Advances in Fuzzy Mathematics
ISSN 0973-533X Volume 4, Number 1 (2009), pp. 41–47

http://www.ripublication.com/afm.htm

An extension of Myhill Nerode Theorem for Fuzzy Automata

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Abstract

In this paper, an attempt has been made to prove the analogue of Myhill Nerode Theorem for fuzzy automata. Though as in finite automata case, it does not help us in minimizing the number of states in a fuzzy automaton, nevertheless it contains very interesting results.

Keywords: Monoid, Non deterministic automaton, equivalence class, fuzzy regular language, fuzzy automaton.

1. Introduction

Myhill Nerode theorem for finite automata

Myhill Nerode theorem for finite automata is a very powerful tool for minimizing the number of states in a finite automaton. The statement of the theorem is as follows. Following statements are equivalent.

(i) L is a regular language.
(ii) L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.
(iii) Define a relation R_L as x R_L y if and only if for all $z \in \Sigma^*$, $xz \in L$ only when $yz \in L$. Then R_L is an equivalence relation of finite index.

Definition of Fuzzy Automata

Let A be a finite non empty set.

A fuzzy automaton over A is a 4-tuple M = (Q, f, I, F) where Q is a finite nonempty set, f is a fuzzy subset of Q x A x Q, I and F are fuzzy subsets of Q.

Let S be a free monoid with identity element e generated by A. If $s \in S$, then $s = a_1a_2\ldots a_n$ where $a_i \in A$. Here n is called the length of s and we write $|s| = n$.

We extend f to a function f*: Q x S x Q → [0,1] which is defined as follows.

f*(q, e, p) = 1 if q = p
= 0 otherwise.

f*(q, sa, p) = $\vee$ [f*(q, s, r) $\wedge$ f (r, a, p)] (s $\in$ S, a $\in$ A)
r $\in$ Q

Theorem

For any two elements s, t $\in$ S and for all p, q $\in$ Q,

f*(p, st, q) = $\vee$ [f*(p, s, r) $\wedge$ f *(r, t, q)].
r $\in$ Q

Proof: Standard

Hereafter, we will assume that S is a free monoid with identity element e generated by a finite non empty set A.

Definition

If M is a fuzzy automaton over S, then we define the language accepted by M denoted by L(M) to be a fuzzy subset of S defined as $\forall$ s $\in$ S , L(M) (s) = I o f_s* o F. Here o denotes max – min composition and f_s*: Q x Q → [0,1] is defined as f_s* (p, q) = f* (p, s, q) for all p, q $\in$ Q.

Definition

A fuzzy subset L of S is said to be a fuzzy regular language if L = L (M) where M is a fuzzy automaton over S.

Lemma

Let X be any set and R_1 and R_2 be two equivalence relations of finite index over X. Then $R_1 \cap$ R2 is again an equivalence relation of finite index.

Proof

Clearly, R1 $\cap$ R2 is an equivalence relation. To prove $R_1 \cap R_2$ is of finite index, assume that the equivalence classes corresponding to R_1 are E_{11}, E_{12},…E_{1m} and the equivalence classes corresponding to R_2 are E_{21}, E_{22},…E_{2n} . Then X = E_{11} U E_{12} U …U E_{1m} = E_{21} U E_{22} U …U E_{2n} . Also $E_{1i} \cap E_{1j} = \phi$ if $i \neq j$ and $E_{2p} \cap E_{2q} = \phi$ if $p \neq q$. If x $\in$ X, then x $\in$ E_{1r} for a unique r and x $\in$ E_{2s} for a unique s ($1 \leq r \leq m$, $1 \leq s \leq n$). If R = $R_1 \cap$ R2, then the equivalence class of x relative to R denoted by $[x]_R$ is nothing but $E_{1r} \cap E_{2s}$. Thus every equivalence class of R appears as the intersection of an E_{1k} with an E_{2l} ($1 \leq k \leq m$, $1 \leq l \leq n$). Since the number of such intersections is finite, it follows that R is of finite index.

The above lemma can be extended to n relations and we can prove that if R_1, R_2,…R_n are equivalence relations of finite index over X, then $R_1 \cap R_2 \cap \ldots \cap R_n$ is again an equivalence relation of finite index.

2. Myhill Nerode Theorem for Fuzzy automata

Let S be a monoid with identity element e and L be a fuzzy subset of S. Then the following statements are equivalent.

(i) L is a fuzzy regular language.

(ii) L can be expressed as a fuzzy union

$$L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L$$

where $\delta_1, \delta_2, \ldots \delta_t \in [0, 1]$. For each $i = 1,2\ldots t$, $(\delta i)_L = \delta i. L_{\delta i}$ where $L_{\delta i} = U\ [s]_{\delta i}$.

This union is a set theoretic union and $[s]_{\delta i}$ denotes the equivalence class of s of a right invariant equivalence relation of finite index in $L_{\delta i}$.

(iii) Define a relation R_L as follows.

If s, t ε S, then s R_L t if and only if for all $u \in S$ and for all $\alpha \in [0,1]$, $L(su) \geq \alpha$ only when $L(tu) \geq \alpha$. Then R_L is a right invariant equivalence relation of finite index.

Proof of (i) → (ii)

Since L is a fuzzy regular language, we have L = L(M) where M = (Q, f*, I, F) is a fuzzy automaton. Consider any α ε [0,1]. With M and α, we associate a non deterministic automaton

$D_\alpha(M) = (Q, d_\alpha, I_\alpha, F_\alpha)$ where d_α: Q x S $\rightarrow 2^Q$ is defined as $d_\alpha(q, s) = \{p \in Q \mid f^*(q, s, p) \geq \alpha\}$, $I_\alpha = \{p \in Q \mid I(p) \geq \alpha\}$ and $F_\alpha = \{p \in Q \mid F(p) \geq \alpha\}$. We will show that $L_\alpha = L(D_\alpha(M))$ which will prove that L_α is a regular language. Let $s \in L_\alpha$.

Then $L(s) = L(M)(s) \geq \alpha$. ie $I \circ f_s^* \circ F \geq \alpha$ which means

$\vee [(f_s^* \circ F)(p) \wedge I(p)] \geq \alpha$ (maximum is taken over all $p \in Q$).

Hence $(f_s^* \circ F)(p) \wedge I(p) \geq \alpha$ for some $p \in Q$ so that $(f_s^* \circ F)(p) \geq \alpha$ and $I(p) \geq \alpha$ proving that $p \in I_\alpha$.

Again $\alpha \leq (f_s^* \circ F)(p)$

$= \vee [f_s^*(p, r) \wedge F(r)]$ (maximum is taken over all $r \in Q$)

and hence $F(r) \wedge f_s^*(p, r) \geq \alpha$ so that $F(r) \geq \alpha$ implying that $r \in F_\alpha$ and $f_s^*(p, r) \geq \alpha$ implying that $f^*(p, s, r) \geq \alpha$. Thus $r \in d_\alpha(p, s)$.

We have thus proved that $\exists\ p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$ proving that $s \in L(D\alpha(M))$.

Thus $L_\alpha \subseteq L(D_\alpha(M))$.

Conversely, let $s \in L(D_\alpha(M))$. Then there exists $p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$.

Let $q \in d_\alpha(p, s) \cap F_\alpha$. Now $p \in I_\alpha$ means $I(p) \geq \alpha$, $q \in d_\alpha(p, s)$ means $f^*(p, s, q) \geq \alpha$.

i.e $f_s^*(p, q) \geq \alpha$. $q \in F_\alpha$ means $F(q) \geq \alpha$.

Hence $f_s^*(p, q) \wedge F(q) \geq \alpha$ so that

$(f_s^* \circ F)(p) = \vee [f_s^*(p, r) \wedge F(r)] \geq \alpha$ (maximum is taken over all $r \in Q$).

Again $I(p) \wedge (f_s^* \circ F)(p) \geq \alpha$ means

$I \circ f_s^* \circ F = \vee [I(t) \wedge (f_s^* \circ F)(t)] \geq \alpha$. (maximum is taken over all $t \in Q$).

Hence $L(M)(s) = L(s) \geq \alpha$ proving that $s \in L_\alpha$. Thus $L(D_\alpha(M)) \subseteq L_\alpha$.

This together with $L_\alpha \subseteq L(D_\alpha(M))$ proves that $L_\alpha = L(D_\alpha(M))$.

Let $Q = \{q_0, q_1, q_2, \ldots q_n\}$. Then for every $s \in S$, the possible values of L(s) are $I(q_0), I(q_1), \ldots I(q_n)$, $f(q_i, a_j, q_k)$ ($q_i, q_k \in Q$, $a_j \in A$), $F(q_0), F(q_1), \ldots F(q_n)$. Denote these fixed values (after arranging them in non decreasing order) by $\delta_1, \delta_2,\ldots, \delta_t$. Then $\delta_1, \delta_2,\ldots, \delta_t \in [0, 1]$ and by what we have proved above, $L_{\delta i} = L(D_{\delta i}(M))$. Since $L_{\delta i}$ is accepted by a finite automaton, by Myhill Nerode Theorem for finite automata, it follows that there exists a right invariant equivalence relation of finite index and $L_{\delta i} = U[s]_{\delta i}$ where $[s]_{\delta i}$ denotes the equivalence class of s in this equivalence relation. This is true for i = 1, 2,… t.

Define $(\delta_i)_L = \delta_i.L_{\delta i.}$ If $s \in S$ such that $L(s) \geq \delta_i$ ($s \in L_{\delta i}$), then $(\delta_i)_L(s) = \delta_i$.

Otherwise, $(\delta_i)_L(s) = 0$. We note that each $(\delta_i)_L$ is a fuzzy set. Let s ε S and assume that

$L(s) = \delta_i$. Now $L(s) = \delta_i \leq \delta_{i+1} \leq \ldots \leq \delta_t$. Again, $L(s) = \delta_i \geq \delta_{i-1} \geq \ldots \geq \delta_1$.

Hence $((\delta_1)_L \cup (\delta_2)_L \cup\ldots\cup (\delta_t)_L)(s) = (\delta_1)_L(s) \vee (\delta_2)_L(s) \vee\ldots(\delta_t)_L(s) = \delta_1 \vee \delta_2 \vee \ldots\vee \delta_i = \delta_i$. This proves that $L = (\delta_1)_L \cup (\delta_2)_L \cup\ldots\cup (\delta_t)_L$

We will now consider the proof of (ii) → (iii).

If $s \in S$, then s R_L s because for all $u \in S$ and for all $\alpha \in [0,1]$, $L(su) \geq \alpha$ only when $L(su) \geq \alpha$ is obviously true. This proves that R_L is reflexive. Clearly, R_L is symmetric. If s R_L t and t R_L v, then for all $u \in S$ and for all $\alpha \in [0,1]$, $L(su) \geq \alpha$ only when $L(tu) \geq \alpha$ only when $L(vu) \geq \alpha$ proving that s R_L v. Hence R_L is transitive. R_L is thus an equivalence relation.

To prove R_L is right invariant, assume that s R_L t and $u \in S$. We have to prove that su R_L tu. For this, we have to prove that for all $v \in S$ and $\alpha \in [0,1]$, $L(suv) \geq \alpha$ only when $L(tuv) \geq \alpha$ which is the same as saying that $L(sz) \geq \alpha$ only when $L(tz) \geq \alpha$ where z = uv. But this is true since s R_L t.

We will now prove that R_L is of finite index. For i = 1,2,…,t, let R_i denote the equivalence relation of finite index in $L_{\delta i}$. Let $R = R_1 \cap R_2 \cap\ldots\cap R_t$. Then R is an equivalence relation of finite index. We will prove that s R t implies s R_L t. This will mean that index $(R_L) \leq$ index (R). Since index (R) is finite, this will prove that index (R_L) is also finite.

Assume that s R t. Consider any $u \in S$ and any $\alpha \in [0, 1]$. Suppose $su \in L_\alpha$. We have to prove that $tu \in L_\alpha$. Now $\alpha \leq L(su) = \delta_j$ (say). Then $su \in L_{\delta j}$ which is a subset of L_α . By definition of R, we have s R_j t. Since R_j is right invariant, it follows that su R_j tu. Since $L_{\delta j}$ is the union of some of the equivalence classes of R_j , su belongs to one of the equivalence classes and hence tu also belongs to the same equivalence class. This proves that $tu \in L_{\delta j}$ and since $L_{\delta j}$ is a subset of L_α , it follows that $tu \in L_\alpha$.

Proof of (iii) → (i)

We have to define a fuzzy automaton M such that L = L(M). For every element $s \in S$, let [s] denote the equivalence class of s under the equivalence relation R_L . Let $Q = \{[s] / s \in S\}$. Since R_L is of finite index, it follows that Q is a finite set. Define I: Q → [0, 1], f*: Q x S x Q → [0, 1] and F: Q → [0, 1] as follows.

I ([s])	= 1 if [s] = [e]
	= 0 otherwise.
f* ([s], t, [u])	= 1 if [u] = [st]
	= 0 otherwise.
F([s])	= L(s).

We will first prove that F is well defined. For this, we have to prove that if [s] = [t],

then L(s) = L(t). Assume that L(s) = β. We will prove that L(t) = β. Since [s] = [t], s R_L t so that L(s) = L(se) ≥ β only when L(t) = L(te) ≥ β. Since L(s) ≥ β, it follows that L[t] ≥ β .

Assume L[t] = γ > β. Take η = (β + γ) / 2. Clearly, β < η < γ = L[t]. Since s R_L t, L[t] > η implies that L[s] ≥ η > β. But this contradicts the fact that L(s) = β. Hence our assumption that L[t] > β is wrong. Since L[t] ≥ β, it follows that L[t] = β.

Take M = (Q, I, f*,F). Then M is a fuzzy automaton and it remains to prove that L = L(M). For this, we have to prove that for all s ∈ S, L(s) = L(M)(s).

We have

$$L(M)(s) = I \circ f^*_s \circ F = \bigvee_{[t]} \{I([t]) \wedge (f^*_s \circ F)([t])\}$$

$$(f^*_s \circ F)([t]) = \bigvee_{[u]} \{f^*_s([t],[u]) \wedge F([u])\} = \bigvee_{[u]} \{f^*([t], s, [u]) \wedge F([u])\} = \bigvee_{[u]} \{f^*([t], s, [u]) \wedge L(u)\} = L(ts)$$

(Note that f* ([t], s, [u]) = 1 if [ts] = [u] and 0 otherwise)

Hence

$$L(M)(s) = \bigvee_{[t]} \{I([t]) \wedge (f^*_s \circ F)([t])\} = \bigvee_{[t]} \{I([t]) \wedge L(ts)\} = L(es) = L(s)$$

(Note that I([t]) = 1 if [t] = [e] and 0 otherwise. If [t] = [e] then since R_L is right invariant, [ts] = [es] so that L(ts) = L(es))

3. Example

Let Σ = {0, 1} and S = Σ*, the set of all strings over the alphabet Σ. Consider the fuzzy automaton M = (Q, f, I, F) where Q = {q_0, q_1, q_2}, f is the fuzzy subset of Q x Σ x Q defined as

$f(q_0, 0, q_1) = 0.5$, $f(q_0, 0, q_2) = 0.6$, $f(q_0, 1, q_1) = 0.3$, $f(q_0, 1, q_2) = 0.4$, $f(q_1, 0, q_2) = 0.7$ and

$f(q_1, 1, q_2) = 1$. I = {q_0} and F is the fuzzy subset of Q defined as $F(q_1) = 0.3$ and $F(q_2) = 0.9$. We have

$$f_{00}^*(q_0, q_2) = f^*(q_0, 00, q_2)$$

$= f(q_0, 0, q_1) \wedge f(q_1, 0, q_2)$
$= 0.5 \wedge 0.7 = 0.5$

Similarly, $f^*(q_0, 01, q_2) = 0.5$ and $f^*(q_0, 10, q_2) = f^*(q_0, 11, q_2) = 0.3$. Also for any string s of length two or more,

$f^*(q_0, s, q_1) = 0$ and for any string s of length three or more, $f^*(q_0, s, q_2) = 0$.

$$\begin{aligned} L(0) = L(M)(0) &= I \text{ o } f_0^* \text{ o } F \\ &= \vee [(f_0^* \text{ o } F)(p) \wedge I(p)] \\ &= (f_0^* \text{ o } F)(q_0) \wedge I(q_0) \\ &= (f_0^* \text{ o } F)(q_0) \\ &= [f_0^*(q_0, q_1) \wedge F(q_1)] \vee \\ &\quad [f_0^*(q_0, q_2) \wedge F(q_2)] \\ &= (0.5 \wedge 0.3) \vee (0.6 \wedge 0.9) \\ &= 0.6. \end{aligned}$$

Similarly, $L(1) = 0.4$, $L(00) = L(01) = 0.5$ and $L(10) = L(11) = 0.3$. For any string s of length three or more, $L(s) = 0$.

Suppose $0 < \alpha \leq 0.3$. Let $D_\alpha(M)$ denote the non deterministic automaton corresponding to α. Then it can be easily seen that $I_\alpha = \{q_0\}$, $F_\alpha = \{q_1, q_2\}$, $d_\alpha(q_0, 0) = d_\alpha(q_0, 1) = \{q_1, q_2\}$, $d_\alpha(q_0, 00) = d_\alpha(q_0, 01) = d_\alpha(q_0, 10) = d_\alpha(q_0, 11) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\}$. Also $L(D_\alpha(M)) = L_\alpha = \{0, 1, 00, 01, 10, 11\}$, $\alpha_L(0) = \alpha_L(1) = \alpha_L(00) = \alpha_L(01) = \alpha_L(10) = \alpha_L(11) = \alpha$. Define $x R_\alpha y$ if and only if $d_\alpha(q_0, x) = d_\alpha(q_0, y)$. Clearly, R_α is a right invariant equivalence relation of finite index. Furthermore, $[0]_\alpha = \{0, 1\}$, $[00]_\alpha = \{00, 01, 10, 11\}$ and $L_\alpha = [0]_\alpha \cup [00]_\alpha$

Now take $0.3 < \alpha \leq 0.4$. Then $I_\alpha = \{q_0\}$, $F_\alpha = \{q_2\}$, $d_\alpha(q_0, 0) = \{q_1, q_2\}$,

$d_\alpha(q_0, 1) = d_\alpha(q_0, 00) = d_\alpha(q_0, 01) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\}$.

Also $L(D_\alpha(M)) = L_\alpha = \{0, 1, 00, 01\}$, $\alpha_L(0) = \alpha_L(1) = \alpha_L(00) = \alpha_L(01) = \alpha$, $\alpha_L(10) = \alpha_L(11) = 0$. Define $x R_\alpha y$ if and only if $d_\alpha(q_0, x) = d_\alpha(q_0, y)$. Clearly, R_α is a right invariant equivalence relation of finite index. Furthermore, $[0]_\alpha = \{0\}$, $[1]_\alpha = \{1, 00, 01\}$ and $L_\alpha = [0]_\alpha \cup [1]_\alpha$

If $0.4 < \alpha \leq 0.5$, then $I_\alpha = \{q_0\}$, $F_\alpha = \{q_2\}$, $d_\alpha(q_0, 0) = \{q_1, q_2\}$,

$d_\alpha(q_0, 00) = d_\alpha(q_0, 01) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\}$. Also $L(D_\alpha(M)) = L_\alpha = \{0, 00, 01\}$,

$\alpha_L(0) = \alpha_L(00) = \alpha_L(01) = \alpha$, $\alpha_L(1) = \alpha_L(10) = \alpha_L(11) = 0$.

Define $x R_\alpha y$ if and only if $d_\alpha(q_0, x) = d_\alpha(q_0, y)$. Clearly, R_α is a right invariant equivalence relation of finite index. Furthermore,

$[0]_\alpha = \{0\}$, $[00]_\alpha = \{00, 01\}$ and $L_\alpha = [0]_\alpha \cup [00]_\alpha$

If $0.5 < \alpha \leq 0.6$, then $I_\alpha = \{q_0\}$, $F_\alpha = \{q_2\}$, $d_\alpha(q_0, 0) = d_\alpha(q_1, 0) = d_\alpha(q_1, 1) = \{q_2\}$.

Also $L(D_\alpha(M)) = L_\alpha = \{0\}$, $\alpha_L(0) = \alpha$, $\alpha_L(1) = \alpha_L(00) = \alpha_L(01) = \alpha_L(10) = \alpha_L(11) = 0$.

Define $x R_\alpha y$ if and only if $d_\alpha(q_0, x) = d_\alpha(q_0, y)$. Clearly, R_α is a right invariant equivalence relation of finite index. Furthermore, $[0]_\alpha = \{0\}$ and $L_\alpha = [0]_\alpha$

If $\alpha > 0.6$, then there is no corresponding non deterministic automaton and $L(D_\alpha(M)) = L_\alpha = \phi$. Also

$(\cup\, \alpha_L)(0) = \vee\, \alpha_L(0)$
$= 0.3 \vee 0.4 \vee 0.5 \vee 0.6 = 0.6 = L(0)$

$(\cup\, \alpha_L)(1) = \vee\, \alpha_L(1)$
$= 0.3 \vee 0.4 = 0.4 = L(1)$

$(\cup\, \alpha_L)(00) = \vee\, \alpha_L(00)$
$= 0.3 \vee 0.4 \vee 0.5 = 0.5 = L(00)$

$(\cup\, \alpha_L)(01) = \vee\, \alpha_L(01)$
$= 0.3 \vee 0.4 \vee 0.5 = 0.5 = L(01)$

$(\cup\, \alpha_L)(10) = \vee\, \alpha_L(10)$
$= 0.3 = L(10)$

$(\cup\, \alpha_L)(11) = \vee\, \alpha_L(11)$
$= 0.3 = L(11)$

This verifies that $L = \cup\, \alpha_L$

The equivalence classes corresponding to R_L are

[0] = {0}, [1] = {1}, [00] = {00, 01}, [10] = {10,11} and [001] which consists of all strings of 0s and 1s whose lengths are greater than or equal to three.

4. Conclusion

In this paper, we have extended Myhill Nerode theorem of finite automata to fuzzy automata. Unlike the finite automata case, in this case, we note that the number of equivalence classes corresponding to R_L is more than the number of states in the given fuzzy automata. Hence this does not help us in reducing the number of states in the fuzzy automata as in the finite automata. We are working on alternative methods for reduction.

References

[1] John E Hopcroft & Jefrey D. Ullman, Introduction to Automata Theory, Languages and Computation. Narosa publishing house.

[2] Jiri Mockor, Fuzzy and Non deterministic Automata, Research Report No. 8, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Chech Republic.