Advances in Fuzzy Mathematics
ISSN 0973-533X Volume 4, Number 2 (2009), pp. 167–179
© Research India Publications
http://www.ripublication.com/afm.htm

Fuzzy Context Free Grammars

[1]V. Ramaswamy and [2]Suvarna G Kanakaraddi

[1]Professor and Head, Dept of Information Science and Engineering, Bapuji Institute of Engineering and Technology, Davangere, Karnataka – 577 004, India.
E-mail id: researchwork04@yahoo.com
[2]Asst.Professor, Department of Computer Science and Engineering, BVB College of Engineering and Technology, Hubli, Karnataka, India.
E-mail id: suvarnabvb@yahoo.co.in

Abstract

Natural language is perhaps the most powerful tool which humans possess for conveying information. We know natural languages in general are full of ambiguities. For example, the same word in English can have different meanings depending on the context in which it is used. Consider for example, the sentences "He can do this job" and "He came with a can". To resolve such ambiguities, fuzzy logic is a more convenient tool.

In this paper, we have introduced the concepts of fuzzy context free grammar and fuzzy regular CFG. We prove that L is a fuzzy regular language if and only if L is generated by a fuzzy regular CFG. We illustrate the constructions with examples.

Keywords: fuzzy context free grammar, fuzzy automata, fuzzy regular language.

Introduction

Language is one of the fundamental aspects of human behavior and plays a very crucial role in our lives. Crucial component of understanding involves computing a representation of the meaning of sentences and texts. A person does not seem to consider all possible senses of a word while using in a sentence which he is able to do intuitively whereas a program must explicitly consider them one by one.

Fuzzy languages generalize the characteristic function $\chi : \Sigma^* \rightarrow \{0,1\}$ of a language L_0 over the alphabet Σ to the membership function $\chi : \Sigma^* \rightarrow [0, 1]$. Note that the set $\{0,1\}$ with just two elements has been replaced by the closed interval $[0,1]$ so that $\chi(x)$ can now take any real value between 0 and 1. Fuzzy context free grammar is a natural choice to generate a fuzzy context-free language. Fuzzy context free grammar G generates apart from the usual “correct strings” x for which $\mu_{L(G)}(x) = 1$, some “incorrect strings” x with $0 < \mu_{L(G)}(x) < 1$ which might arise due to grammatical errors. Erroneous inputs to a parser are assumed to be generated due to grammatical errors. We extend the original context free grammar with some additional rules which will result in a fuzzy context free grammar.

The paper is organized as follows:

Section 2 introduces some notations and basic definitions which are illustrated with examples. In section 3, we introduce fuzzy regular CFG and prove that L is a fuzzy regular language if and only if L is generated by a fuzzy regular CFG. Section 4 includes Appendix, we have illustrated the constructs involved in these proofs with examples includes concluding remarks.

2. Preliminaries

We define a fuzzy context free grammar (fuzzy CFG) to be a 4 – tuple

$G = (V, T, P, S)$ where V is a finite set whose elements are called variables, T is a finite set whose elements are called terminals, S is a special element of V called start symbol and P is a fuzzy subset of $V \times (V \cup T)^*$. Thus $P: V \times (V \cup T)^* \rightarrow [0, 1]$.

Example: Let $G = (V, T, P, S)$ where $V = \{S, A, B\}$, $T = \{a, b\}$ and P is given by $P(S, AB) = 0.6$, $P(A, aA) = 0.5$, $P(A, a) = 0.2$, $P(B, bB) = 0.3$ and $P(B, b) = 0.7$. We can represent P as follows.

$$S \xrightarrow{.6} AB,\ A \xrightarrow{.5} aA,\ A \xrightarrow{.2} a,\ B \xrightarrow{.3} bB,\ B \xrightarrow{.7} b.$$

Consider a derivation $D_1 : A \xrightarrow{\lambda_1} \alpha_1 \xrightarrow{\lambda_2} \alpha_2 \ldots \xrightarrow{\lambda_n} \alpha_n = \alpha$ where $A \in V$ and $\alpha_1, \alpha_2, \ldots \alpha_n \in (V \cup T)^*$. In this case, we say A derives α and denote it by writing $A \overset{*}{\Rightarrow} \alpha$.

We define $\mu(D_1) = \min(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Define

$\mu (A \overset{*}{\Rightarrow} \alpha \quad \max(\mu(D_1), \mu(D_2), \ldots \mu(D_k))$ where $D_1, D_2, \ldots D_k$ denote all possible derivations of α from A.

Example: For the above grammar, consider the following derivations of the string aabbb.

D_1: $S \xrightarrow{.6} AB \xrightarrow{.3} AbB \xrightarrow{.3} AbbB \xrightarrow{.7} Abbb \xrightarrow{.5} aAbbb \xrightarrow{.2} aabbb$. Then $\mu(D_1) = 0.2$.

D_2: $S \xrightarrow{.6} AB \xrightarrow{.5} aAB \xrightarrow{.2} aaB \xrightarrow{.3} aabB \xrightarrow{.3} aabbB \xrightarrow{.7} aabbb$. Again, we see that $\mu(D_2) = 0.2$.

3. Fuzzy Regular Language

Definition: Let $G = (V, T, P, S)$ be a fuzzy CFG. We define the fuzzy language generated by G denoted by $L(G)$ to be a fuzzy subset of T^*. If $t \varepsilon T^*$, then the degree of membership of t in $L(G)$ is given by $L(G)(t) = \mu(S \overset{*}{\Rightarrow} t)$.

Definition: $G = (V, T, P, S)$ is said to be fuzzy right linear if P is a fuzzy subset of $V \times (TV \cup \{\varepsilon\})$. In other words, $P: V \times TV \cup \{\varepsilon\} \rightarrow [0, 1]$. Note that $TV = \{av \,/\, a \varepsilon T \text{ and } v \varepsilon V\}$. G is said to be fuzzy left linear if P is a fuzzy subset of $V \times (VT \cup \{\varepsilon\})$. A fuzzy right linear or a fuzzy left linear CFG is said to be a fuzzy regular CFG.

Theorem 3.1: Let L be a fuzzy regular language. Then L can be generated by a fuzzy regular CFG.

Proof: Since L is a fuzzy regular language, $L = L(M)$ where $M = (Q, \Sigma, f, I, F)$ is a fuzzy automata. Let $Q = \{q_0, q_1, q_2, \ldots q_n\}$, $I = \{q_0\}$. Let $G = (V, T, P, S)$ where

$V = Q$, $T = \Sigma$, $S = q_0$ and P is given by $P(p, aq) = f(p, a, q)$, $P(p, \varepsilon) = F(p)$ for all $p, q \varepsilon Q$ and $a \varepsilon \Sigma$. We will first prove that $L(M)(a) = L(G)(a)$ for an input alphabet a. We have

$$L(M)(a) = [f(q_0, a, q_0) \wedge F(q_0)] \vee [f(q_0, a, q_1) \wedge F(q_1)] \vee [f(q_0, a, q_2) \wedge F(q_2)] \vee \ldots$$
$$[f(q_0, a, q_n) \wedge F(q_n)] = \alpha \text{ (say)}$$

Then $f(q_0, a, q_i) \wedge F(q_i) = \alpha$ for some i and $f(q_0, a, q_j) \wedge F(q_j) \leq \alpha$ for all $j \neq i$. Now consider the following derivations of a from q_0.

D_0: $q_0 \xrightarrow{f(q_0, a, q_0)} aq_0 \xrightarrow{F(q_0)} a\varepsilon = a$ with $\mu(D_0) = f(q_0, a, q_0) \wedge F(q_0) \leq \alpha$.

D_1: $q_0 \xrightarrow{f(q_0, a, q_1)} aq_1 \xrightarrow{F(q_1)} a\varepsilon = a$ with $\mu(D_1) = f(q_0, a, q_1) \wedge F(q_1) \leq \alpha$.

D_2: $q_0 \xrightarrow{f(q_0, a, q_2)} a\,q_2 \xrightarrow{F(q_2)} a\varepsilon = a$ with $\mu(D_2) = f(q_0, a, q_2) \wedge F(q_2) \leq \alpha$.

.

.

.

D_i: $q_0 \xrightarrow{f(q_0, a, q_i)} a\,q_i \xrightarrow{F(q_i)} a\varepsilon = a$ with $\mu(D_i) = f(q_0, a, q_i) \wedge F(q_i) = \alpha$.

.

.

D_n: $q_0 \xrightarrow{f(q_0, a, q_n)} a\,q_n \xrightarrow{F(q_n)} a\varepsilon = a$ with $\mu(D_n) = f(q_0, a, q_n) \wedge F(q_n) \leq \alpha$.

Hence $\mu(q_0 \overset{*}{\Rightarrow} a) = \max(\mu(D_0), \mu(D_1), \ldots \mu(D_i), \ldots \mu(D_n)) = \alpha$ proving that $L(M)(a) = L(G)(a)$.

We will now prove the result for a string of length 2 say ab. We have

$L(M)\ (ab) = [f\ (q_0, ab, q_0) \wedge F\ (q_0)] \vee\ [f\ (q_0, ab, q_1) \wedge F\ (q_1)]\ \nu$
$[f\ (q_0, ab, q_2) \wedge F\ (q_2)] \vee \ldots \vee [f\ (q_0, ab, q_n) \wedge F\ (q_n)] = \beta$ (say). Then
$f\ (q_0, ab, q_i) \wedge F\ (q_i) = \beta$ for some i and $f\ (q_0, ab, q_k) \wedge F\ (q_k) \leq \beta$ for all other k.

$f\ (q_0, ab, q_i) \wedge F\ (q_i) = \beta$ means either $f\ (q_0, ab, q_i) = \beta$ or $F\ (q_i) = \beta$. Assume first that $F\ (q_i) = \beta$. Then $f\ (q_0, ab, q_i) \geq \beta$. This means

$[f\ (q_0, a, q_0) \wedge f\ (q_0, b, q_i)] \vee\ [f\ (q_0, a, q_1) \wedge f\ (q_1, b, q_i)] \vee [f\ (q_0, a, q_2) \wedge f\ (q_2, b, q_i)] \vee \ldots$
$[f\ (q_0, a, q_n) \wedge f\ (q_n, b, q_i)] \geq \beta$. Hence $f\ (q_0, a, q_j) \wedge f\ (q_j, b, q_i)] \geq \beta$ for some j which means $f\ (q_0, a, q_j) \geq \beta$ and $f\ (q_j, b, q_j) \geq \beta$. Now consider the following derivation.

$$D:\ q_0 \xrightarrow{f(q_0,a,\,q_j)} aq_j \xrightarrow{f\,(q_j\,,b,\,q_i\,)} abq_i \xrightarrow{F(q_i)} ab\varepsilon = ab \text{ with } \mu\ (D) = \beta.$$

Any other derivation D' will be of the form

$$q_0 \xrightarrow{f(q_0,a,\,q_k)} aq_k \xrightarrow{f\,(q_k\,,b,\,q_l\,)} abq_l \xrightarrow{F(q_l)} ab\varepsilon = ab.$$

It suffices to show that $\mu\ (D') \leq \beta$. Suppose $\mu\ (D') > \beta$. Since
$\mu\ (D') = f(q_0,a, q_k) \wedge f\ (q_k\ ,b, q_l\ ) \wedge F(q_l)$, it follows that $f(q_0,a, q_k) > \beta$,
$f\ (q_k\ ,b, q_l\ ) > \beta$ and $F(q_l) > \beta$. We now have
$f\ (q_0, ab, q_l) = [f(q_0, a, q_0) \wedge f(q_0, b, q_l)] \vee [f(q_0, a, q_1) \wedge f(q_1, b, q_l)] \vee$
$[f(q_0, a, q_2) \wedge f(q_2, b, q_l)] \vee \ldots \vee [f(q_0, a, q_k) \wedge f(q_k, b, q_l)] \ldots \vee$
$[f(q_0, a, q_n) \wedge f(q_n, b, q_l)] > \beta$ since $f(q_0, a, q_k) \wedge f(q_k, b, q_l) > \beta$.

Hence $f\ (q_0, ab, q_l) \wedge F(q_l) > \beta$ so that $L(M)\ (ab) > \beta$. (Note that L(M) (ab) is the maximum of all such minimums). This is a contradiction since we are assuming that $L(M)\ (ab) = \beta$. This contradiction establishes the fact that $\mu\ (D') \leq \beta$.

Now assume that $f\ (q_0, ab, q_i) = \beta$ so that $F\ (q_i) \geq \beta$. Since
$f\ (q_0, ab, q_i) = \vee\ [f\ (q_0, a, q_r) \wedge f\ (q_r, b, q_i)] = \beta$, it follows that $f\ (q_0, a, q_s) \wedge f\ (q_s, b, q_i) = \beta$. Now consider the following derivation

$$E:\ q_0 \xrightarrow{f(q_0,a,\,q_s)} aq_s \xrightarrow{f\,(q_s\,,b,\,q_i\,)} abq_i \xrightarrow{F(q_i)} ab\varepsilon = ab \text{ with } \mu\ (E) = \beta.$$

The fact that $\mu\ (E') \leq \beta$ for any other derivation E' follows exactly as shown earlier.

It follows that $\mu\ (q_0 \overset{*}{\Rightarrow} ab) = \beta$. We thus obtain L(M) (ab) = L(G) (ab) .

Now take any string $t = t_1t_2\ldots t_mt_{m+1}$. We will prove that L(M) (t) = L(G) (t). We have

$$L(M)(t) = [f(q_0, t_1t_2\ldots t_mt_{m+1}, q_0) \wedge F(q_0)] \vee [f(q_0, t_1t_2\ldots t_mt_{m+1}, q_1) \wedge F(q_1)] \vee [f(q_0, t_1t_2\ldots t_mt_{m+1}, q_2) \wedge F(q_2)] \vee \ldots \vee [f(q_0, t_1t_2\ldots t_mt_{m+1}, q_n) \wedge F(q_n)]$$
$$= \rho \text{ (say)}$$

Then $f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) \wedge F(q_j) = \rho$ for some j and

$f(q_0, t_1t_2\ldots t_mt_{m+1}, q_k) \wedge F(q_k) \leq \rho$ for all other k.

$f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) \wedge F(q_j) = \rho$ means either $f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) = \rho$ or $F(q_j) = \rho$. First assume that $F(q_j) = \rho$ so that $f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) \geq \rho$. Hence

$[f(q_0, t_1t_2\ldots t_m, q_0) \wedge f(q_0, t_{m+1}, q_j)] \vee [f(q_0, t_1t_2\ldots t_m, q_1) \wedge f(q_1, t_{m+1}, q_j)] \vee [f(q_0, t_1t_2\ldots t_m, q_2) \wedge f(q_2, t_{m+1}, q_j)] \vee \ldots \vee [f(q_0, t_1t_2\ldots t_m, q_n) \wedge f(q_n, t_{m+1}, q_j)] \geq \rho$

so that $f(q_0, t_1t_2\ldots t_m, q_k) \wedge f(q_k, t_{m+1}, q_j) \geq \rho$ for some k. We have thus proved that

$f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) \geq \rho$ means $f(q_0, t_1t_2\ldots t_m, q_k) \geq \rho$ and $f(q_k, t_{m+1}, q_j) \geq \rho$. Similarly, we can prove that $f(q_0, t_1t_2\ldots t_m, q_k) \geq \rho$ implies $f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$ and

$f(q_l, t_m, q_k) \geq \rho$ for some l, $f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$ implies $f(q_0, t_1t_2\ldots t_{m-2}, q_r) \geq \rho$ and $f(q_r, t_{m-1}, q_l) \geq \rho$ for some r and so on. Proceeding like this, we obtain $f(q_0, t_1, q_{r1}) \geq \rho$,

$f(q_{r1}, t_2, q_{r2}) \geq \rho$, $f(q_{r2}, t_3, q_{r3}) \geq \rho$ and so on. Now consider the following derivation D.

$$q_0 \xrightarrow{f(q_0,t_1,q_{r1})} t_1q_{r1} \xrightarrow{f(q_{r1},t_2,q_{r2})} t_1t_2q_{r2} \xrightarrow{f(q_{r2},t_3,q_{r3})} t_1t_2t_3q_{r3} \to \ldots\to t_1t_2t_3\ldots t_{m-1}q_l$$
$$\xrightarrow{f(q_l,t_m,q_k)} t_1t_2t_3\ldots t_{m-1}t_mq_k \xrightarrow{f(q_k,t_{m+1},q_j)} t_1t_2t_3\ldots t_{m-1}t_mt_{m+1}q_j \xrightarrow{F(q_j)} t_1t_2t_3\ldots t_{m-1}t_mt_{m+1}\in$$

$$\mu(D) = f(q_0,t_1,q_{r1}) \wedge f(q_{r1},t_2,q_{r2}) \wedge f(q_{r2},t_3,q_{r3}) \wedge \ldots f(q_l, t_m, q_k) \wedge f(q_k,t_{m+1},q_j) \wedge F(q_j)$$
$= \rho$ (Note that $F(q_j) = \rho$ whereas each of the other terms is $\geq \rho$).

Now consider any other derivation D'.

$$q_0 \xrightarrow{f(q_0,t_1,r_1)} t_1r_1 \xrightarrow{f(r_1,t_2,r_2)} t_1t_2r_2 \xrightarrow{f(r_2,t_3,r_3)} t_1t_2t_3r_3 \to \ldots\to t_1t_2t_3\ldots t_mr_m$$
$$\xrightarrow{f(r_m,t_{m+1},r_{m+1})} t_1t_2t_3\ldots t_{m-1}t_mt_{m+1}r_{m+1} \xrightarrow{F(r_{m+1})} t_1t_2t_3\ldots t_{m-1}t_mt_{m+1}\in\; = t_1t_2t_3\ldots t_{m-1}t_mt_{m+1}$$

Suppose μ (D') = $f(q_0,t_1,r_1) \wedge f(r_1,t_2,r_2) \wedge f(r_2,t_3,r_3) \wedge \ldots f(r_m, t_{m+1}, r_{m+1}) \wedge F(r_{m+1}) > \rho$. Then $f(q_0,t_1,r_1) > \rho$, $f(r_1,t_2,r_2) > \rho$, $f(r_2,t_3,r_3) > \rho, \ldots.$ $f(r_m, t_{m+1}, r_{m+1}) > \rho$ and $F(r_{m+1}) > \rho$. Now $f(q_0, t_1t_2\ldots t_mt_{m+1}, r_{m+1}) \wedge F(r_{m+1}) \leq \rho$ implies $f(q_0, t_1t_2\ldots t_mt_{m+1}, r_{m+1}) \leq \rho$

since $F(r_{m+1}) > \rho$. Hence

$[f(q_0, t_1t_2\ldots t_m, q_0) \wedge f(q_0, t_{m+1}, r_{m+1})] \vee [f(q_0, t_1t_2\ldots t_m, q_1) \wedge f(q_1, t_{m+1}, r_{m+1})] \vee$

$[f(q_0, t_1t_2\ldots t_m, q_2) \wedge f(q_2, t_{m+1}, r_{m+1})] \vee \ldots. \vee [f(q_0, t_1t_2\ldots t_m, q_n) \wedge f(q_n, t_{m+1}, r_{m+1})] \leq \rho$ which means

$f(q_0, t_1t_2\ldots t_m, q_0) \wedge f(q_0, t_{m+1}, r_{m+1}) \leq \rho$, $f(q_0, t_1t_2\ldots t_m, q_1) \wedge f(q_1, t_{m+1}, r_{m+1}) \leq \rho$,

$f(q_0, t_1t_2\ldots t_m, q_2) \wedge f(q_2, t_{m+1}, r_{m+1}) \leq \rho, \ldots, f(q_0, t_1t_2\ldots t_m, q_n) \wedge f(q_n, t_{m+1}, r_{m+1}) \leq \rho$. This means the following.

$f(q_0, t_1t_2\ldots t_m, q_0) \leq \rho$ or $f(q_0, t_{m+1}, r_{m+1}) \leq \rho$

$f(q_0, t_1t_2\ldots t_m, q_1) \leq \rho$ or $f(q_1, t_{m+1}, r_{m+1}) \leq \rho$

$f(q_0, t_1t_2\ldots t_m, q_2) \leq \rho$ or $f(q_2, t_{m+1}, r_{m+1}) \leq \rho$

.

.

.

$f(q_0, t_1t_2\ldots t_m, q_n) \leq \rho$ or $f(q_n, t_{m+1}, r_{m+1}) \leq \rho$.

In particular, $f(q_0, t_1t_2\ldots t_m, r_m) \leq \rho$ or $f(r_m, t_{m+1}, r_{m+1}) \leq \rho$. Since $f(r_m, t_{m+1}, r_{m+1}) > \rho$, it follows that $f(q_0, t_1t_2\ldots t_m, r_m) \leq \rho$.

We have thus established that

$f(q_0, t_1t_2\ldots t_mt_{m+1}, r_{m+1}) \leq \rho$ implies $f(q_0, t_1t_2\ldots t_m, r_m) \leq \rho$. In a similar fashion, from

$f(q_0, t_1t_2\ldots t_m, r_m) \leq \rho$, we can conclude that $f(q_0, t_1t_2\ldots t_{m-1}, r_{m-1}) \leq \rho$. Proceeding like this, we will finally end up with $f(q_0,t_1,r_1) \leq \rho$, a contradiction. We thus have a derivation D with μ (D) = ρ and for any other derivation D', μ (D') $\leq \rho$.

Thus max $[\mu(q_0 \overset{*}{\Rightarrow} t)] = \rho$ proving that L(G) (t) = ρ.

Now assume that $f(q_0, t_1t_2\ldots t_mt_{m+1}, q_j) = \rho$ so that $F(q_j) \geq \rho$. Now

$[f(q_0, t_1t_2\ldots t_m, q_0) \wedge f(q_0, t_{m+1}, q_j)] \vee [f(q_0, t_1t_2\ldots t_m, q_1) \wedge f(q_1, t_{m+1}, q_j)] \vee [f(q_0, t_1t_2\ldots t_m, q_2) \wedge f(q_2, t_{m+1}, q_j)] \vee \ldots \vee [f(q_0, t_1t_2\ldots t_m, q_n) \wedge f(q_n, t_{m+1}, q_j)] = \rho$

so that $f(q_0, t_1t_2\ldots t_m, q_k) \wedge f(q_k, t_{m+1}, q_j) = \rho$ for some k. Then either

$f(q_0, t_1t_2\ldots t_m, q_k) = \rho$ or $f(q_k, t_{m+1}, q_j) = \rho$. Suppose $f(q_k, t_{m+1}, q_j) = \rho$.

Then $f(q_0, t_1t_2\ldots t_m, q_k) \geq \rho$. In other words,

$[f(q_0, t_1t_2\ldots t_{m-1}, q_0) \wedge f(q_0, t_m, q_k)] \vee [f(q_0, t_1t_2\ldots t_{m-1}, q_1) \wedge f(q_1, t_m, q_k)] \vee$

$[f(q_0, t_1t_2\ldots t_{m-1}, q_2) \wedge f(q_2, t_m, q_k)]$ v….$[f(q_0, t_1t_2\ldots t_{m-1}, q_n) \wedge f(q_n, t_m, q_k)] \geq \rho$ which means

$f(q_0, t_1t_2\ldots t_{m-1}, q_l) \wedge f(q_l, t_m, q_k) \geq \rho$ for some I. Hence $f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$ and

$f(q_l, t_m, q_k) \geq \rho$. We have thus proved that $f(q_0, t_1t_2\ldots t_m, q_k) \geq \rho$ implies

$f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$ and $f(q_l, t_m, q_k) \geq \rho$ for some I. Now starting with

$f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$, we can conclude $f(q_0, t_1t_2\ldots t_{m-2}, q_s) \geq \rho$ and $f(q_s, t_{m-1}, q_l) \geq \rho$ for some s. Proceeding like this, we obtain $f(q_0,t_1,r_1) \geq \rho$, $f(r_1,t_2,r_2) \geq \rho,\ldots$ Now consider the following derivation D.

$$q_0 \xrightarrow{f(q_0,t_1,r_1)} t_1r_1 \xrightarrow{f(r_1,t_2,r_2)} t_1t_2r_2 \rightarrow \ldots t_1t_2\ldots t_{m-2}q_s \xrightarrow{f(q_s,t_{m+1},q_l)} t_1t_2\ldots t_{m-2}\,t_{m-1}q_l \xrightarrow{f(q_l,t_m,q_k)}$$

$$t_1t_2\ldots t_{m-2}\,t_{m-1}t_mq_k \xrightarrow{f(q_k,t_{m+1},q_j)} t_1t_2\ldots t_{m-2}\,t_{m-1}t_mt_{m+1}\,q_j \xrightarrow{F(q_j)} t_1t_2\ldots t_{m-2}\,t_{m-}1t_mt_{m+1}$$

Then μ (D) = ρ. The fact that μ (D') > ρ for any other derivation D' follows exactly similar to the argument given earlier.

Now assume that $f(q_0, t_1t_2\ldots t_m, q_k) = \rho$ so that $f(q_k, t_{m+1}, q_j) \geq \rho$. Then either

$[f(q_0, t_1t_2\ldots t_{m-1}, q_l) = \rho, f(q_l, t_m, q_k) \geq \rho]$ or $[f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho, f(q_l, t_m, q_k) = \rho]$ for some I. Suppose $f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$, $f(q_l, t_m, q_k) = \rho$. Then it follows that

$f(q_0, t_1t_2\ldots t_{m-2}, q_j) \geq \rho$, $f(q_j, t_{m-1}, q_l) \geq \rho$ for some j. Thus $f(q_0, t_1t_2\ldots t_{m-1}, q_l) \geq \rho$ implies $f(q_0, t_1t_2\ldots t_{m-2}, q_j) \geq \rho$, $f(q_j, t_{m-1}, q_l) \geq \rho$ for some j. Similarly, starting with

$f(q_0, t_1t_2\ldots t_{m-2}, q_j) \geq \rho$, we can conclude that $f(q_0, t_1t_2\ldots t_{m-3}, q_m) \geq \rho$ and $f(q_m, t_{m-2}, q_j) \geq \rho$ for some m. Proceeding like this, we will end up with $f(q_0,t_1,s_1) \geq \rho$, $f(s_1,t_2,s_2) \geq \rho,\ldots$

$f(q_l, t_m, q_k) = \rho$ for some $s_1, s_2\ldots$. Now we can obtain a derivation D for which μ (D) = ρ and μ (D') > ρ for any other derivation D'.

Suppose $f(q_0, t_1t_2\ldots t_{m-1}, q_l) = \rho$, $f(q_l, t_m, q_k) \geq \rho$. Then

$\rho = f(q_0, t_1t_2\ldots t_{m-1}, q_l) = [f(q_0, t_1t_2\ldots t_{m-2}, q_0) \wedge f(q_0, t_{m-1}, q_l)] \vee [f(q_0, t_1t_2\ldots t_{m-2}, q_1) \wedge$

$f(q_1, t_{m-1}, q_l)] \vee [f(q_0, t_1t_2\ldots t_{m-2}, q_2) \wedge f(q_2, t_{m-1}, q_l)] \vee\ldots.[f(q_0, t_1t_2\ldots t_{m-2}, q_n) \wedge$

$f(q_n, t_{m-1}, q_l)]$ so that $f(q_0, t_1t_2\ldots t_{m-2}, q_j) \wedge f(q_j, t_{m-1}, q_l)] = \rho$ and hence

$f(q_0, t_1t_2\ldots t_{m-2}, q_j) = \rho$ or $f(q_j, t_{m-1}, q_l) = \rho$ for some j. If $f(q_0, t_1t_2\ldots t_{m-2}, q_j) = \rho$, then

$f(q_j, t_{m-1}, q_l) \geq \rho$ and we can similarly obtain $f(q_0, t_1t_2\ldots t_{m-3}, q_k) = \rho$ or $f(q_k, t_{m-2}, q_j) = \rho$ for some k. Repeating like this, we will obtain $f(q_r, t_l, q_s) = \rho$ whereas $f(q_i, t_j, q_k) \geq \rho$ for all other i, j and k. Obtaining a derivation D with μ (D) = ρ and μ (D') $\geq \rho$ for any other derivation D' follows exactly similar to the earlier argument.

Theorem 3.2: If L is generated by a fuzzy regular CFG, then L is a fuzzy regular language.

Proof: let G = (V, T, P, S) be a fuzzy regular grammar and let L = L(G). We will prove that L is a fuzzy regular language. For this, we have to define a fuzzy automata

$M = (Q, \Sigma, q_0, f, F)$ such that L = L(M). Take $Q = V$, $\Sigma = T$, $q_0 = S$, $F(q) = P(q, \varepsilon)$ for all $q \in Q$ and $f: Q \times \Sigma \times Q \rightarrow [0, 1]$ as $f(p, a, q) = P(p, aq)$ for all $p, q \in Q$ and $a \in \Sigma$. We will now prove that L = L(M).

Suppose $V = [v_0, v_1, v_2,\ldots v_n\}$ where $v_0 = S$. Consider an input alphabet a. We have

$$L(M)\ (a) = [f\ (v_0, a, v_0) \wedge F\ (v_o)] \vee [f\ (v_0, a, v_1) \wedge F\ (v_1)] \vee [f\ (v_0, a, v_2) \wedge F\ (v_2)] \vee \ldots [f\ (v_0, a, v_n) \wedge F\ (v_n)] = \alpha \text{ (say)}$$

Then $f\ (v_0, a, v_i) \wedge F\ (v_i)] = \alpha$ for some i $(0 \le i \le n)$ and $f\ (v_0, a, v_j) \wedge F\ (v_j) \le \alpha$ for all other j. Now consider the derivation

$$D: v_0 \xrightarrow{f\ (v_0, a, v_i)} av_i \xrightarrow{P(v_i\ , \varepsilon)} a\varepsilon = a.$$ We have $\mu\ (D) = f\ (v_0, a, v_i) \wedge P(v_i\ , \varepsilon)$
$= f\ (v_0, a, v_i) \wedge F\ (v_i)] = \alpha.$

Consider any other derivation say

$$D': v_0 \xrightarrow{f\ (v_0, a, v_j)} av_j \xrightarrow{P(v_j\ , \varepsilon)} a\varepsilon = a.$$ We have $\mu\ (D) = f\ (v_0, a, v_j) \wedge P(v_j\ , \varepsilon)$
$= f\ (v_0, a, v_j) \wedge F\ (v_j)] \le \alpha.$

Hence $L(a) = L(G)\ (a) = \vee\ \mu\ (D)$ (maximum is taken over all possible derivations of a from v_0)
$= \alpha = L(M)\ (a)$

Now consider a string of length m say $a_1a_2\ldots a_m$. We have

$$L(M)\ (a_1a_2\ldots a_m) = [f\ (v_0, a_1a_2\ldots a_m\ , v_0) \wedge F(v_0)] \vee [f\ (v_0, a_1a_2\ldots a_m\ , v_1) \wedge F(v_1)] \vee [f\ (v_0, a_1a_2\ldots a_m\ , v_2) \wedge F(v_2)] \vee \ldots. \vee [f\ (v_0, a_1a_2\ldots a_m\ , v_n) \wedge F(v_n)] = \beta \text{ (say)}$$

Then $f\ (v_0, a_1a_2\ldots a_m\ , v_k) \wedge F(v_k)] = \beta$ for some k and $f\ (v_0, a_1a_2\ldots a_m\ , v_j) \wedge F(v_j)] \le \beta$ for all $j \ne k$. This means either $f\ (v_0, a_1a_2\ldots a_m\ , v_k) = \beta$ or $F(v_k) = \beta$. First assume that $F(v_k) = \beta$ so that $f\ (v_0, a_1a_2\ldots a_m\ , v_k) \ge \beta$. Then

$$\beta \le [f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_0) \wedge f(v_0,a_m,v_k)] \vee [f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_1) \wedge f(v_1,a_m,v_k)] \vee [f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_2) \wedge f(v_2,a_m,v_k)] \vee \ldots \vee [f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_n) \wedge f(v_n, a_m, v_k)]$$

from which it follows that $f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_j) \wedge f(v_j, a_m, v_k) \ge \beta$ so that $f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_j) \ge \beta$ and $f(v_j, a_m, v_k) \ge \beta$ for some j. We have thus proved that $f\ (v_0, a_1a_2\ldots a_m\ , v_k) \ge \beta$ implies $f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_j) \ge \beta$ and $f(v_j, a_m, v_k) \ge \beta$ for some j. Starting with $f\ (v_0, a_1a_2\ldots a_{m-1}\ , v_j) \ge \beta$, we can similarly show that $f\ (v_0, a_1a_2\ldots a_{m-2}\ , v_i) \ge \beta$ and $f(v_i, a_m, v_k) \ge \beta$ for some i. Proceeding like this, we obtain $f(v_0,a_1,v_{r1}) \ge \beta$, $f(v_{r1},a_2,v_{r2}) \ge \beta$ and so on. Now consider the following derivation D.

$$vo \xrightarrow{f(v_0,a_1,v_{r1})} a_1v_{r1} \xrightarrow{f(v_{r1},a_2,v_{r2})} a_1a_2v_{r2} \rightarrow \ldots a_1a_2\ldots a_{m-1}v_j \xrightarrow{f(v_j,\ a_m,\ v_k)} a_1a_2\ldots a_{m-1}\ a_m\, v_k \xrightarrow{F(v_k)} a_1a_2\ldots a_{m-1}\ a_m \in\ = a_1a_2\ldots a_{m-1}\ a_m$$

$\mu\ (D) = f(v_0,a_1,v_{r1}) \wedge f(v_{r1},a_2,v_{r2}) \wedge \ldots \wedge f(v_j, a_m, v_k) \wedge F(v_k) = \beta$. The fact that $\mu\ (D') \ge \beta$ for any other derivation D' follows exactly similar to the earlier argument.

Now assume that $f(v_0, a_1a_2\ldots a_m, v_k) = \beta$ so that $F(v_k) \geq \beta$. Now

$\beta = f(v_0, a_1a_2\ldots a_m, v_k) = [f(v_0, a_1a_2\ldots a_{m-1}, v_0) \wedge f(v_0,a_m,v_k)] \vee [f(v_0, a_1a_2\ldots a_{m-1}, v_1) \wedge f(v_1,a_m,v_k)] \vee [f(v_0, a_1a_2\ldots a_{m-1}, v_2) \wedge f(v_2,a_m,v_k)] \vee\ldots \vee [f(v_0, a_1a_2\ldots a_{m-1}, v_n) \wedge f(v_n, a_m, v_k)]$ so that $f(v_0, a_1a_2\ldots a_{m-1}, v_j) \wedge f(v_j, a_m, v_k) = \beta$ for some j. This means either

$f(v_0, a_1a_2\ldots a_{m-1}, v_j) = \beta$ or $f(v_j, a_m, v_k) = \beta$. If $f(v_0, a_1a_2\ldots a_{m-1}, v_j) = \beta$, then $f(v_j, a_m, v_k) \geq \beta$ and exactly as above, we will get $f(v_0, a_1a_2\ldots a_{m-2}, v_l) = \beta$ or $f(v_l, a_{m-1}, v_j) = \beta$ for some il. Proceeding like this, we will end up with $f(v_i, a_j, v_k) = \beta$ for some i, j, k and for all other p, q, r, $f(v_p, a_q, v_r) \geq \beta$. We can now obtain a derivation D such that μ (D) = β and show that for any other derivation D', μ (D') $\geq \beta$. This completes the proof.

4. Conclusion

In this paper, we have introduced the concepts of fuzzy context-free grammar and fuzzy regular CFG. We have shown that the analog of the result "A language is regular if and only if it is generated by a regular CFG" holds good for our set up. We have illustrated the constructions with examples.

References

[1] Alfred V Aho, "Compilers Principles , Techniques, and Tools", Pearson Education, pp.191-217.

[2] John N Mordeson , " Fuzzy Automata and Languages", Chapman & Hall/CRC Crc Press company ,Washington D.C, pp. 127-137.

[3] James Allen, "Natural Language Understanding" ,Second edition, Pearson Education,2003, pp. 41-72.

[4] Timothy J Ross, Fuzzy Logic,McGraw-Hill,1997

[5] P.R.J Asveld, Fuzzy context free languages –part 2 Dept of Comp.Sci, Twente University of Technology Netherlands, 2005

Appendix

Following example illustrates Theorem 3.1.

Consider a fuzzy automata $M = (Q, \Sigma, f, I, F)$ where $Q = \{p, q, r, s\}$, $\Sigma = \{0, 1\}$, $I = \{p\}$, F is a fuzzy subset of Q given by F(q) = 0.2, F (r) = 0.4 and F (s) = 1. f is a fuzzy subset of $Q \times \Sigma \times Q$ given by the following.

f (p, 0, p) = 0.2, f (p, 0, q) = 0.3, f (p, 0, s) = 0.1, f (p, 1, q) = 0.4,
f (p, 1, r) = 0.5, f (p, 1, s) = 0.2, f (q, 0, p) = 0.1, f (q, 0, r) = 0.1,
f (q, 1, p) = 0.6, f (q, 1, q) = 0.3, f (r, 0, p) = 0.4, f (r, 0, q) = 0.1,
f (r, 0, r) = 0.3, f (r, 0, s) = 0.2, f (r, 1, p) = 0.1, f (r, 1, q) = 0.6,
f (s, 0, p) = 0.2, f (s, 0, q) = 0.1, f (s, 0, s) = 0.5, f (s, 1, p) = 0.4,

$f(s, 1, q) = 0.3$, $f(s, 1, r) = 0.2$, $f(s, 1, s) = 0.1$.

Let $L = L(M)$. We know L is a fuzzy regular language. We will now construct a fuzzy right linear grammar G such that $L = L(G)$. Let $G = (V, T, P, S)$ where
$V = Q$, $T = \Sigma$, $S = p$ and P is given by $P(p, aq) = f(p, a, q)$, $P(p, \varepsilon) = F(p)$ for all $p, q \ \varepsilon\ Q$ and $a \ \varepsilon\ \Sigma$. We have

$$L(M)(0) = [f(p, 0, p) \wedge F(p)] \vee [f(p, 0, q) \wedge F(q)] \vee [f(p, 0, r) \wedge F(r)] \vee [f(p, 0, s) \wedge F(s)]$$
$$= (0.2 \wedge 0) \vee (0.3 \wedge 0.2) \vee (0 \wedge 0.4) \vee (0.1 \wedge 1) = 0.2.$$

To determine $L(G)(0)$, we first determine all possible derivations of 0 from S. Various derivations are

$D_1: p \xrightarrow{.2} 0p \xrightarrow{0} 0\,\varepsilon = 0$ with $\mu(D_1) = 0$.

$D_2: p \xrightarrow{.3} 0q \xrightarrow{.2} 0\,\varepsilon = 0$ with $\mu(D_2) = 0.2$.

$D_3: p \xrightarrow{.1} 0s \xrightarrow{1} 0\,\varepsilon = 0$ with $\mu(D_3) = 0.1$.

It follows that $L(G)(0) = 0.2$.

Again,

$$L(M)(1) = [f(p, 1, p) \wedge F(p)] \vee [f(p, 1, q) \wedge F(q)] \vee [f(p, 1, r) \wedge F(r)] \vee [f(p, 1, s) \wedge F(s)]$$
$$= (0.4 \wedge 0.2) \vee (0.5 \wedge 0.4) \vee (0.2 \wedge 1) = 0.4.$$

Various derivations of 1 from S are

$D_1: p \xrightarrow{0} 1p \xrightarrow{0} 1\,\varepsilon = 1$ with $\mu(D_1) = 0$.

$D_2: p \xrightarrow{.4} 1q \xrightarrow{.2} 1\,\varepsilon = 1$ with $\mu(D_2) = 0.2$.

$D_3: p \xrightarrow{.5} 1r \xrightarrow{.4} 1\,\varepsilon = 1$ with $\mu(D_3) = 0.4$.

$D_4: p \xrightarrow{.2} 1s \xrightarrow{1} 1\,\varepsilon = 1$ with $\mu(D_4) = 0.2$.

It follows that $L(G)(1) = 0.4$.

Now consider the string 011. We have

$$f(p, 01, p) = [f(p, 0, p) \wedge f(p, 1, p)] \vee [f(p, 0, q) \wedge f(q, 1, p)] \vee [f(p, 0, r) \wedge f(r, 1, p)] \vee [f(p, 0, s) \wedge f(s, 1, p)] = 0.3.$$

$f(p, 01, q) = [f(p, 0, p) \wedge f(p, 1, q)] \vee [f(p, 0, q) \wedge f(q, 1, q)] \vee [f(p, 0, r) \wedge f(r, 1, q)] \vee [f(p, 0, s) \wedge f(s, 1, q)] = 0.3.$

$f(p, 01, r) = [f(p, 0, p) \wedge f(p, 1, r)] \vee [f(p, 0, q) \wedge f(q, 1, r)] \vee [f(p, 0, r) \wedge f(r, 1, r)] \vee [f(p, 0, s) \wedge f(s, 1, r)] = 0.2.$

$f(p, 01, s) = [f(p, 0, p) \wedge f(p, 1, s)] \vee [f(p, 0, q) \wedge f(q, 1, s)] \vee [f(p, 0, r) \wedge f(r, 1, s)] \vee [f(p, 0, s) \wedge f(s, 1, s)] = 0.2.$

$f(p, 011, p) = [f(p, 01, p) \wedge f(p, 1, p)] \vee [f(p, 01, q) \wedge f(q, 1, p)] \vee$
$[f(p, 01, r) \wedge f(r, 1, p)] \vee [f(p, 01, s) \wedge f(s, 1, p)] = 0.3.$

$f(p, 011, q) = [f(p, 01, p) \wedge f(p, 1, q)] \vee [f(p, 01, q) \wedge f(q, 1, q)] \vee$
$[f(p, 01, r) \wedge f(r, 1, q)] \vee [f(p, 01, s) \wedge f(s, 1, q)] = 0.3.$

$f(p, 011, r) = [f(p, 01, p) \wedge f(p, 1, r)] \vee [f(p, 01, q) \wedge f(q, 1, r)] \vee$
$[f(p, 01, r) \wedge f(r, 1, r)] \vee [f(p, 01, s) \wedge f(s, 1, r)] = 0.3.$

$f(p, 011, s) = [f(p, 01, p) \wedge f(p, 1, s)] \vee [f(p, 01, q) \wedge f(q, 1, s)] \vee$
$[f(p, 01, r) \wedge f(r, 1, s)] \vee [f(p, 01, s) \wedge f(s, 1, s)] = 0.2.$

Now

$L(M)(011) = [f(p, 011, p) \wedge F(p)] \vee [f(p, 011, q) \wedge F(q)] \vee [f(p, 011, r) \wedge F(r)] \vee$
$[f(p, 011, s) \wedge F(s)] = 0.3$

Some of the derivations of 011 from S = p are

$D_1: p \xrightarrow{.2} 0p \xrightarrow{.4} 01q \xrightarrow{.3} 011q \xrightarrow{.2} 011\varepsilon = 011$ with $\mu(D_1) = 0.2$.

$D_2: p \xrightarrow{.3} 0q \xrightarrow{.6} 01p \xrightarrow{.5} 011r \xrightarrow{.4} 011\varepsilon = 011$ with $\mu(D_2) = 0.3$.

$D_3: p \xrightarrow{.1} 0s \xrightarrow{.4} 01p \xrightarrow{.5} 011r \xrightarrow{.4} 011\varepsilon = 011$ with $\mu(D_3) = 0.1$.

We can see that $L(G)(011) = \mu(p \overset{*}{\Rightarrow} 011) = 0.3$.

Following example illustrates Theorem 3.2.

Consider the fuzzy CFG $G = (V, T, P, S)$ where $V = \{S, A, B\}$, $T = \{0, 1\}$ and $P: V \times TV \cup \{\varepsilon\} \rightarrow [0, 1]$ be given by $P(S, 0S) = 0.3$, $P(S, 1S) = 0.2$, $P(S, 0A) = 0.6$, $P(S, 0B) = 0.1$, $P(S, 1B) = 0.7$, $P(A, 1S) = 0.4$, $P(A, 0A) = 0.3$, $P(A, 0B) = 0.2$, $P(A, 1B) = 0.5$, $P(B, 0S) = 0.1$, $P(B, 0A) = 0.2$, $P(B, 1A) = 0.3$, $P(B, 1B) = 0.1$, $P(S, \varepsilon) = 0.4$, $P(A, \varepsilon) = 0.2$ and $P(B, \varepsilon) = 0.4$.

We define a fuzzy automata $M = (Q, \Sigma, I, f, F)$ where $Q = \{S, A, B\}$, $\Sigma = \{0, 1\}$, $I = \{S\}$, $F(S) = 0.4$, $F(A) = 0.2$ and $F(B) = 0.4$. $f: Q \times \Sigma \times Q \rightarrow [0, 1]$ is defined as $f(p, a, q) = P(p, aq)$ for all $p, q \in Q$ and $a \in \Sigma$. We have

$$L(M)(0) = I \text{ o } f_0 \text{ o } F = \bigvee_{t \, \varepsilon \, Q} [I(t) \wedge (f_0 \text{ o } F)(t)] = (f_0 \text{ o } F)(S).$$

(Note that $I(S) = 1$ and $I(p) = 0$ for all $p \neq S$). Thus

$$\begin{aligned} L(M)(0) &= [f_0(S, S) \wedge F(S)] \vee [f_0(S, A) \wedge F(A)] \vee [f_0(S, B) \vee F(B)] \\ &= [f(S, 0, S) \wedge F(S)] \vee [f(S, 0, A) \wedge F(A)] \vee [f(S, 0, B) \wedge F(B)] \\ &= (0.3 \wedge 0.4) \vee (0.6 \wedge 0.2) \vee (0.1 \wedge 0.4) = 0.3. \end{aligned}$$

Various derivations of 0 from S are

$$S \xrightarrow{.3} 0S \xrightarrow{.4} 0\,\varepsilon = 0$$

$$S \xrightarrow{.6} 0A \xrightarrow{.2} 0\,\varepsilon = 0$$

$$S \xrightarrow{.1} 0B \xrightarrow{.4} 0\,\varepsilon = 0$$

It follows that $L(G)(0) = 0.3$.

Now consider the string 11010. We have

$$\begin{aligned} f(S, 11, S) = {} & [f(S, 1, S) \wedge f(S, 1, S)] \vee [f(S, 1, A) \wedge (f(A, 1, S)] \vee \\ & [f(S, 1, B) \wedge f(B, 1, S)] = 0.2 \wedge 0.2 = 0.2. \\ f(S, 11, A) = {} & [f(S, 1, S) \wedge f(S, 1, A)] \vee [f(S, 1, A) \wedge (f(A, 1, A)] \vee \\ & [f(S, 1, B) \wedge f(B, 1, A)] = 0.7 \wedge 0.3 = 0.3. \end{aligned}$$

$$\begin{aligned} f(S, 11, B) = {} & [f(S, 1, S) \wedge f(S, 1, B)] \vee [f(S, 1, A) \wedge (f(A, 1, B)] \vee \\ & [f(S, 1, B) \wedge f(B, 1, B)] = (0.2 \wedge 0.7) \vee (0.7 \wedge 0.1) = 0.2. \\ f(S, 110, S) = {} & [f(S, 11, S) \wedge f(S, 0, S)] \vee [f(S, 11, A) \wedge (f(A, 0, S)] \vee \\ & [f(S, 11, B) \wedge f(B, 0, S)] = (0.2 \Lambda 0.3) \vee (0.2 \wedge 0.1) = 0.2. \end{aligned}$$

$$\begin{aligned} f(S, 110, A) = {} & [f(S, 11, S) \wedge f(S, 0, A)] \vee [f(S, 11, A) \wedge (f(A, 0, A)] \vee \\ & [f(S, 11, B) \wedge f(B, 0, A)] \\ = {} & (0.2 \wedge 0.6) \vee (0.3 \wedge 0.3) \vee (0.2 \wedge 0.2) = 0.3. \\ f(S, 110, B) = {} & [f(S, 11, S) \wedge f(S, 0, B)] \vee [f(S, 11, A) \wedge (f(A, 0, B)] \vee \\ & [f(S, 11, B) \wedge f(B, 0, B)] = (0.2 \wedge 0.1) \vee (0.3 \wedge 0.2) = 0.2. \end{aligned}$$

$$\begin{aligned} f(S, 1101, S) = {} & [f(S, 110, S) \wedge f(S, 1, S)] \vee [f(S, 110, A) \wedge (f(A, 1, S)] \vee \\ & [f(S, 110, B) \wedge f(B, 1, S)] = 0.3. \\ f(S, 1101, A) = {} & [f(S, 110, S) \wedge f(S, 1, A)] \vee [f(S, 110, A) \wedge (f(A, 1, A)] \vee \\ & [f(S, 110, B) \wedge f(B, 1, A)] = 0.2. \\ f(S, 1101, B) = {} & [f(S, 110, S) \wedge f(S, 1, B)] \vee [f(S, 110, A) \wedge (f(A, 1, B)] \vee \\ & [f(S, 110, B) \wedge f(B, 1, B)] = 0.3. \end{aligned}$$

$f(S, 11010, S) = [f(S, 1101, S) \wedge f(S, 0, S)] \vee [f(S, 1101, A) \wedge (f(A, 0, S)] \vee$
$[f(S, 1101, B) \wedge f(B, 0, S)] = 0.3.$

$f(S, 11010, A) = [f(S, 1101, S) \wedge f(S, 0, A)] \vee [f(S, 1101, A) \wedge (f(A, 0, A)] \vee$
$[f(S, 1101, B) \wedge f(B, 0, A)] = 0.3.$

$f(S, 11010, B) = [f(S, 1101, S) \wedge f(S, 0, B)] \vee [f(S, 1101, A) \wedge (f(A, 0, B)] \vee$
$[f(S, 1101, B) \wedge f(B, 0, B)] = 0.2.$

Now

$L(M)(11010) = [f(S, 11010, S) \wedge F(S)] \vee [f(S, 11010, A) \wedge F(A)] \vee$
$[f(S, 11010, B) \wedge F(B)]$
$= (0.3 \wedge 0.4) \vee (0.3 \wedge 0.2) \vee (0.2 \wedge 0.4) = 0.3.$

To compute L(G) (11010), we consider all possible derivations of 11010 starting with S.

$D_1: S \xrightarrow{.2} 1S \xrightarrow{.2} 11S \xrightarrow{.6} 110A \xrightarrow{.4} 1101S \xrightarrow{.3} 11010S \xrightarrow{.4} 11010\,\varepsilon = 11010$ with
$\mu(D_1) = 0.2.$

$D_2: S \xrightarrow{.7} 1B \xrightarrow{.3} 11A \xrightarrow{.3} 110A \xrightarrow{.4} 1101S \xrightarrow{.3} 11010S \xrightarrow{.4} 11010\,\varepsilon = 11010$ with
$\mu(D_2) = 0.3.$

$D_3: S \xrightarrow{.7} 1B \xrightarrow{.3} 11A \xrightarrow{.3} 110A \xrightarrow{.4} 1101S \xrightarrow{.1} 11010B \xrightarrow{.4} 11010\,\varepsilon = 11010$ with
$\mu(D_3) = 0.1.$

We can see that there is no other derivation whose μ value is greater than 0.3 proving that L(G) (11010) = 0.3 = L(M) (11010).