

## Compatible Mapping of Type $(\alpha)$ and Common Fixed Points in Intuitionistic $\mathcal{M}$ -fuzzy Metric Spaces

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### Abstract

In this paper, we give some new definitions and properties of compatible mappings of type  $(\alpha)$  in intuitionistic  $\mathcal{M}$ -fuzzy metric spaces and prove some common fixed point theorems for four mappings under the condition of compatible mappings of type  $(\alpha)$  in complete intuitionistic  $\mathcal{M}$ -fuzzy metric spaces.

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### 1. Introduction

In 1992, Dhage [2] introduced the notion of generalized metric or  $D$ -metric spaces and proved several fixed point theorems in it. Since  $D$ -metric space not possess some topological properties (see [7, 8, 9]), recently Sedghi and Shobe [12] introduced  $D^*$ -metric space as a probable modification of  $D$ -metric space and studied some topological properties which are not valid in  $D$ -metric spaces. Based on  $D^*$ -metric concepts, they [11, 13] defined  $\mathcal{M}$ -fuzzy metric space and proved a common fixed point theorem in it. Soleimani [14] introduced intuitionistic  $\mathcal{M}$ -fuzzy metric space proved some properties and theorems in it. Mishra et al. [6] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Recently, Jungek et al. [5] introduced the concept of compatible mappings of type (A) in metric spaces, which is equivalent to the concept of compatible mappings under some conditions and proved common fixed point theorems in metric spaces. Cho [1] introduced the concept of compatible mappings of type  $(\alpha)$  in fuzzy metric spaces.

In this paper, we give some definitions of compatible mappings of types  $(\alpha)$  in intuitionistic  $\mathcal{M}$ -fuzzy metric spaces and prove some common fixed point theorems for four mappings under the condition of compatible mappings of types  $(\alpha)$  in complete intuitionistic  $\mathcal{M}$ -fuzzy metric spaces.

## 2. Preliminaries

Now, we give basic definitions and their properties as follows:

**Definition 2.1.** [10] A binary operation  $*$  :  $[0, 1] \times [0, 1] \longrightarrow [0, 1]$  is continuous  $t$ -norm if  $*$  is satisfying the following conditions:

- (1)  $*$  is commutative and associative,
- (2)  $*$  is continuous,
- (3)  $a * 1 = a$  for all  $a \in [0, 1]$ ,
- (4)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ .

**Example 2.2.** Two typical examples of continuous  $t$ -norm are  $a * b = ab$  and  $a * b = \min\{a, b\}$ .

**Definition 2.3.** [10] A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \longrightarrow [0, 1]$  is continuous  $t$ -conorm if  $\diamond$  is satisfying the following conditions:

- (1)  $\diamond$  is commutative and associative,
- (2)  $\diamond$  is continuous,
- (3)  $a \diamond 1 = a$  for all  $a \in [0, 1]$ ,
- (4)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ .

**Example 2.4.** Two typical examples of continuous  $t$ -conorm are  $a \diamond b = \max\{a, b\}$  and  $a \diamond b = \min\{1, a + b\}$ .

**Definition 2.5.** [14] A 5-tuple  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is called an intuitionistic  $\mathcal{M}$ -fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm,  $\diamond$  is a continuous  $t$ -conorm and  $\mathcal{M}, \mathcal{N}$  are fuzzy sets on  $X^3 \times (0, \infty)$  satisfying the following conditions for each  $x, y, z, a \in X$  and  $t, s > 0$ ,

- (1)  $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$ ,
- (2)  $\mathcal{M}(x, y, z, t) > 0$ ,
- (3)  $\mathcal{M}(x, y, z, t) = 1$  if and only if  $x = y = z$ ,
- (4)  $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ , (symmetry) where  $p$  is a permutation function,
- (5)  $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$ ,
- (6)  $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \longrightarrow [0, 1]$  is continuous.

- (7)  $\mathcal{N}(x, y, z, t) > 0$ ,
- (8)  $\mathcal{N}(x, y, z, t) = 0$  if and only if  $x = y = z$ ,
- (9)  $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$ , (symmetry) where  $p$  is a permutation function,
- (10)  $\mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$ ,
- (11)  $\mathcal{N}(x, y, z, \cdot) : (0, \infty) \longrightarrow [0, 1]$  is continuous.

Then  $(\mathcal{M}, \mathcal{N})$  is called an intuitionistic  $\mathcal{M}$ -fuzzy metric on  $X$ . The functions  $\mathcal{M}(x, y, z, t)$  and  $\mathcal{N}(x, y, z, t)$  denote the degree of nearness and the degree of non-nearness between  $x$  and  $y$  with respect to  $t$ , respectively.

**Lemma 2.6.** [14] Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space. For any  $x, y \in X$  and  $t > 0$ , we have

- (1)  $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$ .
- (2)  $\mathcal{N}(x, x, y, t) = \mathcal{N}(x, y, y, t)$ .

**Definition 2.7.** [14] Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space and  $\{x_n\}$  be a sequence in  $X$ .

- (1)  $\{x_n\}$  is said to be convergent to a point  $x \in X$  (denoted by  $\lim_{n \rightarrow \infty} x_n = x$ ) if  $\lim_{n \rightarrow \infty} \mathcal{M}(x, x, x_n, t) = 1$  and  $\lim_{n \rightarrow \infty} \mathcal{N}(x, x, x_n, t) = 0$  for all  $t > 0$ .
- (2)  $\{x_n\}$  is called a Cauchy sequence if for all  $t > 0$  and  $p > 0$ ,

$$\lim_{n \rightarrow \infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(x_{n+p}, x_{n+p}, x_n, t) = 0.$$

- (3) An intuitionistic  $\mathcal{M}$ -fuzzy metric in which every Cauchy sequence is convergent is said to be complete.

**Lemma 2.8.** [14] Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space. Then  $\mathcal{M}(x, y, z, t)$  is non-decreasing and  $\mathcal{N}(x, y, z, t)$  is non-increasing with respect to  $t$ , for all  $x, y, z \in X$ .

**Lemma 2.9.** [15] Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space. If we define  $E_{\lambda, \mathcal{M}} : X^3 \rightarrow \mathbf{R}^+ \cup \{0\}$  by

$$E_{\lambda, \mathcal{M}}(x, y, z) = \inf\{t > 0 : \mathcal{M}(x, y, z, t) > 1 - \lambda\}$$

and  $E_{\lambda, \mathcal{N}} : X^3 \rightarrow \mathbf{R}^+ \cup \{0\}$  by

$$E_{\lambda, \mathcal{N}}(x, y, z) = \sup\{t > 0 : \mathcal{N}(x, y, z, t) < \lambda\}$$

for all  $\lambda \in (0, 1)$  and  $x, y, z \in X$ , then

(1) For all  $\mu \in (0, 1)$  there exists  $\lambda \in (0, 1)$ , such that

$$E_{\mu, \mathcal{M}}(x_1, x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots \\ + E_{\lambda, \mathcal{M}}(x_{n-1}, x_{n-1}, x_n)$$

and

$$E_{\mu, \mathcal{N}}(x_1, x_1, x_n) \geq E_{\lambda, \mathcal{N}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{N}}(x_2, x_2, x_3) + \cdots \\ + E_{\lambda, \mathcal{N}}(x_{n-1}, x_{n-1}, x_n)$$

for all  $x_1, x_2, \dots, x_n \in X$ .

(2) The sequence  $\{x_n\}_{n \in \mathbf{N}}$  is convergent in intuitionistic  $\mathcal{M}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  if and only if  $E_{\lambda, \mathcal{M}}(x_n, x_n, x) \rightarrow 0$  and  $E_{\lambda, \mathcal{N}}(x_n, x_n, x) \rightarrow 0$ . Also the sequence  $\{x_n\}_{n \in \mathbf{N}}$  is a Cauchy sequence if and only if it is a Cauchy sequence with  $E_{\lambda, \mathcal{M}}$  and  $E_{\lambda, \mathcal{N}}$ .

**Theorem 2.10.** [15] Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space. If

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_1, k^n t) \quad \text{and} \quad \mathcal{N}(x_n, x_n, x_{n+1}, t) \leq \mathcal{N}(x_0, x_0, x_1, k^n t)$$

for some  $k > 1$  and for every  $n \in \mathbf{N}$ . Then sequence  $\{x_n\}$  is a Cauchy sequence.

### 3. Main Results

In this section, we present the main result in this paper. And we prove a fixed point theorem for four mappings under the condition of compatible mappings of type  $(\alpha)$  in intuitionistic  $\mathcal{M}$ -fuzzy metric space.

**Definition 3.1.** Let  $A$  and  $B$  be mappings from an intuitionistic  $\mathcal{M}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  in to it self. The mappings are said to be compatible of type  $(\alpha)$  if

$$\lim_{n \rightarrow \infty} \mathcal{M}(ABx_n, BBx_n, BBx_n, t) = 1 \quad \text{and} \quad \mathcal{N}(ABx_n, BBx_n, BBx_n, t) = 0,$$

$$\lim_{n \rightarrow \infty} \mathcal{M}(BAx_n, AAx_n, AAx_n, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{N}(BAx_n, AAx_n, AAx_n, t) = 0,$$

for all  $t > 0$ , whenever  $\{x_n\}$  is a sequence in  $X$ , such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z,$$

for some  $z \in X$ .

**Proposition 3.2.** Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space  $A$  and  $B$  be continuous mappings from  $X$  in to itself. Then  $A$  and  $B$  are compatible of type  $(\alpha)$  and  $Az = Bz$  for some  $z \in X$ , then  $ABz = BBz = BAz = AAz$ .

*Proof.* Let  $\{x_n\}$  be a sequence in  $X$  defined by  $x_n = z$  for some  $z \in X$  and  $n = 1, 2, \dots$  and  $Az = Bz$ . Then, we have

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = Az.$$

Since  $A$  and  $B$  are compatible of type  $(\alpha)$ , we have

$$\lim_{n \rightarrow \infty} \mathcal{M}(ABx_n, BBx_n, BBx_n, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{N}(ABx_n, BBx_n, BBx_n, t) = 0.$$

Thus, we have

$$\mathcal{M}(ABz, BBz, BBz, t) = 1 \quad \text{and} \quad \mathcal{N}(ABz, BBz, BBz, t) = 0.$$

Hence  $ABz = BBz$ . Similarly, we have  $BAz = AAz$ . But,  $Az = Bz$  implies  $BBz = BAz$ . Therefore, we obtain  $ABz = BBz = BAz = AAz$ .  $\blacksquare$

**Proposition 3.3.** Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space and  $A$  and  $B$  be continuous mappings from  $X$  in to itself. Then  $A$  and  $B$  are compatible if and only if they are compatible of type  $(\alpha)$ .

*Proof.* Let  $\{x_n\}$  be a sequence in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$$

for some  $z \in X$ . Since  $A$  is continuous, we have

$$\lim_{n \rightarrow \infty} AAx_n = \lim_{n \rightarrow \infty} ABx_n = Az.$$

Suppose that  $A$  and  $B$  are compatible. Then

$$\lim_{n \rightarrow \infty} \mathcal{M}\left(ABx_n, BAx_n, BAx_n, \frac{t}{2}\right) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{N}\left(ABx_n, BAx_n, BAx_n, \frac{t}{2}\right) = 0,$$

for all  $t > 0$ . From the inequality where, we have

$$\mathcal{M}(AAx_n, BAx_n, BAx_n, t) \geq \mathcal{M}\left(AAx_n, ABx_n, ABx_n, \frac{t}{2}\right) * \mathcal{M}\left(ABx_n, BAx_n, BAx_n, \frac{t}{2}\right)$$

and

$$\mathcal{N}(AAx_n, BAx_n, BAx_n, t) \leq \mathcal{N}\left(AAx_n, ABx_n, ABx_n, \frac{t}{2}\right) \diamond \mathcal{N}\left(ABx_n, BAx_n, BAx_n, \frac{t}{2}\right),$$

for all  $t > 0$ .

Taking limit as  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \mathcal{M}(AAx_n, BAx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{N}(AAx_n, BAx_n, BAx_n, t) = 0.$$

Similarly, we also obtain

$$\lim_{n \rightarrow \infty} \mathcal{M}(BBx_n, ABx_n, ABx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(BBx_n, ABx_n, ABx_n, t) = 0.$$

Hence,  $A$  and  $B$  are compatible of type  $(\alpha)$ .

Conversely, suppose that  $\{x_n\}$  be a sequence in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$$

for some  $z \in X$ . Since  $B$  is continuous, we have

$$\lim_{n \rightarrow \infty} BAx_n = \lim_{n \rightarrow \infty} BBx_n = Bz.$$

Further, since  $A$  and  $B$  are compatible of type  $(\alpha)$ , we have

$$\lim_{n \rightarrow \infty} \mathcal{M}\left(ABx_n, BBx_n, BBx_n, \frac{t}{2}\right) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}\left(ABx_n, BBx_n, BBx_n, \frac{t}{2}\right) = 0$$

$$\lim_{n \rightarrow \infty} \mathcal{M}\left(BAx_n, AAx_n, AAx_n, \frac{t}{2}\right) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}\left(BAx_n, AAx_n, AAx_n, \frac{t}{2}\right) = 0,$$

for all  $t > 0$ .

From the inequality, we have

$$\mathcal{M}(ABx_n, BAx_n, BAx_n, t) \geq \mathcal{M}\left(ABx_n, BBx_n, BBx_n, \frac{t}{2}\right) * \mathcal{M}\left(BAx_n, AAx_n, AAx_n, \frac{t}{2}\right)$$

and

$$\mathcal{N}(ABx_n, BAx_n, BAx_n, t) \leq \mathcal{N}\left(ABx_n, BBx_n, BBx_n, \frac{t}{2}\right) \diamond \mathcal{N}\left(BAx_n, AAx_n, AAx_n, \frac{t}{2}\right)$$

Taking limit as  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \mathcal{M}(ABx_n, BAx_n, BAx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(ABx_n, BAx_n, BAx_n, t) = 0.$$

Hence, the  $A$  and  $B$  are compatible. This completes the proof.  $\blacksquare$

**Proposition 3.4.** Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an intuitionistic  $\mathcal{M}$ -fuzzy metric space and  $A$  and  $B$  be continuous mappings from  $X$  in to itself. Then  $A$  and  $B$  are compatible of type  $(\alpha)$  and  $Az = Bz$  for some  $z \in X$ , then  $ABz = BBz = BAz = AAz$ .

*Proof.* Let  $\{x_n\}$  be a sequence in  $X$  defined by  $x = z$  for some  $z \in X$  and  $n = 1, 2, \dots$  and  $Az = Bz$ . Then, we have

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = Az$$

Since  $A$  and  $B$  are compatible of type  $(\alpha)$ , we have

$$\lim_{n \rightarrow \infty} \mathcal{M}(ABx_n, BBx_n, BBx_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(ABx_n, BBx_n, BBx_n, t) = 0.$$

Thus, we have

$$\mathcal{M}(ABz, BBz, BBz, t) = 1 \text{ and } \mathcal{N}(ABz, BBz, BBz, t) = 0.$$

Hence  $ABz = BBz$ . Similarly, we have  $BAz = AAz$ . But,  $Az = Bz$  implies  $BBz = BAz$ . Therefore, we obtain  $ABz = BBz = BAz = AAz$ . This completes the proof. ■

Now, For the remainder of the Paper Let  $\Phi$  and  $\psi$  are sets of all continuous increasing functions. Let  $\phi : [0, 1]^4 \rightarrow [0, 1]$  is continuous and increasing in each co-ordinate variable. Also  $\phi(s, s, \dots, s) > s$  for every  $s \in [0, 1)$  and  $\psi : [0, 1]^4 \rightarrow [0, 1]$  is continuous and decreasing in each co-ordinate variable. Also  $\psi(s, s, \dots, s) < s$  for every  $s \in [0, 1)$ .

**Example 3.5.** Let  $\phi : [0, 1]^4 \rightarrow [0, 1]$  is define by

$$\phi(x_1, x_2, x_3, x_4) = (\min\{x_i\})^h,$$

for some  $h \in [0, 1)$ . and

Let  $\psi : [0, 1]^4 \rightarrow [0, 1]$  is define by

$$\psi(x_1, x_2, x_3, x_4) = (\max\{x_i\})^h,$$

for some  $h \in [0, 1)$ .

**Theorem 3.6.** Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be a complete intuitionistic  $\mathcal{M}$ -fuzzy metric space. Let  $A, B, S$  and  $T$  be mappings from  $X$  in to itself such that:

- (a)  $A(X) \subseteq T(X), B(X) \subseteq S(X)$ ,
- (b)  $AB = BA, ST = TS, SA = AS, TB = BT, SB = BS$ ,
- (c)  $A, B, S$  and  $T$  are continuous,
- (d) The pairs  $(A, S)$  and  $(B, T)$  are compatible of type  $(\alpha)$ ,
- (e)

$$\mathcal{M}(Ax, By, Bz, t) \geq \phi \left( \begin{array}{cc} \mathcal{M}(Sx, Ty, Tz, kt), & \mathcal{M}(Sx, Ty, Bz, kt), \\ \mathcal{M}(Ty, By, By, kt), & \mathcal{M}(By, Tz, Tz, kt), \end{array} \right),$$

and

$$\mathcal{N}(Ax, By, Bz, t) \leq \psi \left( \begin{array}{cc} \mathcal{N}(Sx, Ty, Tz, kt), & \mathcal{N}(Sx, Ty, Bz, kt), \\ \mathcal{N}(Ty, By, By, kt), & \mathcal{N}(By, Tz, Tz, kt), \end{array} \right),$$

for all  $x, y, z \in X, t > 0$ . Then  $A, S, B$  and  $T$  have a unique common fixed point in  $X$ .

*Proof.* Let  $x_0$  be an arbitrary point of  $X$ . Since  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ , there exist  $x_1, x_2 \in X$  such that  $Ax_0 = Tx_1$ ,  $Bx_1 = Sx_2$ . Inductively, construct the sequences  $\{y_n\}$  and  $\{x_n\}$  in  $X$  such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},$$

for  $n = 0, 1, 2, \dots$ .

If we set  $d_m(t) = \mathcal{M}(x_m, x_m, x_{m+1}, t)$  for all  $t > 0$ , then we prove that  $\{d_m(t)\}$  is increasing with respect to  $m$ . and

we set  $d_{m'}(t) = \mathcal{N}(x_{m'}, x_{m'}, x_{m'+1}, t)$  for all  $t > 0$ , then we prove that  $\{d_{m'}(t)\}$  is decreasing with respect to  $m'$ .

Setting  $m = 2n$ , then by putting  $x = x_{2n}$  and  $y = z = x_{2n+1}$  in (e), we have

$$\begin{aligned} d_{2n}(t) &= \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t) \\ &= \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\geq \phi \left( \begin{array}{cc} \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right) \\ &= \phi \left( \begin{array}{cc} \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n}, kt), & \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), \\ \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, kt), & \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, kt) \end{array} \right) \\ &\geq \phi ( d_{2n-1}(kt), \mathcal{M}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), d_{2n}(kt), d_{2n}(kt) ), \end{aligned}$$

that is,

$$d_{2n}(t) \geq \phi (d_{2n-1}(kt), d_{2n-1}(kt) * d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)).$$

We claim that, for all  $n = 1, 2, \dots$ ,  $d_{2n}(t) \geq d_{2n-1}(kt)$ . In fact, if  $d_{2n}(t) \leq d_{2n-1}(kt)$ , then, we have

$$d_{2n}(t) \geq \phi (d_{2n}(t), d_{2n}(t) * d_{2n}(t), d_{2n}(t), d_{2n}(t)) > d_{2n}(t),$$

which is a contradiction. Hence  $d_{2n}(t) \geq d_{2n-1}(kt)$  for all  $n = 1, 2, \dots$  and  $t > 0$ .

Similarly, for  $m = 2n + 1$ , we have  $d_{2n+1}(t) \geq d_{2n}(kt)$  and so  $\{d_n(t)\}$  is increasing sequence, that is

$$\mathcal{M}(y_n, y_n, y_{n+1}, t) \geq \mathcal{M}(y_{n-1}, y_{n-1}, y_n, kt) \geq \dots \geq \mathcal{M}(y_0, y_0, y_1, k^n t).$$

and

Setting  $m' = 2n$ , then by putting  $x = x_{2n}$  and  $y = z = x_{2n+1}$  in (e), we have

$$\begin{aligned} d_{2n}(t) &= \mathcal{N}(y_{2n}, y_{2n+1}, y_{2n+1}, t) \\ &= \mathcal{N}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\leq \psi \left( \begin{array}{cc} \mathcal{N}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{N}(Sx_{2n}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{N}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{N}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right) \\ &= \psi \left( \begin{array}{cc} \mathcal{N}(y_{2n-1}, y_{2n}, y_{2n}, kt), & \mathcal{N}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), \\ \mathcal{N}(y_{2n}, y_{2n+1}, y_{2n+1}, kt), & \mathcal{N}(y_{2n+1}, y_{2n}, y_{2n}, kt) \end{array} \right) \\ &\leq \psi ( d_{2n-1}(kt), \mathcal{N}(y_{2n-1}, y_{2n}, y_{2n+1}, kt), d_{2n}(kt), d_{2n}(kt) ), \end{aligned}$$

that is,

$$d_{2n}(t) \leq \psi (d_{2n-1}(kt), d_{2n-1}(kt) * d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)).$$

We claim that, for all  $n = 1, 2, \dots$ ,  $d_{2n}(t) \leq d_{2n-1}(kt)$ . In fact, if  $d_{2n}(t) \geq d_{2n-1}(kt)$ , then, we have

$$d_{2n}(t) \leq \psi (d_{2n}(t), d_{2n}(t) * d_{2n}(t), d_{2n}(t), d_{2n}(t)) < d_{2n}(t),$$

which is a contradiction. Hence  $d_{2n}(t) \leq d_{2n-1}(kt)$  for all  $n = 1, 2, \dots$  and  $t > 0$ .

Similarly, for  $m' = 2n + 1$ , we have  $d_{2n+1}(t) \leq d_{2n}(kt)$  and so  $\{d_n(t)\}$  is decreasing sequence, that is

$$\mathcal{N}(y_n, y_n, y_{n+1}, t) \leq \mathcal{N}(y_{n-1}, y_{n-1}, y_n, kt) \leq \dots \leq \mathcal{N}(y_0, y_0, y_1, k^n t).$$

Hence by (2.10),  $\{y_n\}$  is Cauchy sequence and the completeness of  $X$ ,  $\{y_n\}$  converges to a point  $u$  in  $X$ . Let  $\lim_{n \rightarrow \infty} y_n = u$ . Hence we have

$$\begin{aligned} \lim_{n \rightarrow \infty} y_{2n} &= \lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} \\ &= \lim_{n \rightarrow \infty} y_{2n+1} = \lim_{n \rightarrow \infty} Bx_{2n+1} \\ &= \lim_{n \rightarrow \infty} Sx_{2n+2} = u. \end{aligned}$$

Let the pairs  $(A, S)$ ,  $(B, T)$  be compatible of type  $(\alpha)$  and by (c), we have

$$\begin{aligned} ASx_{2n+2} &\rightarrow Su & BTx_{2n+1} &\rightarrow Tu \\ S^2x_{2n+2} &\rightarrow Su & T^2x_{2n+1} &\rightarrow Tu \\ SAx_{2n} &\rightarrow Au & TBx_{2n+1} &\rightarrow Bu \\ A^2x_{2n} &\rightarrow Au & B^2x_{2n+1} &\rightarrow Bu. \end{aligned}$$

By putting  $x = Ax_{2n}$  and  $y = z = x_{2n+1}$ , if  $Au \neq u$ , we have

$$\begin{aligned} &\mathcal{M}(A^2x_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\geq \phi \left( \begin{array}{cc} \mathcal{M}(SAx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{M}(SAx_{2n}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{M}(Au, u, u, t) &\geq \phi \left( \begin{array}{cc} \mathcal{M}(Au, u, u, kt), & \mathcal{M}(Au, u, u, kt), \\ \mathcal{M}(u, u, u, kt), & \mathcal{M}(u, u, u, kt) \end{array} \right) \\ &> \mathcal{M}(Au, u, u, kt), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{N}(A^2x_{2n}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\leq \psi \left( \begin{array}{cc} \mathcal{N}(SAx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{N}(SAx_{2n}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{N}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{N}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{N}(Au, u, u, t) &\leq \psi \left( \begin{array}{cc} \mathcal{N}(Au, u, u, kt), & \mathcal{N}(Au, u, u, kt), \\ \mathcal{N}(u, u, u, kt), & \mathcal{N}(u, u, u, kt) \end{array} \right) \\ &< \mathcal{N}(Au, u, u, kt), \end{aligned}$$

which is a contradiction. Hence  $Au = u$ .

By putting  $x = x_{2n+2}$ ,  $y = Bx_{2n+1}$  and  $z = x_{2n+1}$ , if  $Bu \neq u$ , we have

$$\begin{aligned} &\mathcal{M}(Ax_{2n+2}, B^2x_{2n+1}, Bx_{2n+1}, t) \\ &\geq \phi \left( \begin{array}{cc} \mathcal{M}(Sx_{2n+2}, TBx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{M}(Sx_{2n+2}, TBx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(TBx_{2n+1}, B^2x_{2n+1}, B^2x_{2n+1}, kt), & \mathcal{M}(B^2x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{M}(u, Bu, u, t) &\geq \phi \left( \begin{array}{cc} \mathcal{M}(u, Bu, u, kt), & \mathcal{M}(u, Bu, u, kt), \\ \mathcal{M}(Bu, Bu, Bu, kt), & \mathcal{M}(Bu, u, u, kt) \end{array} \right) \\ &> \mathcal{M}(Bu, u, u, kt), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{N}(Ax_{2n+2}, B^2x_{2n+1}, Bx_{2n+1}, t) \\ &\leq \psi \left( \begin{array}{cc} \mathcal{N}(Sx_{2n+2}, TBx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{N}(Sx_{2n+2}, TBx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{N}(TBx_{2n+1}, B^2x_{2n+1}, B^2x_{2n+1}, kt), & \mathcal{N}(B^2x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{N}(u, Bu, u, t) &\leq \psi \left( \begin{array}{cc} \mathcal{N}(u, Bu, u, kt), & \mathcal{N}(u, Bu, u, kt), \\ \mathcal{N}(Bu, Bu, Bu, kt), & \mathcal{N}(Bu, u, u, kt) \end{array} \right) \\ &< \mathcal{N}(Bu, u, u, kt), \end{aligned}$$

which is a contradiction. Hence  $Bu = u$ .

By putting  $x = Sx_{2n+2}$  and  $y = z = x_{2n+1}$ , if  $Su \neq u$ , we have

$$\begin{aligned} &\mathcal{M}(ASx_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, t) \\ &\geq \phi \left( \begin{array}{cc} \mathcal{M}(S^2x_{2n+2}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{M}(S^2x_{2n+2}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{M}(Su, u, u, t) &\geq \phi \left( \begin{array}{cc} \mathcal{M}(Su, u, u, kt), & \mathcal{M}(Su, u, u, kt), \\ \mathcal{M}(u, u, u, kt), & \mathcal{M}(u, u, u, kt) \end{array} \right) \\ &> \mathcal{M}(Su, u, u, kt), \end{aligned}$$

and

$$\begin{aligned} & \mathcal{N}(ASx_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, t) \\ & \leq \psi \left( \begin{array}{cc} \mathcal{N}(S^2x_{2n+2}, Tx_{2n+1}, Tx_{2n+1}, kt), & \mathcal{N}(S^2x_{2n+2}, Tx_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{N}(Tx_{2n+1}, Bx_{2n+1}, Bx_{2n+1}, kt), & \mathcal{N}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{N}(Su, u, u, t) & \leq \psi \left( \begin{array}{cc} \mathcal{N}(Su, u, u, kt), & \mathcal{N}(Su, u, u, kt), \\ \mathcal{N}(u, u, u, kt), & \mathcal{N}(u, u, u, kt) \end{array} \right) \\ & < \mathcal{N}(Su, u, u, kt), \end{aligned}$$

which is a contradiction. Hence  $Su = u$ .

By putting  $x = x_{2n+2}$ ,  $y = Tx_{2n+1}$  and  $z = x_{2n+1}$ , if  $Tu \neq u$ , we have

$$\begin{aligned} & \mathcal{M}(Ax_{2n+2}, BTx_{2n+1}, Bx_{2n+1}, t) \\ & \geq \phi \left( \begin{array}{cc} \mathcal{M}(Sx_{2n+2}, T^2x_{2n+1}, Tx_{2n+1}, kt), & \mathcal{M}(Sx_{2n+2}, T^2x_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{M}(T^2x_{2n+1}, BTx_{2n+1}, BTx_{2n+1}, kt), & \mathcal{M}(BTx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{M}(u, Tu, u, t) & \geq \phi \left( \begin{array}{cc} \mathcal{M}(u, Tu, u, kt), & \mathcal{M}(u, Tu, u, kt), \\ \mathcal{M}(Tu, Tu, Tu, kt), & \mathcal{M}(Tu, u, u, kt) \end{array} \right) \\ & > \mathcal{M}(Tu, u, u, kt), \end{aligned}$$

and

$$\begin{aligned} & \mathcal{N}(Ax_{2n+2}, BTx_{2n+1}, Bx_{2n+1}, t) \\ & \leq \psi \left( \begin{array}{cc} \mathcal{N}(Sx_{2n+2}, T^2x_{2n+1}, Tx_{2n+1}, kt), & \mathcal{N}(Sx_{2n+2}, T^2x_{2n+1}, Bx_{2n+1}, kt), \\ \mathcal{N}(T^2x_{2n+1}, BTx_{2n+1}, BTx_{2n+1}, kt), & \mathcal{N}(BTx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, kt) \end{array} \right). \end{aligned}$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{N}(u, Tu, u, t) & \leq \psi \left( \begin{array}{cc} \mathcal{N}(u, Tu, u, kt), & \mathcal{N}(u, Tu, u, kt), \\ \mathcal{N}(Tu, Tu, Tu, kt), & \mathcal{N}(Tu, u, u, kt) \end{array} \right) \\ & < \mathcal{N}(Tu, u, u, kt), \end{aligned}$$

which is a contradiction. Hence  $Tu = u$ .

By combining the above results, we have  $Au = Bu = Su = Tu = u$ ; that is  $u$  is a common fixed point of  $A$ ,  $B$ ,  $S$  and  $T$ . For uniqueness, let  $w \neq u$  be another common fixed point of  $A$ ,  $B$ ,  $S$  and  $T$  then, By putting  $x = w$  and  $y = z = u$ , we write

$$\mathcal{M}(Aw, Bu, Bu, t) \geq \phi \left( \begin{array}{cc} \mathcal{M}(Sw, Tu, Tu, kt), & \mathcal{M}(Sw, Tu, Bu, kt), \\ \mathcal{M}(Tu, Bu, Bu, kt), & \mathcal{M}(Bu, Tu, Tu, kt) \end{array} \right),$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{M}(w, u, u, t) &\geq \phi \left( \begin{array}{cc} \mathcal{M}(w, u, u, kt), & \mathcal{M}(w, u, u, kt), \\ \mathcal{M}(u, u, u, kt), & \mathcal{M}(u, u, u, kt) \end{array} \right) \\ &> \mathcal{M}(w, u, u, kt), \end{aligned}$$

and

$$\mathcal{N}(Aw, Bu, Bu, t) \leq \psi \left( \begin{array}{cc} \mathcal{N}(Sw, Tu, Tu, kt), & \mathcal{N}(Sw, Tu, Bu, kt), \\ \mathcal{N}(Tu, Bu, Bu, kt), & \mathcal{N}(Bu, Tu, Tu, kt) \end{array} \right),$$

On making  $n \rightarrow \infty$ , we have

$$\begin{aligned} \mathcal{N}(w, u, u, t) &\leq \psi \left( \begin{array}{cc} \mathcal{N}(w, u, u, kt), & \mathcal{N}(w, u, u, kt), \\ \mathcal{N}(u, u, u, kt), & \mathcal{N}(u, u, u, kt) \end{array} \right) \\ &< \mathcal{N}(w, u, u, kt), \end{aligned}$$

which is a contradiction. Therefore, we have  $u = w$ . This completes the proof of the theorem.  $\blacksquare$

If we put  $S = A = I_x$  (the identity map on  $X$ ) in Theorem (3.6), we have the following:

**Corollary 3.7.** Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be a complete intuitionistic  $\mathcal{M}$ -fuzzy metric space and the condition in theorem (3.6). Let  $B$  and  $T$  be mappings from  $X$  into itself. such that, we have

$$\mathcal{M}(x, By, Bz, t) \geq \phi \left( \begin{array}{cc} \mathcal{M}(x, Ty, Tz, kt), & \mathcal{M}(x, Ty, Bz, kt), \\ \mathcal{M}(Ty, By, By, kt), & \mathcal{M}(By, Tz, Tz, kt), \end{array} \right),$$

and

$$\mathcal{N}(x, By, Bz, t) \leq \psi \left( \begin{array}{cc} \mathcal{N}(x, Ty, Tz, kt), & \mathcal{N}(x, Ty, Bz, kt), \\ \mathcal{N}(Ty, By, By, kt), & \mathcal{N}(By, Tz, Tz, kt), \end{array} \right)$$

for all  $x, y, z \in X, t > 0$ . Then  $B$  and  $T$  have a unique common fixed point in  $X$ .

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## References

- [1] Cho, Y.J., Fixed points in fuzzy metric spaces, J. Fuzzy Math., 5:949–62, 1997.

- [2] Dhage, B.C., Generalised metric spaces and mappings with fixed point, Bulletin of the Calcutta Mathematical Society, 84(4):329–336, 1992.
- [3] Jungck, G., Commuting maps and fixed points, Amer. Math. Monthly, 83:261–3, 1976.
- [4] Jungck, G., Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9:771–9, 1986.
- [5] Jungek, G., Murthy, P.P., Cho, Y.J., Compatible mappings of type (A) and common fixed points, Math. Jpn., 38:381–90, 1993.
- [6] Mishra, S.N., Sharma, S.N., Singh, S.L., Common fixed points of maps in fuzzy metric spaces, Int. J. Math. Math. Sci., 17:253–8, 1994.
- [7] Naidu, S.V.R., Rao, K.P.R., and Srinivasa Rao, N., On the topology of  $D$ -metric spaces and the generation of  $D$ -metric spaces from metric spaces, Internat. J. Math. Math. Sci., 51:2719–2740, 2004.
- [8] Naidu, S.V.R., Rao, K.P.R., and Srinivasa Rao, N., On the concepts of balls in a  $D$ -metric space, Internat. J. Math. Math. Sci., 1:133–141, 2005.
- [9] Naidu, S.V.R., Rao, K.P.R., and Srinivasa Rao, N., On convergent sequences and fixed point theorems in  $D$ -Metric spaces, Internat. J. Math. Math. Sci., 12: 1969–1988, 2005.
- [10] Schweizer, B., and Sklar, A., 1960, Statistical metric spaces. Pacific J. Math., 10:314–34, 1960.
- [11] Sedghi, S., and Shobe, N., 2006, Fixed point theorem in  $\mathcal{M}$ -fuzzy metric spaces with property(E). Advances in Fuzzy Mathematics, 1(1):55–65, 2006.
- [12] Sedghi, S., Shobe, N., and Zhou, H., 2007, A common fixed point theorems in  $D^*$ -metric spaces. Fixed point Theory and Applications, Volume 2007, Artical ID 27906, 13 pages.
- [13] Sedghi, S., Shobe, N., A common fixed point theorem in two  $\mathcal{M}$ -fuzzy metric spaces, Common Korean Math., 22: 513–526, 2007.
- [14] Soleimani, N., Intuitionistic  $\mathcal{M}$ -fuzzy metric space, Advances in Fuzzy Mathematics., 5(3):367—377, 2010.
- [15] Soleimani, N., A fixed point theorem in intuitionistic M-fuzzy metric spaces, Global Journal of Pure and Applied Mathematics., 7(2):159—170, 2011.