

Common Fixed Points for Weakly Compatible Maps in Intuitionistic Fuzzy Metric Spaces

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Abstract

In this paper, we prove a common fixed point theorem for weakly compatible maps in intuitionistic fuzzy metric spaces which generalizes the result of Alaca, Turkoglu and Yildiz[3]. Moreover, we introduce the notions of R-weakly commuting mapping of type (A_g) , R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P) in intuitionistic fuzzy metric spaces and provide various examples to reflect upon the distinctiveness among R-weakly commuting mapping of type (A_g) , R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P). At the end, we discuss some results related to these newly introduced notions and also establish an example in support of our theorem.

Key words: Intuitionistic fuzzy metric space, Weakly compatible mappings, R-weakly commuting mapping of type (A_g) , R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P)

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1. Introduction

An essential feature of metric space is that for any two points in the metric space, there is defined a positive number called the distance between the points. The concept of a probabilistic metric space (PM-space) corresponds to these situations in which we do not know the exact distance between the points but know only the probable values of the distance between the points and this notion was introduced in 1942 by Menger

[26], which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pairs, say (p, q) , denoted by $F(p, q; t)$ where $t > 0$ and interpret this function as the probability that distance between p and q is less than t and whereas in the metric space the distance function is a single positive number. Fuzziness is also a kind of uncertainty. It is applied to those events, whose chances of occurrence are in non-black or non-white state. Zadeh [37] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in $[0, 1]$. The concept of fuzzy set corresponds to the degree of nearness between two objects. Deng[7], Erceg [9], Fang[10], George and Veeramani [12], Kaleva and Seikkala [19], Kramosil and Michalek [22] have introduced the concept of fuzzy metric spaces in different ways.

Atanassove [4] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Intuitionistic fuzzy sets deals with both degree of nearness and non-nearness. Park [29] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [12]. Further, Alaca et al. [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space ,as Park with the help of continuous t-norms and continuous t-conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [22]. Further Coker [6] introduced the concept of intuitionistic fuzzy topological spaces. Turkoglu et. al. [34] gave a generalization of Jungck's common fixed point theorem [16] to intuitionistic fuzzy metric spaces. They first formulated the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [28].

2. Preliminaries

The concepts of triangular norms (t-norm) and triangular cotnorms (t-conorm) are known as the axiomatic skeletons that we use for characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [26] in study of statistical metric spaces.

Definition 2.1[33] A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a$ for all $a \in [0, 1]$;
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2[33] A binary operation \diamond : $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm if \diamond is satisfying the following conditions:

- (i) \diamond is commutative and associative;
- (ii) \diamond is continuous;

- (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;
- (iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al. [2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [22] as follows:

Definition 2.3[2] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, \diamond is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
- (ii) $M(x, y, 0) = 0$ for all $x, y \in X$;
- (iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
- (iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
- (v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
- (vi) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous;
- (vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
- (viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
- (ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
- (x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
- (xi) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
- (xii) for all $x, y \in X$, $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous;
- (xiii) $\lim_{t \rightarrow \infty} N(x, y, t) = 0$ for all x, y in X ;

Then (M, N) is called an intuitionistic fuzzy metric on X . The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t , respectively.

Example 2.1. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with $*$ continuous t-norm and \diamond continuous t-conorm defined by $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} \frac{|x-y|}{t + |x-y|}, & t > 0, \\ 1 & t = 0 \end{cases}$$

Then, $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space, (for $k=1$ [34]).

Remark 2.1. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm $*$ and t-conorm \diamond are associated as

$$x \diamond y = 1 - ((1-x) * (1-y)) \text{ for all } x, y \in X, \text{ see [24].}$$

Remark 2.2. An intuitionistic fuzzy metric spaces with continuous t-norm $*$ and continuous t-conorm \diamond defined by $a * a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$. Then for all $x, y \in X$, $M(x, y, *)$ is non-decreasing and $N(x, y, \diamond)$ is non-increasing.

Proof. Suppose $M(x, y, *)$ is non-increasing, therefore for $t \leq s$, we have $M(x, y, t) \geq M(x, y, s)$.

For all $x, y, z \in X$, we have $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

In particular for $z=y$, we have $M(x, y, t + s) \geq M(x, y, t) * M(y, y, s)$.

$M(x, y, t + s) \geq M(x, y, t) * 1 = M(x, y, t)$, a contradiction, hence $M(x, y, *)$ is non-decreasing.

Again suppose $N(x, y, \diamond)$ is non-decreasing, therefore for $t \leq s$, we have $N(x, y, s) \geq N(x, y, t)$.

For all $x, y, z \in X$, we have $N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s)$ In particular for $z=y$, we have $N(x, y, t + s) \leq N(x, y, t) \diamond N(y, y, s) = N(x, y, t)$, a contradiction, hence $N(x, y, \diamond)$ is non-increasing.

Alaca, Turkoglu, and Yildiz [2] introduced the following notions:

Definition 2.4. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,
 $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0$:

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,
 $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$, $\lim_{n \rightarrow \infty} N(x_n, x, t) = 0$:

Since $*$ and \diamond are continuous, the limit is uniquely determined from (v) and (xi) of Definition 2.3, respectively.

Definition 2.5. An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Turkoglu, Alaca and Yildiz [35] introduced the notions of compatible mappings in intuitionistic fuzzy metric space, akin to the concept of compatible mappings introduced by Jungck [17] in metric spaces.

Definition 2.6. A pair of self-mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be compatible if

$\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) = 1$ and $\lim_{n \rightarrow \infty} N(fgx_n, gfx_n, t) = 0$ for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z \text{ for some } z \in X.$$

Definition 2.7. A pair of self-mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be non-compatible if

$$\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) \neq 1 \text{ or non-existent,}$$

$\lim_{n \rightarrow \infty} N(fgx_n, gfx_n, t) \neq 0$ or non-existent for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z \text{ for some } z \in X.$$

In 1998, Jungck and Rhoades [18] introduced the concept of weakly compatible maps as follows:

Definition 2.8. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Example 2.2. Let $X = \mathbb{R}$ and define $f, g : \mathbb{R} \rightarrow \mathbb{R}$ by $fx = x/3$ and $gx = x^2$ for all $x \in \mathbb{R}$. Here 0 and 1/3 are two coincidence points for the maps f and g . Note that f and g commute at 0, i.e., $fg(0) = gf(0) = 0$, but $fg(1/3) = f(1/9) = 1/27$ and $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible maps on \mathbb{R} .

Remark 2.4. Weakly compatible maps need not be compatible.

Let $X = [0, \infty)$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x, y, t) = \begin{cases} \left(\exp\left(\frac{|x-y|}{t}\right) \right)^{-1} & \text{for all } x, y \in X \text{ and } t > 0 \\ 0 & \text{for all } x, y \in X \text{ and } t = 0. \end{cases}$$

and

$$N(x, y, t) = \begin{cases} \left[\exp\left(\frac{|x-y|}{t}\right) - 1 \right] \left[\exp\left(\frac{|x-y|}{t}\right) \right]^{-1} & \text{for all } x, y \in X \text{ and } t > 0 \\ 1 & \text{for all } x, y \in X \text{ and } t = 0 \end{cases}$$

Clearly, $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with t -norm $*$ and t -conorm \diamond defined by $a*b = \min\{a, b\}$ and $a\diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$, respectively.

Define two self mappings f and g of intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ as follows:

$fx = 1$ for all x in $[0, 1]$, $fx = 1+x$ for all $x \in (1, \infty)$ and

$gx = \begin{cases} 1+x & \text{for all } x \in [0, 1) \\ 1 & \text{for all } x \in [1, \infty). \end{cases}$ Let us consider a sequence $\{x_n\}$ by $x_n = \frac{1}{2n}$.

Then pair (f, g) is not compatible but pair (f, g) is weakly compatible at $x = 1$.

Turkoglu et al. [34] first formulate the definition of weakly commuting and R -weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [28].

Definition 2.9[34]. A pair of self-mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be

(i) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$
and $N(fgx, gfx, t) \leq N(fx, gx, t)$ for all $x \in X$ and $t > 0$.

(ii) R -weakly commuting if there exists some $R > 0$ such that

$M(fgx, gfx, t) \geq M(fx, gx, t/R)$
and $N(fgx, gfx, t) \leq N(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

In 1997, Pathak, Cho and Kang [31] introduced the improved notions of R -weakly commuting mappings in metric spaces and called them R -weakly commuting mappings of type (A_g) and R -weakly commuting mappings of type (A_f) . Further, Imdad et al.[15] introduced the notion of R -weakly commuting mapping of type (P) in fuzzy metric spaces. Now, we introduce the notions of R -weakly commuting mapping

of type (A_g) , R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P) in intuitionistic fuzzy metric spaces akin to the notions R-weakly commuting mapping of type (A_g) , R-weakly commuting mapping of type (A_f) introduced by Pathak et al. in metric spaces and R-weakly commuting mapping of type (P) in fuzzy metric spaces introduced by Imdad et al.[15] .We also provide various examples to reflect upon the distinctiveness among R-weakly commuting mapping of type (A_g) , R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P).

Now we introduce the following notions:

Definition 2.10. A pair of self-mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be

- (i) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that

$$M(gfx, ffx, t) \geq M(fx, gx, t/R)$$

$$\text{And } N(gfx, ffx, t) \leq N(fx, gx, t/R) \text{ for all } x \in X \text{ and } t > 0.$$

- (ii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that

$$M(fgx, ggx, t) \geq M(fx, gx, t/R)$$

$$\text{And } N(fgx, ggx, t) \leq N(fx, gx, t/R) \text{ for all } x \in X \text{ and } t > 0.$$

- (iii) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M(fx, gx, t/R)$$

$$\text{And } N(ffx, ggx, t) \leq N(fx, gx, t/R) \text{ for all } x \in X \text{ and } t > 0.$$

Now, we establish following example which illustrate the independency of R-weakly commuting mappings of different kind:

Example 2.4. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with $X = [0, 1]$, t-norm $*$ and t-conorm \diamond defined by $a*b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, $a, b \in [0, 1]$, respectively. Let (M, N) is the intuitionistic fuzzy set on $X^2 \times (0, \infty)$, defined by

$$M(x, y, t) = \begin{cases} \left(\exp\left(\frac{|x-y|}{t}\right) \right)^{-1} & \text{for all } x, y \in X \text{ and } t > 0 \\ 0 & \text{for all } x, y \in X \text{ and } t = 0. \end{cases}$$

$$\text{and } N(x, y, t) = \begin{cases} \left[\exp\left(\frac{|x-y|}{t}\right) - 1 \right] \left[\exp\left(\frac{|x-y|}{t}\right) \right]^{-1} & \text{for all } x, y \in X \text{ and } t > 0 \\ 1 & \text{for all } x, y \in X \text{ and } t = 0 \end{cases}$$

Then it is well known that $(X, M, N, *, \diamond)$ is a intuitionistic fuzzy metric space. Define $fx = 2x-1$ and $gx = x^2$. Then by a straightforward calculation, one can show that

$$M(fgx, gfx, t) = \left(\exp \left(\frac{2|x-1|^2}{t} \right) \right)^{-1} = M(fx, gx, t/2)$$

$$N(fgx, gfx, t) = \left[\exp \left(\frac{2|x-1|^2}{t} \right) - 1 \right] \left[\exp \left(\frac{2|x-1|^2}{t} \right) \right]^{-1} = N(fx, gx, t/2), \text{ which shows}$$

that the pair (f, g) is R -weakly commuting for $R = 2$. Note that the pair (f, g) is not weakly commuting due to a strict increasing property of the exponential function.

However, various kinds of above mentioned 'R-weak commutativity' notions are independent of one another and none implies the other. To illustrate the independency of 'R-weak commutativity' with 'R-weak commutativity of type (A_f) ' one should consider

$$\begin{aligned} M(fgx, ggx, t) &= \left(\exp \left(\frac{|x^4 - 2x^2 + 1|}{t} \right) \right)^{-1} = \left(\exp \left(\frac{R(x-1)^2 (x+1)^2}{tR} \right) \right)^{-1} \\ &< \left(\exp \left(\frac{R|x-1|^2}{t} \right) \right)^{-1} = M(fx, gx, t/R) \text{ when } x > 1 \end{aligned}$$

$$\begin{aligned} N(fgx, ggx, t) &= \left[\exp \left(\frac{|x^4 - 2x^2 + 1|}{t} \right) - 1 \right] \left[\exp \left(\frac{|x^4 - 2x^2 + 1|}{t} \right) \right]^{-1} \\ &= \left[\exp \left(\frac{R(x-1)^2 (x+1)^2}{tR} \right) - 1 \right] \left[\exp \left(\frac{R(x-1)^2 (x+1)^2}{tR} \right) \right]^{-1} \\ &> \left[\exp \left(\frac{R|x-1|^2}{t} \right) - 1 \right] \left[\exp \left(\frac{R|x-1|^2}{t} \right) \right]^{-1} \end{aligned}$$

$= N(fx, gx, t/R)$ when $x > 1$, which shows that 'R-weak commutativity' does not imply 'R-weak commutativity of type (A_f) '.

Secondly, in order to demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity of type (P) ' we notice that

$$\begin{aligned} M(ffx, ggx, t) &= \left(\exp \left(\frac{|x^4 - 4x + 3|}{t} \right) \right)^{-1} = \left(\exp \left(\frac{R(x-1)^2 (x^2 + 2x + 3)}{tR} \right) \right)^{-1} \\ &< \left(\exp \left(\frac{R|x-1|^2}{t} \right) \right)^{-1} = M(fx, gx, t/R) \text{ for } x > 1. \end{aligned}$$

and

$$\begin{aligned}
N(\text{ffx}, \text{ggx}, t) &= \left[\exp\left(\frac{|x^4-4x+3|}{t}\right) - 1 \right] \left[\exp\left(\frac{|x^4-4x+3|}{t}\right) \right]^{-1} \\
&= \left[\exp\left(\frac{R(x-1)^2(x^2+2x+3)}{tR}\right) - 1 \right] \left[\exp\left(\frac{R(x-1)^2(x^2+2x+3)}{tR}\right) \right]^{-1} \\
&> \left[\exp\left(\frac{R|x-1|^2}{t}\right) - 1 \right] \left[\exp\left(\frac{R|x-1|^2}{t}\right) \right]^{-1}
\end{aligned}$$

= $N(\text{fx}, \text{gx}, t/R)$ for $x > 1$, which shows that ‘R-weak commutativity’ does not imply ‘R-weak commutativity of type (P)’.

Finally, for a change the pair (f, g) is R-weakly commuting of type (A_g) as

$$\begin{aligned}
M(\text{gfx}, \text{ffx}, t) &= \left(\exp\left(\frac{|(2x-1)^2-4x+3|}{t}\right) \right)^{-1} = \left(\exp\left(\frac{4|x-1|^2}{t}\right) \right)^{-1} \\
&= M(\text{fx}, \text{gx}, t/4) \\
N(\text{gfx}, \text{ffx}, t) &= \left[\exp\left(\frac{|(2x-1)^2-4x+3|}{t}\right) - 1 \right] \left[\exp\left(\frac{|(2x-1)^2-4x+3|}{t}\right) \right]^{-1} \\
&= \left[\exp\left(\frac{4|x-1|^2}{t}\right) - 1 \right] \left[\exp\left(\frac{4|x-1|^2}{t}\right) \right]^{-1}
\end{aligned}$$

= $N(\text{fx}, \text{gx}, t/4)$, which shows that (f, g) is R-weakly commuting of type (A_g) for $R = 4$. This situation may also be utilized to interpret that an R-weakly commuting pair of type (A_g) need not be R-weakly commuting pair of type (A_f) or type (P). We can easily find examples to establish the independence of one of these definitions from the others which shows that there exist situations to suit a definition but not the others.

3. Main Results

Lemma 3.1[1]. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1)$,

$$(3.1) \quad M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t) \text{ then } x = y.$$

Proof. Suppose that there exists a number $k \in (0, 1)$ such that

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t) \text{ for all } x, y \in X \text{ and } t > 0.$$

$$\text{Then } M(x, y, t) \geq M(x, y, t/k) \geq M(x, y, t/k^2) \dots \geq M(x, y, t/k^n)$$

and $N(x, y, t) \leq N(x, y, t/k) \leq N(x, y, t/k^2) \dots \leq N(x, y, t/k^n)$, for some positive integer n .

Proceeding \lim as $n \rightarrow \infty$, we have $x = y$.

Lemma 3.2 [1]. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X . If there exists a number $k \in (0, 1)$ such that

(3.2) $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$ for all $t > 0$ and $n = 1, 2, \dots$, then $\{y_n\}$ is a Cauchy sequence in X .

Alace et al [3] proved the following theorem:

Theorem 3.1. Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces $(X, M, N, *, \diamond)$ with continuous t-norm $*$ and continuous t-conorm \diamond defined by $a*a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying the following conditions:

(3.3) $A(X) \subset T(X)$, $B(X) \subset S(X)$

(3.4) S and T are continuous

(3.5) the pairs $\{A, S\}$ and $\{B, T\}$ are compatible maps

(3.6) for all $x, y \in X$, $t > 0$ and $k \in (0, 1)$ such that

$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ * M(By, Sx, 2t) * M(Ax, Ty, t)$$

$$N(Ax, By, kt) \leq N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond N(By, Ty, t) \\ \diamond N(By, Sx, 2t) \diamond N(Ax, Ty, t).$$

Then A, B, S and T have a unique common fixed point in X .

Now we prove our main result which generalizes the above theorem.

Theorem 3.2. Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces $(X, M, N, *, \diamond)$ with continuous t-norm $*$ and continuous t-conorm \diamond defined by $a*a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying (3.3), (3.6), then

(i) A and S have a point of coincidence

(ii) B and T have a point of coincidence.

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible maps, then A, B, S and T have a unique common fixed point in X .

Proof. By (3.3), since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1 , we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2, \dots$$

By Theorem 17 of Alaca et al. [3], we have $\{y_n\}$ is Cauchy sequence in X . Since $(X, M, N, *, \diamond)$ is a complete intuitionistic fuzzy metric spaces, therefore, there exists a point z in X such that $\lim_{n \rightarrow \infty} y_n = z$. So $\lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = z$ and $\lim_{n \rightarrow \infty} Bx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+2} = z$, i.e., $\lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Bx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+2} = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$.

Now using (v) and (xi) of Definition of 2.3 and inequality (3.6), one obtain

$$M(Au, z, (k+1)t) \geq M(Au, Bx_{2n-1}, kt) * M(Bx_{2n-1}, z, t) \\ \geq M(Su, Tx_{2n-1}, t) * M(Au, Su, t) * M(Bx_{2n-1}, Tx_{2n-1}, t) \\ * M(Bx_{2n-1}, Su, 2t) * M(Au, Tx_{2n-1}, t) * M(Bx_{2n-1}, z, t)$$

and

$$N(Au, z, (k+1)t) \leq N(Au, Bx_{2n-1}, kt) \diamond N(Bx_{2n-1}, z, t) \\ \leq N(Su, Tx_{2n-1}, t) \diamond N(Au, Su, t) \diamond N(Bx_{2n-1}, Tx_{2n-1}, t)$$

$$\diamond N(Bx_{2n-1}, Su, 2t) \diamond N(Au, Tx_{2n-1}, t) \diamond N(Bx_{2n-1}, z, t)$$

Taking limit as $n \rightarrow \infty$, we get $Au = z = Su$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then again using (3.6), we have

$$\begin{aligned} M(z, Bv, kt) &= M(Au, Bv, kt) \\ &\geq M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) \\ &\quad * M(Bv, Su, 2t) * M(Au, Tv, t) \end{aligned}$$

$$\begin{aligned} \text{and} \quad N(z, Bv, kt) &= N(Au, Bv, kt) \\ &\leq N(Su, Tv, t) \diamond N(Au, Su, t) \diamond N(Bv, Tv, t) \\ &\quad \diamond N(Bv, Su, 2t) \diamond N(Au, Tv, t) \end{aligned}$$

Therefore, we obtain $z = Bv = Tv$. Thus $Au = Su = Bv = Tv = z$, this implies that both pairs have a point of coincidence.

Since pair of maps A and S are weakly compatible, therefore, $ASu = SAu$, i.e., $Az = Sz$. Now we prove that z is a common fixed point of A and S . Then again from (3.6), we have

$$\begin{aligned} M(Az, z, kt) &= M(Az, Bv, kt) \\ &\geq M(Sz, Tv, t) * M(Az, Sz, t) * M(Bv, Tv, t) \\ &\quad * M(Bv, Sz, 2t) * M(Az, Tv, t) \\ &= M(Sz, z, t) * 1 * 1 * M(z, Sz, 2t) * M(Az, z, t) \end{aligned}$$

$$\begin{aligned} \text{and} \quad N(Az, z, kt) &= N(Az, Bv, kt) \\ &\leq N(Sz, Tv, t) \diamond N(Az, Sz, t) \diamond N(Bv, Tv, t) \\ &\quad \diamond N(Bv, Sz, 2t) \diamond N(Az, Tv, t), \text{ which implies } Az = z. \end{aligned}$$

Hence $Az = Sz = z$.

Similarly, pair of maps B and T are weakly compatible, we have $Bz = Tz = z$, since

$$\begin{aligned} M(z, Bz, kt) &= M(Az, Bz, kt) \geq M(Sz, Tz, t) * M(Az, Sz, t) * M(Bz, Tz, t) \\ &\quad * M(Bz, Sz, 2t) * M(Az, Tz, t) \end{aligned}$$

$$\begin{aligned} \text{and} \quad N(z, Bz, kt) &= N(Az, Bz, kt) \leq N(Sz, Tz, t) \diamond N(Az, Sz, t) \diamond N(Bz, Tz, t) \\ &\quad \diamond N(Bz, Sz, 2t) \diamond N(Az, Tz, t), \text{ which implies } Bz = z. \end{aligned}$$

Thus $Az = Bz = Sz = Tz = z$. Hence z is a common fixed point of A, B, S and T .

Theorem 3.3. Theorem 3.2 remains true if a weakly compatible mappings property is replaced by any one (retaining the rest of the hypotheses) of the following:

- (i) R-weakly commuting property,
- (ii) R-weakly commuting of type (A_f) property,
- (iii) R-weakly commuting of type (A_g) property,
- (iv) R-weakly commuting of type (P) property,
- (v) weakly commuting property.

Proof: Since all the conditions of Theorem 3.2 are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (A, S) , then using R-weak commutativity one gets

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R) = 1$$

$N(ASx, SAx, t) \leq N(Ax, Sx, t/R) = 0$, which amounts to say that $ASx = SAx$. Thus the pair (A, S) is weakly compatible. Similarly (B, T) commutes at all of its

coincidence points. Now applying Theorem 3.2, one concludes that A,B, S and T have a unique common fixed point.

In case (A, S) is an R-weakly commuting of type (A_f) pair, one obtain

$$M(ASx, S^2x, t) \geq M(Ax, Sx, t/R) = 1$$

$N(ASx, S^2x, t) \leq N(Ax, Sx, t/R) = 0$, which amounts to say that $ASx = S^2x$. Now

$$\begin{aligned} M(ASx, SAx, t) &\geq M(ASx, S^2x, \frac{t}{2}) * M(S^2x, SAx, \frac{t}{2}) \\ &= 1 * 1 = 1 \end{aligned}$$

$$N(ASx, SAx, t) \leq N(ASx, S^2x, \frac{t}{2}) \diamond N(S^2x, SAx, \frac{t}{2})$$

$= 0 \diamond 0 = 0$, yielding thereby $ASx = SAx$. Similarly, if pair is R-weakly commuting mappings of type (A_g) or type (P) or weakly commuting, then (A, S) also commutes at their points of coincidence. Similarly, one can show that the pair (B, T) is also weakly compatible. Now in view of Theorem 3.2, in all four cases A, B, S and T have a unique common fixed point. Hence completes the proof.

As an application of Theorem 3.2, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 3.4. Let $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$, $\{S_1, S_2, \dots, S_p\}$ and $\{T_1, T_2, \dots, T_q\}$ be four finite families of self-mappings of complete intuitionistic fuzzy metric spaces $(X, M, N, *, \diamond)$ with continuous t-norm $*$ and continuous t-conorm \diamond defined by $a*a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ such that $A = A_1A_2 \dots A_m$, $B = B_1B_2 \dots B_n$, $S = S_1S_2 \dots S_p$ and $T = T_1T_2 \dots T_q$ satisfying the conditions (3.3) and (3.6), then

- (i) A and S have a point of coincidence,
- (ii) B and T have a point of coincidence.

Moreover, if $A_iA_j = A_jA_i$, $B_kB_l = B_lB_k$, $S_rS_s = S_sS_r$, $T_tT_u = T_uT_t$, $A_iS_r = S_rA_i$ and $B_kT_t = T_tB_k$ for all $i, j \in I_1 = \{1, 2, \dots, m\}$, $k, l \in I_2 = \{1, 2, \dots, n\}$, $r, s \in I_3 = \{1, 2, \dots, p\}$ and $t, u \in I_4 = \{1, 2, \dots, q\}$, then (for all $i \in I_1, k \in I_2, r \in I_3$ and $t \in I_4$) A_i, S_r, B_k and T_t have a common fixed point.

Proof: The conclusions (i) and (ii) are immediate as A, B, S and T satisfies all the conditions of Theorem 3.2. Now appealing to component wise commutativity of various pairs, one can immediately prove that $AS = SA$ and $BT = TB$ and hence, obviously both pairs (A, S) and (B, T) are weakly compatible. Note that all the conditions of Theorem 3.2 are satisfied ensuring the existence of a unique common fixed point say z. Now one need to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} A(A_i z) &= ((A_1A_2 \dots A_m)A_i)z = (A_1A_2 \dots A_{m-1})((A_mA_i)z) \\ &= (A_1 \dots A_{m-1})(A_iA_m z) = (A_1 \dots A_{m-2})(A_{m-1}A_i(A_m z)) \\ &= (A_1 \dots A_{m-2})(A_iA_{m-1}(A_m z)) = \dots \\ &= A_1A_i(A_2A_3A_4 \dots A_m z) = A_iA_1(A_2A_3 \dots A_m z) \\ &= A_i(Az) = A_i z. \end{aligned}$$

Similarly, one can show that

$$A(S_r z) = S_r(Az) = S_r z, S(S_r z) = S_r(Sz) = S_r z,$$

$$\begin{aligned} S(A_i z) &= A_i(Sz) = A_i z, B(B_k z) = B_k(Bz) = B_k z, \\ B(T_t z) &= T_t(Bz) = T_t z, T(T_t z) = T_t(Tz) = T_t z \\ \text{and } T(B_k z) &= B_k(Tz) = B_k z, \end{aligned}$$

which show that (for all i, r, k and t) $A_i z$ and $S_r z$ are other fixed points of the pair (A, S) whereas $B_k z$ and $T_t z$ are other fixed points of the pair (B, T) . Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$z = A_i z = S_r z = B_k z = T_t z$, which shows that z is a common fixed point of A_i, S_r, B_k and T_t for all i, r, k and t .

By setting $A = A_1 = A_2 = \dots = A_m, B = B_1 = B_2 = \dots = B_n, S = S_1 = S_2 = \dots = S_p$ and $T = T_1 = T_2 = \dots = T_q$, one deduces the following f or certain iterates of maps, which runs as follows:

Corollary 3.1. Let A, B, S and T be four self-mappings of complete intuitionistic fuzzy metric spaces $(X, M, N, *, \diamond)$ with continuous t -norm $*$ and continuous t -conorm \diamond defined by $a * a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ such that A_m, B_n, S_p and T_q satisfy the conditions (3.3) and (3.6). If one of $A_m(X), B_n(X), S_p(X)$ or $T_q(X)$ is a complete subspace of X , then A, B, S and T have a unique common fixed point provided (A, S) and (B, T) commute.

Example 3.1. Let $X = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$ with the usual metric and, for all $t > 0$ and $x, y \in X$, define (M, N) by

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

and

$$N(x, y, t) = \begin{cases} \frac{|x-y|}{t + |x-y|}, & t > 0, \\ 1 & t = 0 \end{cases}$$

Clearly, $(X, M, N, *, \diamond)$ is a complete intuitionistic fuzzy metric space, where $*$ and \diamond are defined by $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ respectively. Let A, B, S

and T be define by $Ax = \frac{x}{4}, Sx = \frac{x}{2}, Bx = \frac{x}{6}, Tx = \frac{x}{3}$ for all $x \in X$.

Then we have

$$A(X) = \left\{ \frac{1}{4n} : n = 1, 2, \dots \right\} \cup \{0\} \subseteq \left\{ \frac{1}{2n} : n = 1, 2, \dots \right\} \cup \{0\} = S(X)$$

$$B(X) = \left\{ \frac{1}{6n} : n = 1, 2, \dots \right\} \cup \{0\} \subseteq \left\{ \frac{1}{3n} : n = 1, 2, \dots \right\} \cup \{0\} = T(X)$$

Also, the condition (3.3) of Theorem 3.2 is satisfied. Further, the pairs (A, S) and (B, T) are weakly compatible maps at $x = 0$, which is the coincidence points of the

maps A, B, S and T. Thus all the conditions of Theorem 3.2 are satisfied and also 0 is the unique common fixed point of A, B, S and T.

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