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Edgecoloring of a Fuzzy Graph

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Abstract

In this paper we consider edge coloring of a fuzzy graph. Analogous to vertex coloring given in [4], we introduce the concepts of (d, f) extended k-coloring and (d, f) edge chromatic number. With examples, we show that some of the results of edge coloring of a crisp graph do not carry over to our set up. We also develop an algorithm for determining the (d, f) edge chromatic number of a fuzzy graph.

Keywords: Fuzzy graph, Edge coloring of a fuzzy graph.

1. Introduction

Graph coloring is one of the most important problems of combinatorial optimization. Many problems of practical interest can be modeled as coloring problems. Edge coloring of a graph [1] refers to assigning colors to edges so that adjacent edges are differently colored. It is known that every graph G can be edge colored with at most δ (G) + 1 colors and at least δ (G) colors are always necessary where δ (G) denotes the maximum degree of G. Given a graph G = (V, E), a coloring function is a function C: $E \rightarrow N$ such that C (i, j) $\neq$ C (i, k) and C (i, j) $\neq$ C (l, j) for all edges (i, j), (i, k) and (l, j) $\in$ E. A k-coloring C^k is a coloring function in which no more than k different colors are used. In other words, $C^k: E \rightarrow \{1, 2, \ldots\ldots k\}$.

A graph is said to be edge k-colored, if it admits a k-coloring. The minimum value of k such that G is edge k-colored is called edge chromatic number of G and is denoted by χ (G). The edge coloring problem is concerned with determining the edge chromatic number of a graph and the associated edge coloring function. This paper deals with edge coloring of a fuzzy graph.

Fuzzy set theory, introduced by Zadeh [2] is a mathematical tool for handling uncertainties like vagueness, ambiguity and imprecision in linguistic variables. The first definition of fuzzy graph was proposed by Kaufmann [3] using the fuzzy relations introduced by Zadeh. Rosenfeld introduced yet another definition involving fuzzy nodes and fuzzy edges. In this paper, we consider graphs whose nodes are crisp but whose edges are fuzzy.

Let **X** be a universal set and **A** be a subset of X. A fuzzy set on A is a function $\mu_A : X \to [0, 1]$. If $x \in X$, then $\mu_A(x)$ denotes the degree to which x belongs to A.A fuzzy graph $G = (V, \mu, \rho)$ is a non empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\rho : V \times V \to [0,1]$ such that for all $x, y \in V$,

$$\rho(x, y) \le \min\{\mu(x), \mu(y)\}.$$

In this paper, we consider fuzzy graphs in which $\mu(v) = 1$ for all $v \in V$. Hence a fuzzy graph is a pair (V, ρ) where ρ is a fuzzy subset of $V \times V$.

Note that a fuzzy graph is a generalization of crisp graph in which

$$\mu(v) = 1 \quad \text{for all } v \in V$$

and

$$\rho(i, j) = 1 \quad \text{if } (i, j) \in E$$
$$= 0 \quad \text{otherwise.}$$

2. The (d, f) – extended coloring function of a fuzzy graph.

Following [4], we define dissimilarity or distance measure on the color set S as follows.

A dissimilarity measure defined on S is a function $d: S \times S \to [0,\infty)$ which satisfies the following properties for all $r, s \in S$.

1. $d(r, s) = 0 \Leftrightarrow r = s$
2. $d(r, s) = d(s, r)$

This dissimilarity measure d can take into account the incompatibility degree in the sense that the more incompatible two edges are, the more distant their associated colors are.

Consider a fuzzy graph $G = (V, \rho)$. Note that here $\mu(v) = 1$ for all $v \in V$. Also for $u, v \in V$, $\rho(u, v)$ need not take only real values between 0 and 1. $\rho(u, v)$ can assume fuzzy values such as low, high etc. Let I denote the image set of ρ. In other words, I denotes the set of all membership grades assigned to the edges. We assume there is a relation < defined on the elements of I.

Let $f: I \to [0, \infty)$ be a non decreasing function which means $f(\mu) \le f(\mu') \; \forall \mu, \mu' \in I$ such that $\mu < \mu'$, f is called a scale function.

For our purpose, we assume that a fuzzy graph G has five components namely (V, ρ, S, d, f) where V denotes the vertex set, ρ is a fuzzy subset of $V \times V$, which can assume fuzzy values, S denotes the available color set, d denotes the dissimilarity measure and f denotes the scale function. We take I = Image (ρ).

The dissimilarity measure and scale function introduced above lead to the following definition.

2.1 Definition

Given a fuzzy graph G = (V, ρ, S, d, f), a (d, f) extended coloring function of G, denoted by $C_{d,f}$ or simply as C is a mapping C : E → S satisfying the following.

$d(C(i, j), C(i, l)) \geq \wedge \{ f(\rho_{i,j}), f(\rho_{i,l}) \}$ for all edges (i, j) and (i, l) .

$d(C(i, j), C(l, j)) \geq \wedge \{ f(\rho_{i,j}), f(\rho_{l,j}) \}$ for all edges (i, j) and (l, j) .

A (d, f) extended k- coloring $C^k_{d,f}$ or simply C^k, is a (d, f) extended coloring function which takes maximum k different colors.

In other words,

$C^k : E \rightarrow S$ where S = {1, 2,……….k} which satisfies the following

$$\left.\begin{array}{l} d(C^k(i, j), C^k(i, l)) \geq \wedge \{f(\rho_{i,j}), f(\rho_{i,l})\} \text{ for all edges (i, j) and (i, l) .} \\ d(C^k(i, j), C^k(l, j)) \geq \wedge \{f(\rho_{i,j}), f(\rho_{l,j})\} \text{ for all edges (i, j) and (l, j) .} \end{array}\right\} \quad (1)$$

Note that the (d, f) extended k- coloring of a fuzzy graph G is nothing but a generalization of the

k-coloring of a crisp graph G = (V, E).Take I = {0, 1}, f (0) = 0, f(1) = 1 and d = d° where d°is defined as

$$d^\circ(r, s) = \begin{cases} 1 & \text{if } r \neq s \\ 0 & \text{if } r = s \end{cases}$$

Edge coloring of a fuzzy graph differs from edge coloring of a crisp graph in the following sense.

Given a crisp graph G = (V, E), an edge coloring function for G always exists (note that $\chi(G) \leq \delta(G) + 1$) whereas given a fuzzy graph G = (V, ρ, S, d, f), a (d, f) – extended coloring function for G **need not** exist as the following example shows.

Example

Let G = (V, ρ, S, d, f) be a fuzzy graph where V = {A, B, C, D}. Let I = Image (ρ) = {n, l, m, h} where

n, l, m and h denote null, low, medium and high respectively. We assume that n < l < m < h.

ρ is defined by the following matrix.

$$\rho = \begin{pmatrix} \_ & h & m & m \\ h & \_ & l & h \\ m & l & \_ & l \\ m & h & l & \_ \end{pmatrix}$$

Let S = {1, 2, 3, 4} be the color set and **d** be the dissimilarity measure defined as d (r, s) = |r – s|. Let the scale function f be defined as follows

Table 1: Scale function.

I	n	l	m	h
f(I)	0	1	2	3

This fuzzy graph is shown in the figure given below.

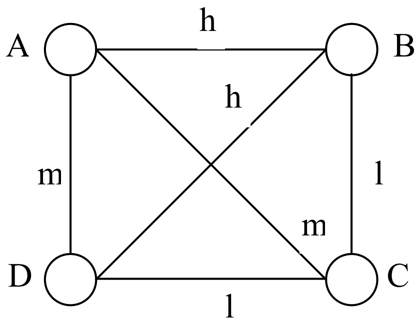

Figure 1: Fuzzy graph.

Consider the following cases.

Case 1: C^4 (A, B) = 1 and C^4 (A, C) = 3. This assignment of colors is possible because d (C^4 (A, B) , C^4 (A, C)) = 2 whereas $\wedge$ (f ($\rho_{A,B}$) , f ($\rho_{A,C}$)) = 2.
It is not possible to color (A, D) with either 2 or 4 because d(C^4 (A, C) , C^4 (A, D)) = 1 when C^4 (A, D) = 2 or when C^4 (A, D) = 4.
However $\wedge$ (f ($\rho_{A, C}$) , f ($\rho_{A, D}$)) is 2 . We thus note that equation (1) is not satisfied.

Case 2: C^4 (A, B) = 1 and C^4 (A, C) = 4, This assignment of colors is possible because d (C^4 (A, B) , C^4 (A, C)) = 3 whereas $\wedge$ (f ($\rho_{A, B}$) , f ($\rho_{A, C}$)) is 2
It is not possible to color (A, D) with either 2 or 3.
Note that d (C^4(A, C), C^4 (A, D)) = 1 when C^4 (A, D) = 3 whereas $\wedge$ (f ($\rho_{A, C}$), f ($\rho_{A, D}$)) is 2
d (C^4 (A, B), C^4 (A, D)) = 1 when C^4(A, D) = 2 whereas $\wedge$(f ($\rho_{A, B}$), f ($\rho_{A, D}$)) is 2 violating equation (1).

Case 3: C^4 (A, B) = 2 and C^4 (A, C) = 4, This assignment of colors is possible because d (C^4 (A, B), C^4 (A, C)) = 2 whereas $\wedge$(f ($\rho_{A, B}$), f ($\rho_{A, C}$)) = 2.
Now it is not possible to color (A, D) with either 1 or 3.
Note that d (C^4 (A, C), C^4(A, D)) = 1 when C^4(A, D) = 3 whereas $\wedge$(f ($\rho_{A,C}$), f ($\rho_{A,D}$)) is 2
d (C^4 (A, B), C^4 (A, D)) = 1 when C^4(A, D) = 1, whereas $\wedge$(f ($\rho_{A,B}$), f ($\rho_{A,D}$)) is 2 again violating equation (1).

Case 4: C^4(A, B) =1 and C^4(A, C) = 2 or C^4(A, B) = 2 and C^4(A, C) = 3 or C^4(A, B) = 3 and C^4(A, C) = 4.

Above assignments of colors are not possible because in each case

$d(C^4(A, B), C^4(A, C)) = 1$ whereas $\wedge(f(\rho_{A,B}), f(\rho_{A,C})) = 2$ which violates equation (1).

Thus it is not possible to color the above graph.

In a (d, f) – extended k coloring, it might happen that there are some colors which are not assigned to any of the edges as the following example shows.

Example

Let G = (V, ρ, S, d, f) be a fuzzy graph where V = {A, B, C, D, E}, I = Image (ρ) = {n, l, m, h} where n, l, m and h have the same definitions as in the above example. ρ is defined by the following matrix.

$$\rho = \begin{pmatrix} - & l & l & n & m \\ l & - & m & n & h \\ l & m & - & h & n \\ n & n & h & - & m \\ m & h & n & m & - \end{pmatrix}$$

Let S = {1, 2, 3, 4, 5} be the color set and **d** be the dissimilarity measure defined as $d(r, s) = |r - s|$

Let the scale function f be defined as follows

Table 2: Scale function.

I	n	l	m	h
f(I)	0	1	2	3

This fuzzy graph is shown in the figure given below.

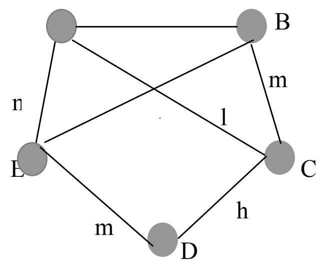

Figure 2: Fuzzy graph.

A (d, f) –extended 5-coloring is $C^5(A, B) = 1$;$C^5(A, C) = 2$;$C^5(A, E) = 3$; $C^5(B, C) = 3$; $C^5(B, E) = 5$;$C^5(C, D) = 5$; $C^5(D, E) = 1$.

Note that color 4 is not assigned to any of the edges of G.

The (d, f) chromatic number of a fuzzy graph is defined similar to the crisp case.

Definition: For a given fuzzy graph G = (V, ρ, S, d, f), the **minimum** value of k for which a (d, f) –extended k –coloring exists is called the (d, f) edge chromatic number of G and is denoted by $\chi_{d,f}$ (G).

Let G = (V, ρ, S, d, f) be a fuzzy graph where V = {A, B, C, D, E}, I = Image (ρ) = {n, l, m, h}, where n, l m, h are defined exactly the same way as in the above examples.

ρ is defined by the following matrix.

$$\rho = \begin{pmatrix} - & m & m & l & m \\ m & - & l & m & m \\ m & l & - & h & l \\ l & m & h & - & l \\ m & m & l & l & - \end{pmatrix}$$

Let S = {1, 2, 3, 4, 5} be the color set and **d** be the dissimilarity measure defined as follows.

Table 3: Color table.

	1	**2**	**3**	**4**	**5**
1	0	1	2	3	2
2	1	0	1	2	3
3	2	1	0	1	2
4	3	2	1	0	1
5	2	3	2	1	0

The scale function f is defined below.

Table 4: Scale function.

I	n	l	m	h
f(I)	0	1	2	3

The fuzzy graph is represented in the figure below.

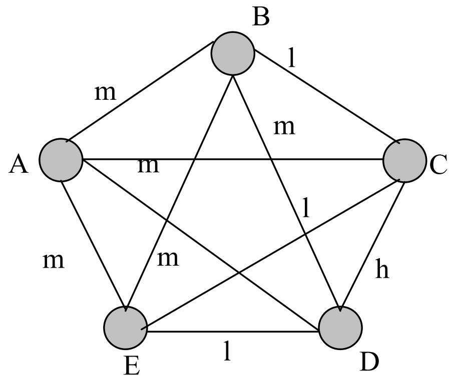

Figure 3: Fuzzy graph.

A (d, f) –extended 5-coloring is
C^5(A, B) = 1 ; C^5(A, C) = 3 ; C^5(A, D) = 2 ; C^5(A, E) = 5
C^5(B, C) = 4; C^5(B, D) = 5 ; C^5(B, E) = 3
C^5(C, D) = 1 ; C^5(C, E) = 2; C^5(D, E) = 4

Note that all the **five colors** are used and the (d, f) edge chromatic number $\chi_{d,f}$ (G) is 5.

3. Algorithm to determine the (d, f) edge chromatic number

A program has been written in VC++ to determines the (d, f) edge chromatic number χd,f (G) of the graph G of **n** vertices.

The graph is read in the form of adjacent matrix **adjMatrix [ ][ ].** The membership grades on the edges of the graph is read in terms of two dimensional array **weight [ ][ ].** The number of colors **k** provided to color the edges of the graph and their color difference table (**colorTable[ ][ ]**) are read.

Class **node** with **'index'** and **'deg'** as member variables is created. class **edge** with 'a','b' and 'color' as member variables is created. The vertices are sorted in descending order and stored in vertex array. An array of edges **e [totaldeg]** is created and array **e[enumber]** is obtained after removing the redundant edges.

The program calls the procedure **grapgColor (adjMatrix, n, k)** to properly color the graph. This procedure in turn calls the procedure **edgeColor (e, eNumber, 0),** to start coloring the edges of the graph. The importance of this procedure is that, it is **recursively** called to color all the edges of the graph. For each edge, it get list of colors from procedure getavailcol ( ),after removing all the color already used to color the edges incident on the vertex considered and by removing those color which does not satisfy the color difference table .Color the edge with first available color (preferred color). If not possible to color any particular edge, it **backtrack** to change the color of the already colored **previous edge** with next preferred color. The procedure edgecolor ( ) returns true to grapgcolor ( ) only when all edges are colored. Colors used to color the edges are now stored in the **finalcolorset [ ]** array.

sub edgeColor (edge e, eNumber, index) // To color all the edges of the graph
get number of available colors availcolor
for (ic = 1 up to (ic ≤ availColor **and** !colored (e,eNumber) **and** index < eNumber))
for (i = index to i < eNumber)
e[i].color = -1 // Undo wrong choice
end for
c = getAvailcolor (e, index, n, ic, k+1)
if (c = = -1) break
e [index] .color = c // color e[index] with color c and update colMatrix
edgeColor (e, eNumber, index + 1) // call recursively edgeColor by incrementing index
end for
if (edge in e is colored)
return true // returns to **grapgColor**
else
return false // back track to edgeColor to color previous edge
end sub

Example1: Consider a graph with 5 vertices whose adjacency matrix is given below:

Table 5: Adjacency Matrix.

	A	**B**	**C**	**D**	**E**
A	0	1	0	1	1
B	1	0	1	0	0
C	0	1	0	1	1
D	1	0	1	0	1
E	1	0	1	1	0

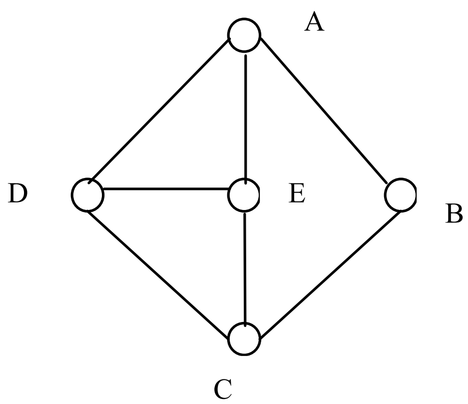

Figure 4: Fuzzy graph.

The membership grades on the edges and the color difference table are given by the following tables:

Table 6: Membership value table.

	A	**B**	**C**	**D**	**E**
A	0.0	0.2	0.0	0.4	0.6
B	0.2	0.0	0.4	0.0	0.0
C	0.0	0.4	0.0	0.6	0.4
D	0.4	0.0	0.6	0.0	0.2
E	0.6	0.0	0.4	0.2	0.0

Table 7: Color table.

	1	**2**	**3**	**4**	**5**
1	0.0	0.8	0.5	0.2	0.4
2	0.8	0.0	1.0	0.4	0.2
3	0.5	1.0	0.0	0.2	0.6
4	0.2	0.4	0.2	0.0	0.9
5	0.4	0.2	0.6	0.9	0.0

Outputs

The edge (A, B) and (C, D) are colored with 1
The edge (A, D) and (C, E) are colored with 2
The edge (A, E) and (C, B) are colored with 5
The edge (D, E) is colored with 3
The (d, f) edge chromatic number is **4** and the colors used are 1, 2, 3 and 5.

4. Conclusion

We note that the (d, f) edge chromatic number heavily depends on the d function and f function. We are now studying the relationship between (d, f) edge chromatic number, d function and f function. We are also attempting to develop edge coloring of a fuzzy graph, chromatic number etc without considering the d and f functions.

References

[1] PEARLS in GRAPH THEORY, A comprehensive Introduction Nora Hartsfield, Gerhard Ringel, Academic press, Inc.

[2] Zadeh LA. Similarity relations and fuzzy ordering. Information Sciences 971; 3(2):177–200.

[3] Kaufmann A. Introduction a la theorie des sous-ensembles flous, Elements theoriques de base. Paris: Masson et Cie;1976.

[4] Susana Mu˜noz,M. Teresa Ortu˜no, Javier Ramirez, Javier Ya˜nez Coloring fuzzy graphs, Omega 33 (2005) 211 – 221