Advances in Fuzzy Mathematics
ISSN 0973-533X Volume 4, Number 1 (2009), pp. 1–8
© Research India Publications
http://www.ripublication.com/afm.htm

Product of Intuitionistic Fuzzy Bck-Algebras

Bavanari Satyanarayana and Ravutla Durga Prasad

Department of Applied Mathematics, Acharya Nagarjuna University,
Dr. M.R. Appa Row Campus,
Nuzvid-521 201, Krishna (District), Andhra Pradesh, INDIA

Abstract

The purpose of this paper is to introduce the notion of product of two intuitionistic fuzzy sets in BCK-algebras and we provide some results on it.

Key Words: BCK-algebras, Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy ideals

1. Introduction

Iseki and Tanaka [4] introduced BCK-algebras. Atanassov [2] introduced intuitionistic fuzzy sets. Satyanarayana, Kondala Rao and Krishna [3] studied BCK-algebras and investigated some relations. Akram and Dar [1] introduced the Cartesian product two fuzzy sets. In this paper we introduce the Cartesian product of two Intuitionistic fuzzy BCK-algebras and investigated some results.

2. Preliminaries

Definition: 2.1. Algebra $(X, *, 0)$ of type (2, 0) is called a BCK – algebra, if for all $x, y \in X$, the following axioms hold:

(1) $(x * y) * (x * z) \le (z * y)$
(2) $x * (x * y) \le y$
(3) $x \le x$
(4) $x \le y, y \le x \Rightarrow x = y$
(5) $0 \le x$
Where $x \le y$ is defined by $x * y = 0$.

Definition: 2.2: A subset I of a BCK-algebra $(X, *, 0)$ is called an ideal of X, if for any $x, y \in X$

(i) $0 \in I$

(ii) $x * y$ and $y \in I \Rightarrow x \in I$.

Definition: 2.3: An ideal I of a BCK-algebra $(X,*,0)$ is called closed, if $0 * x \in I$, for all $x \in I$.

Proposition: 2.4: If y is in ideal of I and $x \leq y$ then $x \in I$.

Proof: Let y is in ideal of I and $x \leq y$

=> $y \in I$ and $x * y = 0 \in I$

=> $x \in I$.

Definition: 2.5: Let X be a non empty set. A fuzzy sub set μ of the set X is a mapping μ: X → [0, 1]. The complement of a fuzzy set μ of a set X is

denoted by $\bar{\mu}(x) = 1 - \mu(x)$, for all $x \in X$.

Definition: 2.6.: Let μ and λ be the fuzzy sets of X. For s, t ε [0, 1], the set U(μ,s) = { $x \in X$ / μ(x) ≥ s} is called upper level of μ and the set L(λ, t) = { $x \in X$ / λ(x) ≤ t} is called Lower level of λ.

Definition: 2.7.: An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x))/x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denoted the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we use the symbol form $A = (X, \mu_A, \lambda_A)$ or $A = (\mu_A, \lambda_A)$.

Definition: 2.8.: Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in X. Then

(i) $\Pi A = \{(x, \mu_A(x), \bar{\mu}_A(x))/x \in X\}$

(ii) $\Diamond A = \{(x, \bar{\lambda}_A(x), \lambda_A(x))/x \in X\}$.

Definition: 2.9.: An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in X is called an intuitionistic fuzzy ideal of X, if it satisfies the following axioms:

(IF1) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$

(IF2) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$

(IF3) $\lambda_A(x) \leq \max\{\lambda_A(x * y), \lambda_A(y)\}$, for all $x, y \in X$.

Definition: 2.10.: An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in BCK-algebra X is called an intuitionistic fuzzy closed ideal of X, if it satisfies (IF2), (IF3) and the following:

(IF4) $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$ for all $x \in X$.

Definition: 2.11.: Let μ and λ be the fuzzy sets in a set X. The Cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(x)\}$, for all $x, y \in X$.

Definition: 2.12.: Let $A = (X, \mu_A, \lambda_A)$ and $B = (X, \mu_B, \lambda_B)$ are two IFS of X. The Cartesian product $A \times B = (X \times X, \mu_A \times \mu_B, \lambda_A \times \lambda_B)$ is defined by $(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\lambda_A \times \lambda_B)(x, y) = \max\{\lambda_A(x), \lambda_B(x)\}$, where $\mu_A \times \mu_B : X \times X \to [0,1]$ and $\lambda_A \times \lambda_B : X \times X \to [0,1]$, $\forall\, x, y \in X$.

3. Main Results

Proposition: 3.1.: Let $A = (X, \mu_A, \lambda_A)$ and $B = (X, \mu_B, \lambda_B)$ are IF ideals of X then $A \times B$ is IF-ideal of $X \times X$.

Proof: For any $(x, y) \in X \times X$, we have

$$\begin{aligned}(\mu_A \times \mu_B)(0,0) &= \min\{\mu_A(0), \mu_B(0)\} \\ &\geq \min\{\mu_A(x), \mu_B(y)\}, \text{for all } x, y \in X, \\ &= (\mu_A \times \mu_B)(x, y).\end{aligned}$$

And

$$\begin{aligned}(\lambda_A \times \lambda_B)(0,0) &= \max\{\lambda_A(0), \lambda_B(0)\} \\ &\leq \max\{\lambda_A(x), \lambda_B(y)\}, \text{for all } x, y \in X, \\ &= (\lambda_A \times \lambda_B)(x, y).\end{aligned}$$

Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$(\mu_A \times \mu_B)(x_1, y_1) = \min\{\mu_A(x_1), \mu_B(y_1)\}$$

$$\begin{aligned}
&\geq \min\{\min\{\mu_A(x_1 * x_2), \mu_A(x_2)\}, \min\{\mu_B(y_1 * y_2), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)\}, \min\{\mu_A(x_2), \mu_B(y_2)\}\} \\
&= \min\{(\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2),\ (\mu_A \times \mu_B)(x_2, y_2)\} \\
&= \min\{(\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)),\ (\mu_A \times \mu_B)(x_2, y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A \times \lambda_B)(x_1, y_1) &= \max\{\lambda_A(x_1), \lambda_B(y_1)\} \\
&\leq \max\{\max\{\lambda_A(x_1 * x_2), \lambda_A(x_2)\}, \max\{\lambda_B(y_1 * y_2), \lambda_B(y_2)\}\} \\
&\leq \max\{\max\{\lambda_A(x_1 * x_2), \lambda_B(y_1 * y_2)\}, \max\{\lambda_A(x_2), \lambda_B(y_2)\}\} \\
&= \max\{(\lambda_A \times \lambda_B)(x_1 * x_2, y_1 * y_2),\ (\lambda_A \times \lambda_B)(x_2, y_2)\} \\
&= \max\{(\lambda_A \times \lambda_B)((x_1, y_1) * (x_2, y_2)), (\lambda_A \times \lambda_B)(x_2, y_2)\}.
\end{aligned}$$

Hence $A \times B$ is IF-ideals of $X \times X$.

Proposition: 3.2.: Let $A = (X, \mu_A, \lambda_A)$ and $B = (X, \mu_B, \lambda_B)$ are IF closed ideals of X then $A \times B$ is IF closed ideal of $X \times X$.

Proof: Since

$$\begin{aligned}
(\mu_A \times \mu_B)((0,0) * (x, y)) &= (\mu_A \times \mu_B)(0 * x, 0 * y) \\
&= \min\{\mu_A(0 * x), \mu_B(0 * y)\} \\
&\geq \min\{\mu_A(x), \mu_B(y)\} \\
&= (\mu_A \times \mu_B)(x, y).
\end{aligned}$$

Similarly we can prove $(\lambda_A \times \lambda_B)((0,0) * (x, y)) \leq (\lambda_A \times \lambda_B)(x, y)$. Hence $A \times B$ is IF closed ideal of $X \times X$.

Lemma: 3.3.: If $A = (X, \mu_A, \lambda_A)$ and $B = (X, \mu_B, \lambda_B)$ are IF ideals of X then so is $\Pi(A \times B) = (X, \mu_A \times \mu_B, \bar{\mu}_A \times \bar{\mu}_B)$.

Proof: Since

$$(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

$$\Rightarrow \quad 1 - (\bar{\mu}_A \times \bar{\mu}_B)(x, y) = \min\{1 - \bar{\mu}_A(x),\ 1 - \bar{\mu}_B(y)\}$$

$$\Rightarrow 1 - \min\{1 - \bar{\mu}_A(x), 1 - \bar{\mu}_B(y)\} = (\bar{\mu}_A \times \bar{\mu}_B)(x, y)$$

$$\Rightarrow \quad (\bar{\mu}_A \times \bar{\mu}_B)(x, y) = \max\{\bar{\mu}_A(x), \bar{\mu}_B(y)\}.$$

Hence $\Pi(A\times B)=(X,\mu_A\times\mu_B,\bar{\mu}_A\times\bar{\mu}_B)$ is an IF ideal of $X\times X$.

Lemma: 3.4.: If $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF ideals of X then so is $\Diamond(A\times B)=(X\times X,\bar{\lambda}_A\times\bar{\lambda}_B,\lambda_A\times\lambda_B)$.

Proof: Consider

$$(\lambda_A\times\lambda_B)(x,y)=\max\{\lambda_A(x),\lambda_B(y)\}$$

$$\Rightarrow\quad 1-(\bar{\lambda}_A\times\bar{\lambda}_B)(x,y)=\max\{1-\bar{\lambda}_A(x),\ 1-\bar{\lambda}_B(y)\}$$

$$\Rightarrow 1-\max\{1-\bar{\lambda}_A(x),1-\bar{\lambda}_B(y)\}=(\bar{\lambda}_A\times\bar{\lambda}_B)(x,y)$$

$$\Rightarrow\quad (\bar{\lambda}_A\times\bar{\lambda}_B)(x,y)=\min\{\bar{\lambda}_A(x),\bar{\lambda}_B(y)\}.$$

Thus $\Diamond(A\times B)=(X\times X,\bar{\lambda}_A\times\bar{\lambda}_B,\lambda_A\times\lambda_B)$ is IF ideal of BCK-algebra $X\times X$.

By the above two lemmas, it is not difficult to verify that the following theorem is valid.

Theorem: 3.5.: If $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF ideals of X if and only if $\Pi(A\times B)=(X,\mu_A\times\mu_B,\bar{\mu}_A\times\bar{\mu}_B)$ and $\Diamond(A\times B)=(X\times X,\bar{\lambda}_A\times\bar{\lambda}_B,\lambda_A\times\lambda_B)$ are IF ideals of $X\times X$.

Lemma: 3.6.: If $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF closed ideals of X then so is $\Pi(A\times B)=(X,\mu_A\times\mu_B,\bar{\mu}_A\times\bar{\mu}_B)$.

Proof: Let $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF closed ideals of X implies $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF ideals of X implies $A\times B$ is IF ideal of X.

Since $\quad(\mu_A\times\mu_B)((0,0)*(x,y))\geq(\mu_A\times\mu_B)(x,y)$

$\Rightarrow\quad 1-(\bar{\mu}_A\times\bar{\mu}_B)((0,0)*(x,y))\geq 1-(\bar{\mu}_A\times\bar{\mu}_B)(x,y)$

$\Rightarrow\quad (\bar{\mu}_A\times\bar{\mu}_B)((0,0)*(x,y))\leq(\bar{\mu}_A\times\bar{\mu}_B)((x,y))$.

Hence $\Pi(A\times B)=(X,\mu_A\times\mu_B,\bar{\mu}_A\times\bar{\mu}_B)$ is an IF closed ideal of $X\times X$.

Lemma: 3.7.: If $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF closed ideals of X then so is $\Diamond(A\times B)=(X\times X,\bar{\lambda}_A\times\bar{\lambda}_B,\lambda_A\times\lambda_B)$.

Proof: The proof is similar to the proof of above lemma.

By the above two lemmas, it is not difficult to verify that the following theorem is valid.

Theorem: 3.8.: If $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF closed ideals of X if and only if $\Pi(A\times B)=(X,\mu_A\times\mu_B,\bar{\mu}_A\times\bar{\mu}_B)$ and $\Diamond(A\times B)=(X\times X,\bar{\lambda}_A\times\bar{\lambda}_B,\lambda_A\times\lambda_B)$ are IF closed ideals of $X\times X$.

4. Upper and Lower Level Cuts

Definition: 4.1.**:** Let $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy BCK-sub algebra. For $s,t\in[0,1]$, the set $U(\mu_A,s)=\{x\in X/\mu_A(x)\geq s\}$ is called upper level of $\mu_A(x)$ and $L(\lambda_A,t)=\{x\in X/\lambda_A(x)\leq t\}$ is called lower level of $\lambda_A(x)$.

Definition: 4.2.: Let $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are IF sub sets of X. For $s,t\in[0,1]$, the set $U(\mu_A\times\mu_B;s)=\{(x,y)\in X\times X/(\mu_A\times\mu_B)(x,y)\geq s\}$ is called upper level of $(\mu_A\times\mu_B)(x,y)$ and $L(\lambda_A\times\lambda_B;t)=\{(x,y)\in X\times X/(\lambda_A\times\lambda_B)(x,y)\leq t\}$ is called lower level of $(\lambda_A\times\lambda_B)(x,y)$.

Theorem: 4.3: An intuitionistic fuzzy sets $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are intuitionistic fuzzy closed ideals of X if and only if the non-empty upper s-level cut $U(\mu_A\times\mu_B;\ s)$ and the non-empty lower t-level cut $L(\lambda_A\times\lambda_B;\ t)$ are closed ideals of $X\times X$ for any s, t ε [0, 1].

Proof: Let $A=(X,\mu_A,\lambda_A)$ and $B=(X,\mu_B,\lambda_B)$ are intuitionistic fuzzy closed ideals of X, therefore for any $(x,y)\in X\times X$,

$$(\mu_A\times\mu_B)((0,0)*(x,y))\geq(\mu_A\times\mu_B)(x,y)$$

and $$(\lambda_A\times\lambda_B)((0,0)*(x,y))\leq(\lambda_A\times\lambda_B)(x,y).$$

For, $s\in[0,1]$, if $(\mu_A\times\mu_B)(x,y)\geq s$ => $(\mu_A\times\mu_B)((0,0)*(x,y))\geq s$

=> $(0,0)*(x,y)\in U(\mu_A\times\mu_B;\ s)$. Let $(x,y),(x',y')\in X\times X$ such that $(x,y)*(x',y')\in U(\mu_A\times\mu_B;s)$ and $(x',y')\in U(\mu_A\times\mu_B;\ s)$. Now $(\mu_A\times\mu_B)(x,y)\geq\min\{(\mu_A\times\mu_B)((x,y)*(x',y')),\ (\mu_A\times\mu_B)(x',y')\}$

$\geq\min\{s, s\} = s$

=> $(x,y)\in U(\mu_A\times\mu_B);\ s)$. Thus $U(\mu_A\times\mu_B;\ s)$ is closed ideal of $X\times X$.

Similar to above $L(\lambda_A\times\lambda_B;\ t)$ is closed ideal of $X\times X$.

Converse: let $(x,y)\in X\times X$ such that $(\mu_A\times\mu_B)(x,y)=s$ and $(\lambda_A\times\lambda_B)(x,y)=t$ implies $(x,y)\in U(\mu_A\times\mu_B;\ s)$ and $(x,y)\in L(\lambda_A\times\lambda_B;\ t)$.

Since $(0,0)*(x,y)\in U(\mu_A\times\mu_B;\ s)$ and $(0,0)*(x,y)\in L(\lambda_A\times\lambda_B;t)$ (by definition of closed ideal)

=> $(\ \mu_A\times\mu_B)((0,0)*(x,y))\geq s$ and $(\lambda_A\times\lambda_B)((0,0)*(x,y))\leq t$

=> $(\mu_A\times\mu_B)((0,0)*(x,y))\geq(\mu_A\times\mu_B)(x,y)$ and
$(\lambda_A\times\lambda_B)((0,0)*(x,y))\leq(\lambda_A\times\lambda_B)(x,y)$

Claim:

(1) $(\mu_A\times\mu_B)(x,y)\geq\min\{(\mu_A\times\mu_B)((x,y)*(x_1,y_1)),(\mu_A\times\mu_B)(x_1,y_1)\}$,

(2) $(\lambda_A\times\lambda_B)(x,y)\leq\max\{(\lambda_A\times\lambda_B)((x,y)*(x_1,y_1)),\lambda_A\times\lambda_B(x_1,y_1)\}$.

If possible, we assume that $(a,b),(c,d)\in X\times X$ such that $(\mu_A\times\mu_B)(a,b)\ \leq\min\{(\mu_A\times\mu_B)((a,b)*(c,d)),\ \mu_A\times\mu_B(c,d)\}$. Let

$s=\frac{1}{2}[(\mu_A\times\mu_B)(a,b)+\min\{(\mu_A\times\mu_B)((a,b)*(c,d)),(\mu_A\times\mu_B)(c,d)\}]$

=> $s\leq\min\{(\mu_A\times\mu_B)((a,b)*(c,d)),(\mu_A\times\mu_B)(c,d)\}$

=> $(\mu_A\times\mu_B)((a,b)*(c,d))\geq\ s$ and $(\mu_A\times\mu_B)(c,d)\geq s$

=> $(a,b)*(c,d)\in U(\mu_A\times\mu_B;\ s)$ and $(c,d)\in U(\mu_A\times\mu_B;\ s)$, but $(a,b)\notin U(\mu_A\times\mu_B;\ s)$, which is contradiction to closed ideal. Hence the claim (1). Similarly we can prove claim (2). This completes the proof of the theorem.

Acknowledgement

The authors take this opportunity to express their sincere thanks to Prof. K. Panduranga Rao and Prof. M.S.N.Murty, Department of Applied Mathematics, Acharya Nagarjuna University- Nuzvid Campus, Nuzvid for their encouragement and valuable suggestions.

References

[1] Akram, M. and Dar, K.H., On fuzzy d-algebras, Punjab Uni. J. Math. Vol.37(2005), PP.67-76.

[2] Atanassov, K.T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1)(1983), 87-96.

[3] Satyanarayana, B., Kondala Rao, E.V. and Krishna, L., Intuitionistic Fuzzy BCK-algebras (Communicated).

[4] Kiyoshi iseki and Shotaro Tanaka, An introduction to the theory of BCK-algebras, Math. Japon., 23(1978), 1-26.

[5] Kiyoshi Iseki and Shotaro Tanaka, Ideal Theory of BCK-algebras, Math. Japonica. 21(1976), 351-366.

[6] Jun, Y.B. and Kim, K.H., Intuitionistic fuzzy ideals of BCK-algebras, Internat. J. Math. and Math. Sci., 24(12) (2000), 839-849.

[7] Jun, Y.B., Closed fuzzy ideals in BCI-algebra, Math. Japonica, 3(1993), 199-202.

[8] Ougen, Xi, Fuzzy BCK-algebra, Math. Japanica 36, 5(1991), 935-942.

[9] Takeuti, G. and Titants, S. Intuitionistic fuzzy logic and Intuitionistic fuzzy set theory, Journal of Symbolic Logic, 49(1984), 851-866.

[10] L. A. Zadeh, L. A., Fuzzy sets, Information Control, 8(1965), 338-353.