x_2x_n & \ldots & \sum_{i=1}^n x_i{}^2 - 2x_n{}^2 \end{pmatrix}$$

where

$$\psi(X) = \frac{r^2 X}{< X, X >}, X \in M^n.$$

Theorem 2.10: The inversion map preserves the line of curvatures.

Proof:

$$\psi : M^n \longrightarrow M^n$$

$$X \longrightarrow \psi(X) = \frac{r^2 X}{\|X\|^2}$$

The normal vector field $N = (a_1 \text{þ} a_2, \ldots, a_n)$ of Mobius space M^n defined by

$$M^n = H_n \bigcup \{P_\infty\}$$

is a constant vector field. Hence, for every $X \in \aleph(M^n)$,

$$\begin{aligned} S(X) &= D_X N \\ &= X[N] \\ &= 0 \\ S &= 0_n. \end{aligned}$$

This shape operator is the same at every point of Mobius space M^n. When $\alpha : I \longrightarrow M^n$ is a curvature line and T is tangent vector field of α, it follows that

$$S(T) = 0.T.$$

Hence,since

$$S(\psi_*(T)) = 0.\psi_*(T)$$

for $\psi_*(T)$, we obtain that every line is a principal direction and every curve is a curvature line.

3.2. Conform Map

Definition 2.11: Suppose M and M' be two manifolds and let

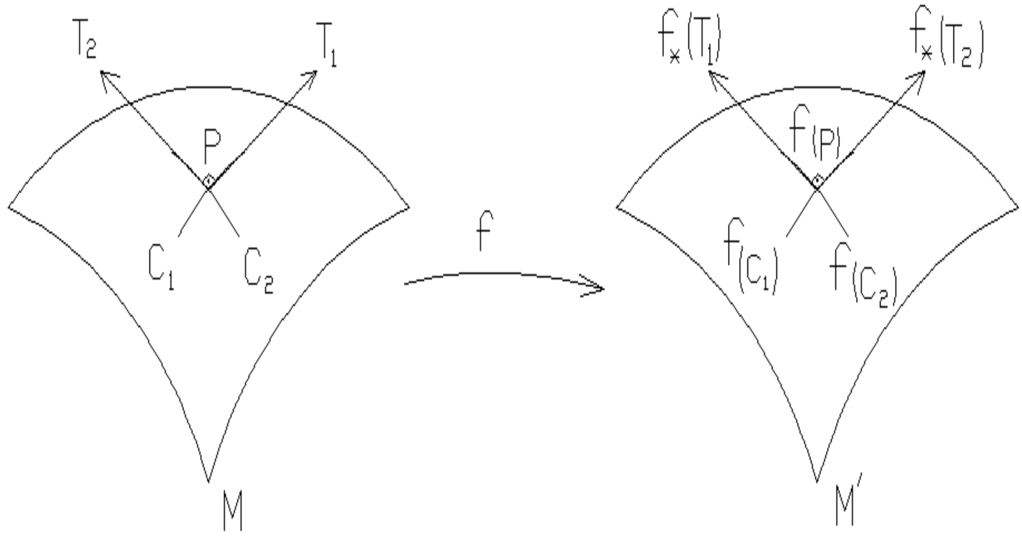
$$f : M \longrightarrow M'$$

be a C^∞ maps. If $\forall p \in M$, there exists a function λ in a positive reel C^∞ class on M

$$< f_*X, f_*Y > |_P = \lambda^2(p) < X, Y >, \forall X, Y \in \chi(M)$$

then the function f is called a conform map. λ is called a scaler function [5].

Theorem 2.12: Conform maps preserves angles.



Proof: The intersection of the curves c_1 and c_2 at the point p is the angle between tangent of these curves on the manifold from Figure 2.3, we get

$$cos\varphi = \frac{< f_*(T_1), f_*(T_2) >}{\| f_*(T_1)\| \| f_*(T_2)\|} = cos\theta$$

if θ than $90°$,then the angles φ between the intersection of the corresponding curves $f(c_1)$ and $f(c_2)$ becomes φ. In other words, we get

$$< f_*(T_1), f_*(T_2) >= 0.$$

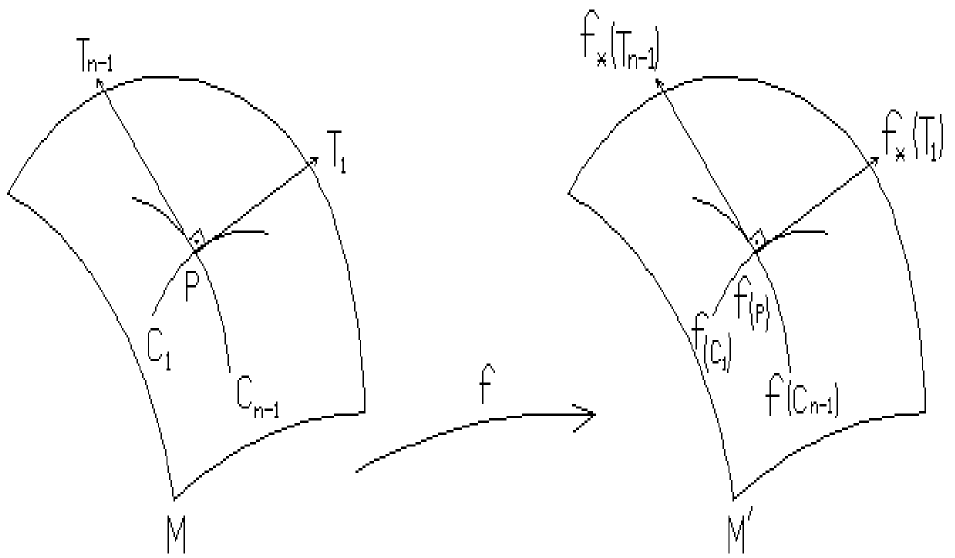
Thus,this preserves orthogonality.

Theorem 2.13: If a differentiable f is a conform mapping of a hypersurface M into a hypersurface M', then f preserves the property of curve lines.

Proof:

$$f : M \longrightarrow M', \, for \forall p \in M$$

$$< f_*(T_1), f_*(T_{n-1}) >= \lambda^2(P) < T_1, T_{n-1} >, \forall T_1, T_{n-1} \in \chi(M).$$



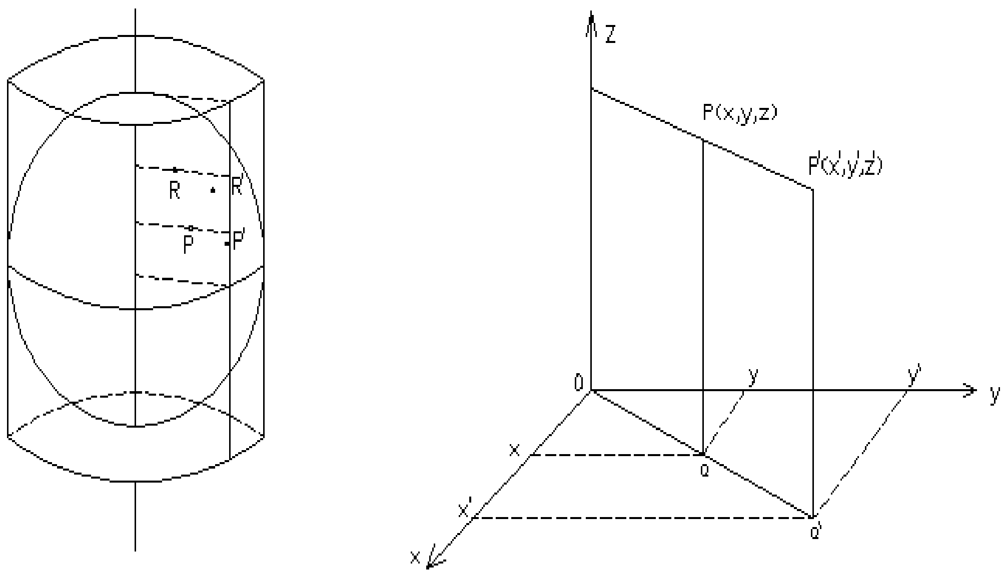
Let p be a point on the hyperspace M and the orthogonal curves family $c_1, c_2, \ldots, c_{n-1}$ be a curve lines passing at the point P. Then, corresponding vector fields $T_1, T_2, \ldots, T_{n-1}$ become fundamental direction (Figure 2.4)

Since f differentiable, f_* exists and the figure of T_i is $f_*(T_i)$. $T_1, T_2, \ldots, T_{n-1}, N$ form an n tuple orthogonal system on the M.Since f is a conform maps, it preserves angles. So, $\{f_*(T_1), f_*(T_2) \ldots, f_*(T_{n-1}), f_*(N)\}$ constitute an orthogonal system is preserved. By the theorem Dupin, the curves $\{f(c_1), f(c_2), \ldots, f(c_{n-1})\}$ become curve lines on the M'. So, a conform maps f preserves the property of curve lines.

Corollary 2.14: The direction of the lines of the curvature whose images of tangent vector fields under f_* are perpendicular to e_3 under f is the lines of curvature on the C^2.

3.3. Lambert Projection

Lambert projection is a projection from E^3 into cylinder (Figure 2.5)[4]. Consider an equation of a cylinder:$x^2 + y^2 = 1, (z = t, arbitrary)$ and an equation of a sphere: $x^2 + y^2 + z^2 = 1$.



The equation of a Lambert projection is obtained as: By similarity triangles $OX'Q'$ OXQ

$$\frac{\bar{OX}}{\bar{OX'}} = \frac{\bar{OQ}}{\bar{OQ'}}, \bar{OQ} = 1$$

so,

$$\frac{x}{x'} = \frac{\sqrt{x^2 + y^2}}{1}, x' = \frac{x}{\sqrt{1 - z^2}}$$

and

$$x'^2 + y'^2 = 1, \frac{x^2}{1 - z^2} + y'^2 = 1, y' = \frac{y}{\sqrt{1 - z^2}}.$$

From this

$$f : S^2 \longrightarrow C^2$$

$$f(x, y, z) = (\frac{x}{\sqrt{1 - z^2}}, \frac{y}{\sqrt{1 - z^2}}, z).$$

The derivative map of f can be obtained as follows:

$$f_* = \begin{pmatrix} \frac{1}{\sqrt{1 - z^2}} & 0 & \frac{zx}{(1 - z^2)^{3/2}} \\ 0 & \frac{1}{\sqrt{1 - z^2}} & \frac{zy}{(1 - z^2)^{3/2}} \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 2.15: The Lambert projection preserves the line of curvature that consist of meridian and parallel circles.

Proof: Consider

$$S_k(T) = \lambda, \lambda = constant$$

for a curvature line, $x : I \longrightarrow S^2$, and a tangent vector field T. Then it is enough to find lines such that

$$S_C = (f_*(T)) = \mu f_*(T)$$

where S_C is a shape operator of cylinder. We will find the,lines such that

$$e_3 = f_*(X)$$

for $X \in \chi(S^2)$. Since

$$f_*(X) = (0, 0, 1)$$

for $X = (x_1, x_2, x_3)$, it follows that

$$X = \left(-\frac{zx}{1-z^2}, -\frac{zy}{1-z^2}, 1\right).$$

Curvature lines whose image are e_3 :

Since

$$X = \lambda X$$

$$\left(-\frac{zx}{1-z^2}, -\frac{zy}{1-z^2}, 1\right) = \left(-\frac{zx}{1-z^2}, -\frac{zy}{1-z^2}, 1\right)$$

we find that

$$\lambda = 1.$$

For the operator $S_C = O_2$ of a cylinder, we write

$$S_C(f_*(X)) = 0 f_*(X).$$

Corollary 2.16: The curves whose images of tangent vector fields under f_* are e_3, are lines of curvature on a cylinder.

Here are the lines of curvature whose images of tangent vector fields under f_* are perpendicular to e_3.

Since $S_C = I_2$, in the direction, perpendicular to e_3, we have

$$S_C(f_*(X)) = f_*(X).$$

$$f_*(X) = (x_1, x_2, 0).$$

This shows that the curvature lines whose principal directions are perpendicular to e_3, are lines of curvature.

Acknoledgement

Authors thank to the referees for all their helpful suggestion.

References

[1] Hacisalihoglu, H. H., (1980), "Introduction to Advanced Differential Geometry," Firat Universty, faculty of Sciences.

[2] Guggenheimer, H. W., (1963), "Differential Geometry," College of Liberal Arts, University of Minnesota, pp. 222–223.

[3] Hicks, N. J., (1974), "Notes on Differential Geometry," Van Nostrant Reinhold Company, London, pp. 1–49.

[4] Goetz, A., (1970), "Introduction to Differential Geometry," University of Notre Dame, Indiana, 208, 212, pp. 298–299.

[5] Hacisalihoglu, H.H., (1976), "Maps and Geometry in High Dimensional space," Ankara University, Faculty of Sciences, pp. 183–185.

[6] O'Neill, B., (1966), "Elementary Differential Geometry," Academic Press, New York, 6–11, pp. 268–269.

[7] Ozdemir, M., (1985), "The Lines of Curvature and The Manifold Maps," Doga Bilim Dergisi, Vol. 9, No. A-1, pp. 19–25.