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Strongly Almost Convergent Classes of Sequences of Fuzzy Numbers Generated by Infınıte Matrıces Defıned By A Modulus Functıon

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Absract

In this paper, we introduce and examine some new almost convergent classes of sequences of fuzzy numbers by using the A-transforms and a modulus function. We also examine topological properties and some inclusion relations for these new classes of sequences of fuzzy numbers.

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1. Introductıon

The concept of fuzzy sets was first introduced by Zadeh [1]. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2]. Matloka show that every convergent sequences of fuzzy numbers is bounded.Later on sequences of fuzzy numbers have been discussed by Nanda [3], Nuray and Savas [4], Choudhary andTripathy [5], Esi [6] and many others.

Let D denote the set of all closed bounded intervals A=$\left[\underline{A},\bar{A}\right]$ on the real line R.

For A, B$\in$ D define A$\leq$B if and only if $\underline{A}\leq\underline{B}$ and $\bar{A}\leq\bar{B}$,

$$\bar{d}\ (A,B)=\max\left\{\left|\underline{A}-\underline{B}\right|,\left|\bar{A}-\bar{B}\right|\right\}.$$

It is easy to see that $\bar{d}$ defines a metric on D and $\left(D,\bar{d}\right)$ is complete metric space. Also $\leq$ is a partial order on D. A fuzzy number is a fuzzy subset of real line R which is bounded , convex and normal. Let $L(R)$ denote the set of all fuzzy numbers those are upper semicontinuous and have compact support. In other words, if X∈ then for any $\alpha \in [0,1]$, X^{α} is compact , where

$X^{\alpha}(t)$ = {t∈R:$X(t)$≥α} for α>0. The $\overline{0}$-cut is
defined as the closure of the strong $\overline{0}$-cut i.e. closure{ t∈R:$X(t)$> α}.

The absolute value of $X \in L(R)$ is defined as

$$|X(t)| = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad [7].$$

For each $0 < \alpha \leq 1$, the α-level set X^{α} is a non-empty compact subset of R. The linear structure of $L(R)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in$ R, in terms of α-level sets defined by

$$[X+Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha} \quad and \quad [\lambda X]^{\alpha} = \lambda [X]^{\alpha}$$

for each $0 \leq \alpha \leq 1$.

Define d : $L(R)$ x $L(R)$ $\rightarrow R$ by $d(X,Y) = \sup_{0\leq\alpha\leq 1} \overline{d}(X^{\alpha}, Y^{\alpha})$. For X,Y∈ $L(R)$. Define X ≤ Y if and only if X^{α} ≤ Y^{α} for any $\alpha \in [0,1]$. It is know that $L(R)$ is a complete metric space with the metric d (see for instance [2]).

A metric on $L(R)$ is said to be a translation invariant if $d(X+Z, Y+Z) = d(X,Y)$ for $X,Y,Z \in$ $L(R)$. The metric d has the following properties:

$$d(cX, cY) = |c| d(X,Y) \quad \ldots(1)$$

for $c \in R$ and

$$d(X+Y, Z+W) \leq d(X,Z) + d(Y,W) \quad \ldots(2)$$

A sequence of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$.The fuzzy number X_k denotes the value of the function at $k \in N$ [2].We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$of fuzzy numbers is bounded [2]. We denote by $l_{\infty}(F)$ the set of all bounded sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_o, if for every $\varepsilon > 0$ there is a positive integer n_0 such that $d(X_k, X_o) < \varepsilon$

for $k > n_o$ [2]. We denote by $c(F)$ the set of all convergent sequences $X = (X_k)$ of fuzzy numbers.

It is straightforward to see that $c(F) \subset l_\infty(F) \subset w(F)$. In [3], it is shown that $c(F)$ and $l_\infty(F)$ are complete metric spaces. For further studies,one may refer to [8] and [1].

Let $p=(p_k)\in l_\infty$, then the following well-known inequality will be used in the paper:

For sequences (a_k) and (b_k) of complex numbers we have [9]

$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k})$ …(3)

where C= *max* $(1, 2^{H-1})$ and $H = \sup_k p_k$.

Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers and $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $(AX)_n$ defined by $(AX)_n = \sum_k a_{nk} X_k$ for all $n \in N$ and is called the A-transform of $X = (X_k)$ whenever this series converges for each n. Throughout $A = (a_{nk})$ denotes an infinite matrix of non-negative real numbers and let $p = (p_k)$ be a bounded sequence of positive real numbers.

The notion of modulus was introduced by Nakano [10] and followed by Ruckle [11] and many others. We recall that a function f: $[0,\infty) \to [0,\infty)$ is called a *modulus* if

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \le f(x) + f(y)$,

(iii) f is increasing ,

(iv) f is continuous from right at 0.

It is immediate from (*ii*) and (*iv*) that f is continuous on $[0,\infty)$.

Lemma. Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$, we have $f(x) \le 2 f(1)\delta^{-1} x$, [12].

Remark. If f is a modulus function, then the composition $f^s = f.f...f\ (s - times)$ is also a modulus function, where s is a positive integer.

In this paper, we introduce and examine some new almost convergent classes of sequences of fuzzy numbers by using the A-transforms and a modulus function. We also examine topological properties and some inclusion relations for these new classes of sequences of fuzzy numbers.

Let $A=(a_{nk})$ be an infinite matrix and $p=(p_n)$ be a bounded sequence of real numbers.

We now define:

$$F_o[A,f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(A_{n,m}(X),\bar{0}\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\}$$

$$F[A,f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(A_{n,m}(X),X_o\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\},$$

$$F_\infty[A,f,p]=\left\{X=(X_k):\ \sup_{n,m} f\left(d\left(A_{n,m}(X),\bar{0}\right)\right)^{p_n}<\infty\right\}.$$

where

$A_{n,m}(X_k)=\sum_k a_{nk}X_{k+m}$ and the series converges for each n and m.

If $X=(X_k)\in F[A,f,p]$, we say that $X=(X_k)$ is strongly almost A-convergent to X_0 with respect to modulus function f.

We can specialize these classes as follows:

(i) If $A=(C,1)$ *Cesaro matrix, i.e.* $a_{nk}=\begin{cases}1, & 1\le k\le n\\ 0 & k>n\end{cases}$

then, we obtain the classes of sequences of fuzzy numbers as follows:

$$F_o[C,f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(\sum_{k=1}^{n}X_{k+m},\bar{0}\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\}$$

$$F[C,f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(\sum_{k=1}^{n}X_{k+m},X_o\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\},$$

$$F_\infty[C,f,p]=\left\{X=(X_k):\ \sup_{n,m} f\left(d\left(\sum_{k=1}^{n}X_{k+m},\bar{0}\right)\right)^{p_n}<\infty\right\}$$

If $X=(X_k)\in F_o[C,f,p]$, we say that $X=(X_k)$ is strongly almost convergent to X_0 with respect to modulus function f.

(ii) If $A=I$, the unit matrix , then we get the classes of sequences of fuzzy numbers with respect to modulus f as follows:

$$F_o[f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(X_{n+m},\bar{0}\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\}$$

$$F[f,p]=\left\{X=(X_k):\ \lim_{n\to\infty} f\left(d\left(X_{n+m},X_o\right)\right)^{p_n}=0,\quad uniformly\ in\ m\right\},$$

$$F_\infty[f,p]=\left\{X=(X_k):\ \sup_{n,m} f\left(d\left(X_{n+m},\bar{0}\right)\right)^{p_n}<\infty\right\}.$$

(iii) If $f(x)=x$, then we get the classes of sequences of fuzzy numbers as follows:

$F_o[A,p]=\left\{X=(X_k):\ \lim_{n\to\infty} d\left(A_{n,m}(X),\overline{0}\right)^{p_n}=0,\ \ \textit{uniformly in } m\right\}$,

$F[A,p]=\left\{X=(X_k):\ \lim_{n\to\infty} d\left(A_{n,m}(X),X_o\right)^{p_n}=0,\ \ \textit{uniformly in } m\right\}$,

$F_\infty[A,p]=\left\{X=(X_k):\ \sup_{n,m} d\left(A_{n,m}(X),\overline{0}\right)^{p_n}<\infty\right\}$,

which were defined and studied by Savaş [13].

2. Maın Results

Theorem 1.(i) Let $p=(p_k)$ be a bounded sequence of strictly positive real numbers. Then, the classes of $F_o[A,f,p], F[A,f,p]$ and $F_\infty[A,f,p]$ are closed under the operations addition and scalar multiplication if the metric d translation invariant.

(ii) $F_o[A,f,p]\subset F[A,f,p]$, $F_o[A,f,p]\subset F[A,f,p]_\infty$ and $F[A,f,p]\subset F[A,f,p]_\infty$.

Proof.(i) Let $X=(X_k),Y=(Y_k)\in F[A,f,p]$. Then by taking into account the properties (1) and (2) of the metric d , the property (ii) of modulus f and inequality (3)

$$f\left(d\left(A_{n,m}(X)+A_{n,m}(Y),X_o+Y_o\right)\right)^{p_n}\le f\left(d\left(A_{n,m}(X),X_o\right)+d\left(A_{n,m}(Y),Y_o\right)\right)^{p_n}$$
$$\le\left[f\left(d\left(A_{n,m}(X),X_o\right)\right)+f\left(d\left(A_{n,m}(Y),Y_o\right)\right)\right]^{p_n}$$
$$\le Cf\left(d\left(A_{n,m}(X),X_o\right)\right)^{p_n}+Cf\left(d\left(A_{n,m}(Y),Y_o\right)\right)^{p_n}.$$

Hence $X+Y=(X_k)+(Y_k)\in F[A,f,p]$.

Now, let $X=(X_k)\in F[A,f,p]$. For $\lambda\in R$, there exists an integer T such that $|\lambda|\le T$ .By taking into account the properties (2) and the properties of modulus f, we have

$$f\left(d\left(A_{n,m}(\lambda X),\lambda X_o\right)\right)^{p_n}\le T^H f\left(d\left(A_{n,m}(X),X_o\right)\right)^{p_n},$$

where $H=\sup_n p_n<\infty$. Therefore $\lambda X=(\lambda X_k)\in F[A,f,p]$. The others can be treated similarly.

(ii) It is evident that $F_o[A,f,p]\subset F[A,f,p]$ and $F_o[A,f,p]\subset F[A,f,p]_\infty$. For $F[A,f,p]\subset F[A,f,p]_\infty$, we use the triangle inequality and inequality (3),

$$f\left(d\left(A_{n,m}(X),\overline{0}\right)\right)^{p_n}\le f\left[d\left(A_{n,m}(X),X_o\right)+d\left(X_o,\overline{0}\right)\right]^{p_n}$$
$$\le\left[f\left(d\left(A_{n,m}(X),X_o\right)\right)+f\left(d\left(X_o,\overline{0}\right)\right)\right]^{p_n}$$

$\leq C\left[f\left(d\left(A_{n,m}(X),X_o\right)\right)\right]^{p_n}+C\left[f\left(d\left(X_o,\bar{0}\right)\right)\right]^{p_n}$

$\leq C\left[f\left(d\left(A_{n,m}(X),X_o\right)\right)\right]^{p_n}+C\left[f\left(\max\left(1,\left|X_o\right|\right)\right)\right]^{p_n}$.

So, $X=(X_k)\in F[A,f,p]$ implies $X=(X_k)\in F[A,f,p]_\infty$. This completes the proof.

Theorem 2. (i) $F_o[A,f,q]\subset F_o[A,f,p]$ if $\liminf\left(\frac{p_n}{q_n}\right)>0$.

(ii) $F_\infty[A,f,q]$ is closed subset of $F_\infty[A,f,p]$ if $0<p_n\leq q_n\leq 1$.

Proof.(i) Suppose that $\liminf\left(\frac{p_n}{q_n}\right)>0$ and $X=(X_k)\in F_o[A,f,q]$. Then there exists an $\lambda>0$

such that $p_n>\lambda q_n$ for large n. Hence for large n

$$f\left[d\left(A_{n,m}(X),\bar{0}\right)\right]^{p_n}\leq\left(f\left[d\left(A_{n,m}(X),\bar{0}\right)\right]^{q_n}\right)^{\lambda}$$

since $f\left[d\left(A_{n,m}(X),\bar{0}\right)\right]^{p_n}<1$ for each n and m. Hence $X=(X_k)\in F_o[A,f,p]$.

(ii) Suppose that $0<p_n\leq q_n\leq 1$ for all $n\in N$ and $X=(X_k)\in F_\infty[A,f,q]$. Then there is a constant K>1 such that

$$f\left(d\left(A_{n,m}(X),\bar{0}\right)\right)^{q_n}\leq K$$

for all n and m. This implies that

$$f\left(d\left(A_{n,m}(X),\bar{0}\right)\right)^{P_n}\leq K$$

for all n and m.Hence $X=(X_k)\in F_\infty[A,f,p]$. To show that $F_\infty[A,f,q]$ is closed, suppose that $X^i=\left(X_k^i\right)\in F_\infty[A,f,q]$, $X^i\to X_o$ and $X_o\in F_\infty[A,f,p]$. Then for every ε, $0<\varepsilon<1$, there is $i_o\in N$ such that for all n and m.

$$f\left(d\left(A_{n,m}\left(X^i-X_o\right),\bar{0}\right)\right)^{p_n}<\varepsilon$$

for $i>i_o$. Now

$$f\left(d\left(A_{n,m}\left(X^i-X_o\right),\bar{0}\right)\right)^{q_n}<f\left(d\left(A_{n,m}\left(X^i-X_o\right),\bar{0}\right)\right)^{p_n}<\varepsilon$$

for $i > i_o$. Therefore $X = (X_k) \in F_\infty[A, f, q]$, i.e. $F_\infty[A, f, q]$ is closed subset of $F_\infty[A, f, p]$.

Theorem 4. Let $\inf p_n = h > 0$. Then, $F[A] \subset F[A, f, p]$.
Where
$F[A] = \{X = (X_k):\ \lim_{n\to\infty} d(A_{n,m}(X), X_o) = 0,\ \ \textit{uniformly in } m\}$.

Proof. Suppose that $X = (X_k) \in F[A]$. Since f modulus, then

$\lim_{n\to\infty} f(d(A_{n,m}(X), X_o)) = f(\lim_{n\to\infty} d(A_{n,m}(X), X_o)) = 0,\ \textit{uniformly in } m$.
Since $\inf p_n = h > 0$ then,
$\lim_{n\to\infty} f(d(A_{n,m}(X), X_o))^h = 0,\ \textit{uniformly in } m$.
So, for $0 < \varepsilon < 1, \exists n_o \ni$ for all $n > n_o$ and all m,
$f(d(A_{n,m}(X), X_o))^h < \varepsilon < 1$
and since $p_n \geq h$ for all n,
$f(d(A_{n,m}(X), X_o))^{p_n} \leq f(d(A_{n,m}(X), X_o))^h < \varepsilon < 1$,
then we get,
$\lim_{n\to\infty} f(d(A_{n,m}(X), X_o))^{p_n} = 0,\ \textit{uniformly in } m$.

Therefore $X = (X_k) \in F[A, f, p]$. This completes the proof.

Theorem 5. $F[A, f, p]$ is complete metric space with the metric
$g(X, Y) = \sup_{n,m} f(d(A_{n,m}(X), A_{n,m}(Y)))^p$ for $0 < p < 1$.

Proof. Let (X^s) be a Cauchy sequence in $F[A, f, p]$, where

$(X^s) = (X_i^s) = (X_1^s, X_2^s, X_3^s, \ldots) \in F[A, f, p]$ for each $s \in N$. Then

$g(X^s, X^t) = \sup_{n,m} f(d(A_{n,m}(X^s), A_{n,m}(X^t)))^p \to 0 \ \textit{as}\ s, t \to \infty$.
Hence for each n and m, *as* $s, t \to \infty$, we have $f(d(A_{n,m}(X^s), A_{n,m}(X^t))) \to 0$. By continuity of f for all n and m
$\lim_{s,t\to\infty} f(d(A_{n,m}(X^s), A_{n,m}(X^t)))^p = f[\lim_{s,t\to\infty}(d(A_{n,m}(X^s), A_{n,m}(X^t)))]^p = 0$.
It follows that $\lim_{s,t\to\infty}(d(A_{n,m}(X^s), A_{n,m}(X^t))) = \lim_{s,t\to\infty} d\left(\sum_k a_{nk} X_{k+m}^s, \sum_k a_{nk} X_{k+m}^t\right) = 0$. From the properties (1) of the metric d,

$\lim_{s,t\to\infty} d\left(\sum_k a_{nk} X^s_{k+m}, \sum_k a_{nk} X^t_{k+m}\right) = \sum_k a_{nk} \lim_{s,t\to\infty} d\left(X^s_{k+m}, X^t_{k+m}\right) = 0$. This implies that $\lim_{s,t\to\infty} d\left(X^s_{k+m}, X^t_{k+m}\right) = 0$ for each m. Hence $\left(X^s\right)_s$ is a Cauchy sequence in $L(R)$. Since $L(R)$ is complete, it is convergent $\lim_{s\to\infty} X^s_k = X_k$ say, for each $k \in N$. Since $\left(X^s\right)_s$ is a Cauchy sequence, for

each $\varepsilon > 0$, there exists $s_o = s_o(\varepsilon)$ such that
$g\left(X^s, X^t\right) < \varepsilon$, for all $s,t \geq s_o$.
So, we have
$\lim_{t\to\infty} f\left(d\left(A_{n,m}\left(X^s\right), A_{n,m}\left(X^t\right)\right)\right) = f\left(\lim_{t\to\infty} d\left(A_{n,m}\left(X^s\right), A_{n,m}\left(X^t\right)\right)\right)$
$= f\left(d\left(A_{n,m}\left(X^s\right), A_{n,m}\left(X\right)\right)\right) \leq \varepsilon$
for all m and $s \geq s_o$. This implies that $g\left(X^s, X\right) < \varepsilon$, for all $s \geq s_o$, that is $X^s \to X$ *as* $s \to \infty$, where $X = (X_k)$. Since
$f\left(d\left(A_{n,m}(X), X_o\right)\right)^p \leq f\left[d\left(A_{n,m}\left(X^{s_o}\right), X_o\right) + d\left(A_{n,m}\left(X^{s_o}\right), A_{n,m}(X)\right)\right]^p$
$\leq f\left(d\left(A_{n,m}\left(X^{s_o}\right), X_o\right)\right)^p + f\left(d\left(A_{n,m}\left(X^{s_o}\right), A_{n,m}(X)\right)\right)^p \to 0$
as $n \to \infty$, uniformly in m. So, we obtain $X = (X_k) \in F[A, f, p]$. Therefore $F[A, f, p]$ is complete metric space. This completes the proof.

Theorem 6. Let f be a modulus function and s be a positive integer. Then

$$F[A, f, p] \subset F\left[A, f^s, p\right].$$

Proof. Let $X = (X_k) \in F[A, f, p]$. Then we have

$$T_{n,m} = f\left(d\left(A_{n,m}(X), X_o\right)\right)^{p_n} \to 0 \text{ as } n \to \infty, \quad \text{uniformly in } m.$$

Let $\varepsilon > 0$ be given and choose δ with such that $f(t) < \varepsilon$, for $0 \leq t \leq \delta$. Write

$y_{n,m} = f^{s-1}\left(d\left(A_{n,m}(X), X_o\right)\right)$ and consider

$$f^s\left(d\left(A_{n,m}(X), X_o\right)\right)^{p_n} = \left[f\left(f^{s-1}\left(d\left(A_{n,m}(X), X_o\right)\right)\right)\right]^{p_n} = f\left(y_{n,m}\right)^{p_n}$$

$$= f\left(y_{n,m}\right)^{p_n}_{y_{n,m\leq\delta}} + f\left(y_{n,m}\right)^{p_n}_{y_{n,m>\delta}}$$
$$\leq \max\left(1, 2f(1)\delta^{-1}\right)^H T_{n,m} + \varepsilon^H$$

from Lemma , where $H = \sup_n p_n < \infty$. Therefore we obtain $X = (X_k) \in F[A, f^s, p]$.

3. Conclusıon

In this paper we have introduced some classes of sequences of fuzzy numbers using the matrix A and modulus function f. Giving particular values the matrix A and modulus f, we obtain some classes of sequences of fuzzy numbers which are the special cases of classes that we have defined. The most of the results proved in the previous section will be true for these classes.

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