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Fuzzy ℓ–Ideals of an ℓ–Ring

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Abstract

The concept of fuzzy ℓ – ideal of an ℓ – ring is introduced and some examples are presented. Some of their properties are studied.

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Introduction

The notion of fuzzy sets was introduced by Lofti. A. Zadeh [11] in 1965. Zadeh had initiated fuzzy set theory as a modification of the ordinary set theory. In 1982, Liu [9] developed the concept of fuzzy subrings as well as fuzzy ideals in rings. Since then the literature of these fuzzy algebraic concepts has been growing very rapidly. The concept of fuzzy lattices was developed by Sudarsan Nanda [8], Yuan Bo and Wu Wangming [7], N. Ajmal and K. V. Thomas [6], Andreja Tepavcevic and Goran Trajkovski [4] and many others. R. Natarajan and S. Mohanavalli [1] introduced the concept of fuzzy sub lattice ordered rings.

In classical theory, ℓ – ideals associated to any ℓ – ring, play a central role. On considering this, in this paper, we made an attempt to study the algebraic nature of fuzzy ℓ – ideals of an ℓ – ring.

Preliminaries

In this section the basic concepts of lattice ordered rings and fuzzy subsets are recalled.

Definition 2.1

A partial order set $(L, \leq)$ is said to form a **lattice** if for every a, b $\in$ L, Sup {a, b} and

Inf {a, b} exist in L. In that case, we write Sup {a, b} = $a \vee b$ and Inf {a, b} = $a \wedge b$.

A lattice L is called a **distributive lattice** if

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all a, b, c $\in$ L.

Definition 2.2

A **partial order ring** is a ring R which is also a partial order set under a relation $\geq$, in which

- $x \geq y$ implies $a + x \geq a + y$ for all $a \in R$
- $x \geq 0$ and $y \geq 0$ imply $xy \geq 0$ in R.

Definition 2.3

A **lattice ordered ring** or ***ℓ* – ring** R is a partial order ring in which any two elements x, y in R, have a least upper bound and a greatest lower bound.

Definition 2.4

A non – empty set R together with the binary operations denoted by +, ., $\vee$ and $\wedge$ is called an **lattice ordered ring** or ***ℓ* – ring** if it satisfies the following axioms:

- (R, +, .) is a ring.
- (R, $\vee$, $\wedge$) is a lattice.
- $x + (y \vee z) = (x + y) \vee (x + z)$

$x + (y \wedge z) = (x + y) \wedge (x + z)$

$(y \vee z) + x = (y + x) \vee (z + x)$

$(y \wedge z) + x = (y + x) \wedge (z + x)$, for all x, y, z $\in$ R.

- $x \,.\, (y \vee z) = (x \,.\, y) \vee (x \,.\, z)$

$x \,.\, (y \wedge z) = (x \,.\, y) \wedge (x \,.\, z)$

$(y \vee z) \,.\, x = (y \,.\, x) \vee (z \,.\, x)$

$(y \wedge z) \,.\, x = (y \,.\, x) \wedge (z \,.\, x)$, for all x, y, z $\in$ R.

Example 2.5

(Z, +, ., $\vee$, $\wedge$) is an ***ℓ*** – ring.

Example 2.6

Now (R = {a, b, c, d}, +, ., $\vee$, $\wedge$) is an ***ℓ*** – ring under the operations +, ., $\vee$ and $\wedge$ defined by the following tables.

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

.	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	d	c	d
d	d	d	d	d

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Example 2.7

($S = \{m, n\}, *, \bullet, \vee_1, \wedge_1$) is an ℓ – ring, under the operations $*$, $\bullet$, $\vee_1$ and $\wedge_1$ defined by the following table.

$*$	m	n
m	m	n
n	n	m

$\bullet$	m	n
m	m	m
n	m	n

$\vee_1$	m	n
m	m	n
n	n	n

$\wedge_1$	m	n
m	m	m
n	m	n

Definition 2.8

Let X be a non – empty set. A mapping $\mu: X \rightarrow [0, 1]$ is called a **fuzzy subset** of X.

Definition 2.9

Let μ be any fuzzy subset of a set X and $\mu = (x_i, t_i)$ / i =1 to n and $t_i \in [0, 1]\}$. Then, $\{t_i$ / i =1 to n$\}$ is called the **image set** of μ and is denoted by Im μ.

Definition 2.10

A fuzzy subset μ of a lattice ordered ring (or ℓ – ring in short) R, is called a **fuzzy sub ℓ – ring** of R, if the following conditions are satisfied:

- $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$
- $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$
- $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$
- $\mu(xy) \geq \min \{\mu(x), \mu(y)\}$,

for all $x, y \in R$.

Example 2.11

Consider the fuzzy subset μ_1 of the ℓ – ring (Z, +, ., $\vee$, $\wedge$).

$$\mu_1(x) = \begin{cases} .9 \ if \ \ x \in \langle 5 \rangle \\ .2 \quad Z \sim \langle 5 \rangle \end{cases}$$

Then μ_1 is a fuzzy sub ℓ – ring of Z.

Example 2.12

Consider the fuzzy subset μ_2 of the ℓ – ring (Z, +, ., $\vee$, $\wedge$).

$$\mu_2(x) = \begin{cases} .4 \ if \ \ x \in \langle 4 \rangle \\ .9 \quad Z \sim \langle 4 \rangle \end{cases}$$

Then μ_2 is not a fuzzy sub ℓ – ring of Z.

Fuzzy ℓ – ideal

Here the concept fuzzy ℓ – ideal of an ℓ – ring is introduced.

Definition 3.1
A fuzzy subset μ of an ℓ – ring R, is called a fuzzy ℓ – ring ideal or **fuzzy ℓ – ideal** of R if, for all x, y∈R the following conditions are satisfied:

- $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$
- $\mu(x \wedge y) \geq \max\{\mu(x), \mu(y)\}$
- $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
- $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$

Example 3.2
Consider the fuzzy subset μ of the ℓ – ring R, defined in Example 2.6

$$\mu(x) = \begin{cases} .9 & if \quad x = a \\ .6 & if \quad x = b \\ .4 & if \quad x = c, d \end{cases}$$

Then μ is a fuzzy ℓ – ideal of R.

Example 3.3
Consider the ℓ – ring given in Example 2.7. Then the fuzzy subset σ of S defined by

$$\sigma(x) = \begin{cases} .4 & if \quad x = m \\ .1 & if \quad x = n \end{cases}$$

Then σ is a fuzzy ℓ – ideal of R.

Remark 3.4
Every fuzzy ℓ – ideal of an ℓ – ring R, is a fuzzy sub ℓ – ring of R. But the converse need not be true.

Proof: By the Example,
Consider the fuzzy subset μ_1 of the ℓ – ring (Z, +, ., ∨, ∧).

$$\mu_1(x) = \begin{cases} .6 \; if \;\; x \in \, < 2 > \\ .3 \; otherwise \end{cases}$$

Then μ_1 is a fuzzy sub ℓ – ring of Z. But μ_1 is not a fuzzy ℓ – ideal of Z.

Properties of fuzzy ℓ – ideal

Some of the properties of fuzzy ℓ – ideal of an ℓ – ring is studied here.

Proposition 4.1
If μ is any fuzzy ℓ – ideal of an ℓ – ring R, then $\mu(1) \leq \mu(x) \leq \mu(0)$, for all $x \in R$ where 0 is the least element and 1 is the greatest element in R.

Proof

Given μ is any fuzzy ℓ – ideal of an ℓ – ring R with least element 0 and greatest element 1.

To prove $\mu(1) \leq \mu(x) \leq \mu(0)$, for all $x \in R$.

Let $x \in R$ be arbitrary. Then,

$\mu(x) = \mu(1 \wedge x) \geq \max \{\mu(1), \mu(x)\} \geq \mu(1)$ ----------→ (1)

And $\mu(0) = \mu(x - x) \geq \min \{\mu(x), \mu(x)\} \geq \mu(x)$ -------------→ (2)

From (1) and (2), we have $\mu(1) \leq \mu(x) \leq \mu(0)$, for all $x \in R$.

Proposition 4.2

Let μ be any fuzzy ℓ – ideal of an ℓ – ring R. Then $\mu(x) = \mu(-x)$ for all $x \in R$.

Proof

Let μ be any fuzzy ℓ – ideal of an ℓ – ring R, let $x \in R$ be arbitrary.

Then, $-x = 0 + (-x) = 0 - x$

$\Rightarrow \mu(-x) = \mu(0 - x) \geq \min \{\mu(0), \mu(x)\} \geq \mu(x)$ ------→ (1)

Again $\mu(x) = \mu(-(-x)) \geq \mu(-x)$, by (1)

$\geq \mu(x)$, by (1)

$\Rightarrow \mu(x) = \mu(-x)$.

Hence $\mu(x) = \mu(-x)$, for all $x \in R$.

Proposition 4.3

Let μ be any fuzzy ℓ – ideal of an ℓ – ring R. $\mu(x) \geq \mu(y)$ whenever $x \leq y$, where $x, y \in R$.

Proof

Given μ is any fuzzy ℓ – ideal of an ℓ – ring R. Let $x, y \in R$ be arbitrary.

Assume that $x \leq y$.

$\Rightarrow x \wedge y = x$ and $x \vee y = y$ ----------→(1)

Now, $\mu(x) = \mu(x \wedge y)$, by (1)

$\geq \max \{\mu(x), \mu(y)\} \geq \mu(y)$

Proposition 4.4

Every constant fuzzy subset of an ℓ – ring R is a fuzzy ℓ – ideal of R. But the converse need not be true. That is, every fuzzy ℓ – ideal of R need not be a constant function.

Proof

Assume that μ is a constant fuzzy subset of an ℓ – ring R.

$\Rightarrow \mu(x) = c$, for all $x \in R$.

Then it is easy to prove the four inequalities.

We prove the second part by giving a counter example. Consider the ℓ – ring R defined in Example 2.6.

Then θ defined below is a fuzzy ℓ – ideal of R, but θ is not a constant function.

$$\theta(x) = \begin{cases} .9 & if \quad x = a, b \\ .4 & if \quad x = c, d \end{cases}$$

Thus every fuzzy ℓ – ideal of R need not be a constant function.

Remark 4.5

Let μ be a fuzzy ℓ – ideal of an ℓ – ring R. As $x \wedge y \leq x \leq x \vee y$, then by the Proposition 4.3, we have $\mu(x \wedge y) \geq \mu(x) \geq \mu(x \vee y)$, for all $x \in R$.

Thus μ is either a decreasing function or a constant function.

Proposition 4.6

Let μ be a fuzzy ℓ – ideal of R. If $\mu(x) < \mu(y)$ for some $x, y \in R$, then $\mu(x - y) = \mu(x)$.

Proof

Let μ be a fuzzy ℓ – ideal of R.

Let $x, y \in R$ be arbitrary.

Assume that $\mu(x) < \mu(y)$ -------→ (1)

Now $\mu(x - y) \geq \min \{\mu(x), \mu(y)\} = \mu(x)$, by (1)

$\Rightarrow \mu(x - y) \geq \mu(x)$ --------------→ (2)

And $x = x + 0 = x + (-y + y) = (x - y) + y$

Therefore, $\mu(x) = \mu((x - y) + y) \geq \min \{\mu(x - y), \mu(y)\} = \mu(x - y)$ as $\mu(x) < \mu(y)$

$\Rightarrow \mu(x) \geq \mu(x - y)$ --------------→ (3)

From (2) and (3), we have $\mu(x) = \mu(x - y)$.

Proposition 4.7

Let μ be any fuzzy ℓ – ideal of an ℓ – ring R. If $\mu(x) < \mu(y)$ for some $x, y \in R$, then $\mu(x \vee y) = \mu(x)$.

Proof

Let μ be any fuzzy ℓ – ideal of an ℓ – ring R.

Let $x, y \in R$ be arbitrary.

Assume that $\mu(x) < \mu(y)$ -------------→ (1)

Now $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\} = \mu(x)$, by (1) ----------→ (2)

Again $\mu(x) = \mu((x \vee y) \wedge x) \geq \max \{\mu(x \vee y), \mu(x)\} \geq \mu(x \vee y)$ -----→ (3)

From (2) and (3), we have, $\mu(x \vee y) = \mu(x)$.

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