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Bifuzzy (General) Topology on Max-Min General Fuzzy Automata

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Abstract

In this paper, by considering the notion of general fuzzy automata, we define the concepts of max-min general fuzzy automata, response function, accessible state with threshold c and connected with threshold c. Then we construct a bifuzzy topology on the set of fuzzy start states of a max-min general fuzzy automaton and a bifuzzy topology on a subset of the set words on the set of input symbols of a max-min general fuzzy automaton. Also, we define the concepts of connectedness (semi-connectedness) between two states p and q with respect to a fuzzy subset of the set of states of a max-min general fuzzy automaton, component of a max-min general fuzzy automaton and concordance between two states p and q with threshold c and with respect to a fuzzy subset of the set of states of a max-min general fuzzy automaton. Finally, by using these notions, we construct a general topology on the set of fuzzy start states of a max-min general fuzzy automaton.

Keywords: (General) Fuzzy automata; Bifuzzy (General) topology; Accessibility; Connectedness

1. Introduction and preliminaries

In 1965, L.A. Zadeh introduced the notion of a fuzzy subset of a set as a method for representing uncertainty [17]. His ideas have been applied to a wide range of scientific areas. One such area is automata theory first introduced by W.G. Wee in [16]. Automata have a long history both in theory and application [1,2,3,6,7,14,15].

A fuzzy finite-state automaton (FFA) is an six-tuple denoted as $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0,1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0,1]$ and $\omega : Q \to Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in $[0,1]$. We call this membership value the weight of the transition.

The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refer to the weight of the transition. Otherwise, it specifies the transition itself. Also, the set of all transitions of $\tilde{F}$ is denoted as Δ. The above definition is generally accepted as a formal definition for FFA [8,9,11,12].

There is the important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how to assign a membership value to a next state upon the completion of a transition. Secondly, how should we deal with the cases where a state is forced to take several membership values simultaneously via overlapping transition?

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [4]. Now, we follow [4] and give some new notions and results as mentioned in the abstract.

Let X be a set. A word of X is the product of a finite sequence of elements in X, Λ is empty word and X^* is the set of all words on X. In fact, X^* is the free monoid on X. The length $\ell(x)$ of word $x \in X^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ denoted the set of all fuzzy sets on X and $P(X)$ denoted the set of all subsets on X. For any $\lambda_i \in \tilde{P}(X)$, $i \in I$, and for any $x \in X$,

$$\text{(i)}\ (\bigcap_{i\in I} \lambda_i)(x) = \wedge\{\lambda_i(x) : i \in I\},$$

$$\text{(ii)}\ (\bigcup_{i\in I} \lambda_i)(x) = \vee\{\lambda_i(x) : i \in I\}.$$

Definition 1. 1. [4] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$,

(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, ..., a_m\}$,

(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subset \tilde{P}(Q)$,

(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$,

(v) $\omega : Q \to Z$ is the output function,

(vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function,

(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function .

Function $F_1(\mu,\delta)$ as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as :

$$\mu^{t+1}(q_j)=\tilde{\delta}((q_i,\mu^t(q_i)),a_k,q_j)=F_1(\mu^t(q_i),\delta(q_i,a_k,q_j)).$$

Which means that membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition .

There are many options which can be used for the function $F_1(\mu,\delta)$. It can be for example $\max\{\mu,\delta\}$, $\min\{\mu,\delta\}$, $\mu+\delta/2$ or any other applicable mathematical function.

(viii) $F_2:[0,1]^*\to[0,1]$ is called multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

We let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i\geq 0$. We have $Q_{act}(t_0)=\tilde{R}$ and $Q_{act}(t_i)=\{\ (q,\mu^{t_i}(q)):\exists q'\in Q_{act}(t_{i-1}),\exists a\in\Sigma,\delta(q',a,q)\in\Delta\ \},\forall i\geq 1$.

Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q\in Domain(Q_{act}(t_i))$ and $T\subseteq Domain(Q_{act}(t_i))$. Hereafter, we simply denote them as:

$$q\in Q_{act}(t_i)\text{ and }T\subseteq Q_{act}(t_i).$$

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 1. 2. [4] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to that :

(1) Each transition weight $\delta(q_i,a_k,q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i,

$$v_i=\tilde{\delta}((q_i,\mu^t(q_i)),a_k,q_j)=F_1(\mu^t(q_i),\delta(q_i,a_k,q_j)).$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous mv of the active state q_j.

$$\mu^{t+1}(q_j)=\overset{n}{\underset{i=1}{F_2}}[v_i]=\overset{n}{\underset{i=1}{F_2}}[F_1(\mu^t(q_i),\delta(q_i,a_k,q_j))].$$

Where

- n: is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$: is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Definition 1. 3. [5] Let X be a nonempty set. A subset τ of $P(X)$ is called a general topology on X if

(i) $\emptyset, X \in \tau$,

(ii) If $A_1, A_2 \in \tau$, then $A_1 \bigcap A_2 \in \tau$,

(iii) If $A_i \in \tau, \forall i \in I$, then $\bigcup_{i \in I} A_i \in \tau$.

Definition 1. 4. [13] Let X be a nonempty set and $\eta \in \tilde{P}(\tilde{P}(X))$. η is called a bifuzzy topology on X if

(i) $\eta(1) = \eta(0) = 1$,

(ii) If $\lambda_1, \lambda_2 \in \tilde{P}(X)$, then $\eta(\lambda_1) \wedge \eta(\lambda_2) \leq \eta(\lambda_1 \bigcap \lambda_2)$,

(iii) If $\lambda_i \in \tilde{P}(X), \forall i \in I$, then $\bigwedge_{i \in I} \eta(\lambda_i) \leq \eta(\bigcup_{i \in I} \lambda_i)$.

Definition 1. 5. [10] Let S, T be two sets and $\tilde{A}$, $\tilde{B}$ be fuzzy subsets of S and T, respectively. Then a fuzzy relation $\tilde{R}$ of $\tilde{A}$ to $\tilde{B}$ is a fuzzy subset of $S \times T$ such that $\tilde{R}(x, y) \leq \tilde{A}(x) \wedge \tilde{B}(y)$.

2. Two Bifuzzy Topology on Max-Min General Fuzzy Automata

Definition 2. 1. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata of the form $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that $\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0,1]$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ and let for every i, $i \geq 0$ $\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \quad , \\ 0, & otherwise \end{cases}$ and for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p) \ ,$$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p) \ ,$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 ... u_n, p) = \vee \{ \ \tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots$$

$$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) \ \Big| \ \ p_1 \in Q_{act}(t_1), \ p_2 \in Q_{act}(t_2), \ldots, \ p_{n-1} \in Q_{act}(t_{n-1}) \ \ \},$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \le i \le n-1$.

Definition 2. 2. Let $\tilde{F}^*$ be a max-min general fuzzy automaton. The response function $r^{\tilde{F}^*} : \Sigma^* \times Q \to [0,1]$ of $\tilde{F}^*$, for any $x \in \Sigma^*, q \in Q$, is defined by

$$r^{\tilde{F}^*}(x,q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q).$$

Definition 2. 3. Let $q \in Q$, $0 \le c < 1$. Then q is called an accessible state of $\tilde{F}^*$ with threshold c if there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x,q) > c$.

Theorem 2. 4. If q is an accessible state of $\tilde{F}^*$ with threshold c, then there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x_0,q) > c$ and $\ell(x_0) < n$, where $n = |Q|$.

Proof. Since q is an accessible state of $\tilde{F}^*$, then there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x,q) > c$. If $\ell(x) < n$, then $x_0 = x$. Otherwise, Let $x = u_1 u_2 \ldots u_m$, $m \ge n$. Then there exists $q_0 \in Q_{act}(t_0)$ such that $\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), x, q) > c$. Thus there exist $q_1 \in Q_{act}(t_1)$, $\ldots$, $q_{m-1} \in Q_{act}(t_{m-1})$ such that $\tilde{\delta}((q_0, \mu^{t_0}(q_0)), u_1, q_1) > c$, $\ldots$, $\tilde{\delta}((q_i, \mu^{t_i}(q_i)), u_{i+1}, q_{i+1}) > c$, $\ldots$, $\tilde{\delta}((q_{m-1}, \mu^{t_{m-1}}(q_{m-1})), u_m, q) > c$. Since $m \ge n$, then there exist i, j such that $q_i = q_j$, $i < j.$ Let $x_1 = u_1 u_2 \ldots u_i u_{j+1} \ldots u_m$. Then $r^{\tilde{F}^*}(x_1, q) > c$. If $\ell(x_1) < n$, then $x_0 = x_1$. If $\ell(x_1) \ge n$ repeat the same argument. □

Theorem 2. 5. Let q be an accessible state of $\tilde{F}^*$ with threshold c, for every c, $0 \le c < 1$. Then there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x,q) = 1$, $\ell(x) < n$, where $n = |Q|$.

Proof. The proof is similar to Theorem 2. 4. 2 of [8], by using the suitable modification . □

Theorem 2. 6. If q is an accessible state of $\tilde{F}^*$ with threshold c, for every c, $0 \le c < 1$, then there exists

$$0 \le j < |Q| - 1,\ q' \in Q_{act}(t_j) \text{ and } u \in \Sigma \text{ such that } \tilde{\delta}((q', \mu^{t_j}(q')), u, q) > c.$$

Proof. By Theorem 2.4, there exists $x_0 \in \Sigma^*$ such that $r^{\tilde{F}^*}(x_0, q) > c$, $\ell(x) < n$, where $n = |Q|$.

If $x_0 = u_1 u_2 ... u_m \ne \Lambda$, $m < n$, then there exists $q_0 \in Q_{act}(t_0)$ such that $\tilde{\delta}^*((q_0, \mu^{t_0}(q_0), u_1 u_2 ... u_m, q) >$ c. Thus, there exist $p_1 \in Q_{act}(t_1)$, $p_2 \in Q_{act}(t_2)$, $\ldots$, $p_{m-1} \in Q_{act}(t_{m-1})$ such that $\tilde{\delta}((q_0, \mu^{t_0}(q_0)), u_1,$ $p_1) > c$, $\tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) > c$, $\ldots$, $\tilde{\delta}((p_{m-1}, \mu^{t_{m-1}}(p_{m-1})), u_m, q) > c$. Hence, we should let $j = m-1, q' = p_{m-1} \in Q_{act}(t_{m-1})$, $u = u_m$. If $x_0 = \Lambda$, then there exists $q_0 \in Q_{act}(t_0)$ such that $\tilde{\delta}^*((q_0,$ $\mu^{t_0}(q_0)), \Lambda, q) > c$. Then $q_0 = q$. Consequently, we should let $j = 0$, $q' = q_0 = q \in Q_{act}(t_0), u = \Lambda$.□

Definition 2. 7. $\tilde{F}^*$ is said to be connected with threshold c, $0 \le c < 1$, if $Q = \overline{Q_C}$, where $\overline{Q_C}$ is the set of all accessible states with threshold c.

Theorem 2. 8. Let $x \in \Sigma^*, x \neq \Lambda$, $q_0 \in Q_{act}(t_0)$ and $C(q,x) = \underset{q' \in Q - Q_{act}(t_0)}{\wedge} \tilde{\delta}^*((q, \mu^{t_0}(q)), x, q')$.

If $C(q,x) \neq 0$, then $\tilde{F}^*$ is connected with threshold c, for every c, $0 \leq c < C(q,x)$.

Proof. Let $Q - Q_{act}(t_0) = \{q_1, q_2, ..., q_n\}$. Since $C(q,x) > c$, then $\tilde{\delta}^*((q, \mu^{t_0}(q)), x, q_i) > c, \forall 1 \leq i \leq n$.

Consequently, $r^{\tilde{F}^*}(x, q_i) = \underset{q \in Q_{act}(t_0)}{\vee} \tilde{\delta}^*((q, \mu^{t_0}(q)), x, q_i) > c, \forall 1 \leq i \leq n$. On the other hand, $r^{\tilde{F}^*}(\Lambda, q') = 1 > c, \forall q' \in Q_{act}(t_0)$. Thus, $\tilde{F}^*$ is connected with threshold c, $\forall c, 0 \leq c < C(q,x)$. □

Example 2. 9. Consider the max-min GFA in Fig.1 with several transition overlaps. It is specified as:

$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ where $Q = \{q_0, q_1, q_2, q_3, q_4\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $\tilde{R} = \{(q_0, 1)\}$, $Z = \varnothing$ and ω is not applicable, $Q_{act}(t_0) = \{q_0\}$, $Q_{act}(t_1) = \{q_1, q_4\}$, $Q_{act}(t_2) = \{q_1, q_2, q_4\}$, $Q_{act}(t_3) = \{q_2, q_3\}$, … , and

$${}^1F_1(\mu,\delta) = \delta, \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\wedge}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$${}^2F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\wedge}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$${}^3F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\vee}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$${}^4F_1(\mu,\delta) = Max(\mu,\delta), \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\wedge}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$${}^5F_1(\mu,\delta) = Max(\mu,\delta), \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\vee}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

$${}^6F_1(\mu,\delta) = Min(\mu,\delta), \quad F_2(\,) = \mu^{t+1}(q_m) = \sum_{i=1}^{n} F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)) / n,$$

$${}^7F_1(\mu,\delta) = \frac{\mu+\delta}{2}, \quad F_2(\,) = \mu^{t+1}(q_m) = \overset{n}{\underset{i=1}{\vee}} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))),$$

where n is the number of simultaneous transitions to the active state q_m at time $t+1$.

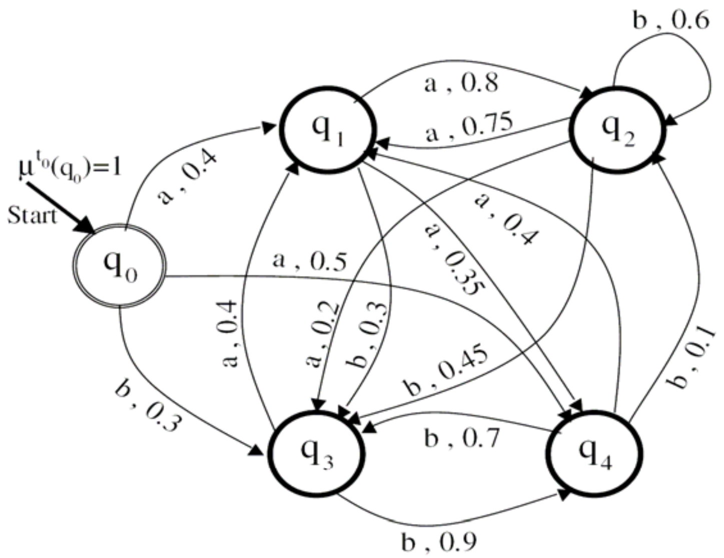

Figure 1: The GFA of Example 2.9

If we choose

$^1F_1(\mu,\delta)=\delta,\ F_2(\ )=\mu^{t+1}(q_m)=\overset{n}{\underset{i=1}{\wedge}}(F_1(\mu^t(q_i),\delta(q_i,a_k,q_m)))$, then we have :

$$\mu^{t_0}(q_0)=1,$$

$$\mu^{t_1}(q_1)=F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))=F_1(1{,}0.4)=0.4,$$

$$\mu^{t_1}(q_4)=F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_4))=F_1(1{,}0.5)=0.5,$$

$$\mu^{t_2}(q_1)=F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))=F_1(0.5{,}0.4)=0.4,$$

$$\mu^{t_2}(q_2)=F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_2))=F_1(0.4{,}0.8)=0.8,$$

$$\mu^{t_2}(q_4)=F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_4))=F_1(0.4{,}0.35)=0.35,$$

$$\mu^{t_3}(q_2)=F_1(\mu^{t_2}(q_4),\delta(q_4,b,q_2))\wedge F_1(\mu^{t_2}(q_2),\delta(q_2,b,q_2))=$$

$$=F_1(0.4{,}0.1)\wedge F_1(0.8{,}0.6)=0.1\wedge 0.6=0.1,$$

$$\mu^{t_3}(q_3)=F_1(\mu^{t_2}(q_1),\delta(q_1,b,q_3))\wedge F_1(\mu^{t_2}(q_2),\delta(q_2,b,q_3))\wedge$$

$$F_1(\mu^{t_2}(q_4),\delta(q_4,b,q_3))=$$

$$=F_1(0.4{,}0.3)\wedge F_1(0.8{,}0.45)\wedge F_1(0.35{,}0.7)=0.3\wedge 0.45\wedge 0.7=0.3,$$

which there are two simultaneous transitions to the active state q_2 at time t_3 and there are three simultane- ous transitions to the active state q_3 at time t_3. So, we can draw the following table :

Table 1: Active states and their membership values (mv) at different times in Example 2.9

time	t_0	t_1		t_2			t_3	
input	Λ	a		a			b	
$Q_{act}(t_i)$	q_0	q_1	q_4	q_1	q_2	q_4	q_2	q_3
mv^1	1.0	0.4	0.5	0.4	0.8	0.35	0.1	0.3
mv^2	1.0	0.4	0.5	0.4	0.4	0.35	0.1	0.3
mv^3	1.0	0.4	0.5	0.4	0.4	0.35	0.4	0.4
mv^4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^6	1.0	0.4	0.5	0.4	0.4	0.35	0.25	0.35
mv^7	1.0	0.7	0.75	0.575	0.75	0.525	0.763	0.613

If[5] $F_1(\mu,\delta)=Max(\mu,\delta),\ F_2(\ )=\mu^{t+1}(q_m)=\overset{n}{\underset{i=1}{\vee}}(F_1(\mu^t(q_i),\delta(q_i,a_k,q_m)))$, then we have

$$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0),a^2,q_1)=\underset{q'\in Q_{act}(t_1)}{\vee}[\tilde{\delta}((q_0,\mu^{t_0}(q_0),a,q')\wedge\tilde{\delta}((q',\mu^{t_1}(q'),a,q_1)]$$

$$=[\tilde{\delta}((q_0,\mu^{t_0}(q_0),a,q_1)\wedge\tilde{\delta}((q_1,\mu^{t_1}(q_1),a,q_1)]\vee[\tilde{\delta}((q_0,\mu^{t_0}(q_0),a,q_4)\wedge\tilde{\delta}((q_4,\mu^{t_1}(q_4),a,q_1)]$$

$$=[F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))\wedge F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_1))]\vee[F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_4))\wedge F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))]=$$

$$=[F_1(1,0.4)\wedge F_1(1,0)]\vee[F_1(1,0.5)\wedge F_1(1,0.4)]=[1\wedge 1]\vee[1\wedge 1]=1\vee 1=1.$$

Thus $\tilde{\delta}^*((q_0,\mu^{t_0}(q_0),a^2,q_1)=1$. Similarly, we have :

$$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0),a^2,q_2)=1,\ \tilde{\delta}^*((q_0,\mu^{t_0}(q_0),a^2,q_3)=1,\ \tilde{\delta}^*((q_0,\mu^{t_0}(q_0),a^2,q_4)=1.$$

Consequently, $C(q_0,a^2)=\underset{q'\in Q-\{q_0\}}{\wedge}\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q')=1\neq 0$.

Thus, by Theorem 2.8, $\tilde{F}^*$ is connected with threshold $c,\forall c,\ 0\le c<C(q_0,a^2)=1$. Also, we have

$$r^{\tilde{F}^*}(a^2,q_1)=\underset{q'\in Q_{act}(t_0)}{\vee}\tilde{\delta}^*((q',\mu^{t_0}(q')),a^2,q_1)=\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_1)=1,$$

$$r^{\tilde{F}^*}(a^2,q_2)=\underset{q'\in Q_{act}(t_0)}{\vee}\tilde{\delta}^*((q',\mu^{t_0}(q')),a^2,q_2)=\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_2)=1,$$

$$r^{\tilde{F}^*}(a^2,q_3)=\underset{q'\in Q_{act}(t_0)}{\vee}\tilde{\delta}^*((q',\mu^{t_0}(q')),a^2,q_3)=\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_3)=1,$$

$$r^{\tilde{F}^*}(a^2,q_4)=\underset{q'\in Q_{act}(t_0)}{\vee}\tilde{\delta}^*((q',\mu^{t_0}(q')),a^2,q_4)=\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_4)=1.$$

Definition 2. 10. Let $\tilde{F}^*$ be a max-min general fuzzy automaton. $\overline{D}$ is called bifuzzy operator on Q from $\tilde{P}(Q)$ to $\tilde{P}(Q)$ and it defined as follows:

$$\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}, \forall \lambda \in \tilde{P}(Q),\ \forall p \in Q.$$

Remark 2. 11. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and $p \in Q_{act}(t_0)$. Then we have $\lambda(p) \leq \overline{D}(\lambda)(p)$. In fact, we have $\lambda(p) = \lambda(p) \wedge 1 = \lambda(p) \wedge r^{\tilde{F}^*}(\Lambda, p) \leq \overline{D}(\lambda)(p)$.

Theorem 2. 12. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $\lambda_i \in \tilde{P}(Q), \forall i \in I$. Then

$$\text{(i) } \overline{D}(\bigcap_{i\in I} \lambda_i)(p) \leq \bigwedge_{i\in I} \overline{D}(\lambda_i)(p) \text{ for any } p \in Q,$$

$$\text{(ii) } \overline{D}(\bigcup_{i\in I} \lambda_i)(p) = \bigvee_{i\in I} \overline{D}(\lambda_i)(p) \text{ for any } p \in Q.$$

Proof. (i) $\overline{D}(\bigcap_{i\in I} \lambda_i)(p) = \vee\{(\bigwedge_{i\in I} \lambda_i(p)) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$

$$\leq \bigwedge_{i\in I}(\vee\{\lambda_i(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\})$$

$$= \bigwedge_{i\in I} \overline{D}(\lambda_i)(p).$$

(ii) In the process of the above proof, using $\bigcup_{i\in I}$ and $\bigvee_{i\in I}$ to replace $\bigcap_{i\in I}$ and $\bigwedge_{i\in I}$, respectively, we can obtain (ii) and therefore complete the proof of theorem. □

Let X be a nonempty set. Now, we introduced the following notations similar to the semantic expressions in logic, which are utilized in the sequel.

(i) $\nu(\lambda_1 \subseteq \lambda_2) = \bigwedge_{x\in X}(1 \wedge 1 - \lambda_1(x) + \lambda_2(x)), \forall \lambda_1, \lambda_2 \in \tilde{P}(X)$,

(ii) $\nu(\lambda_1 \equiv \lambda_2) = \nu(\lambda_1 \subseteq \lambda_2) \wedge \nu(\lambda_2 \subseteq \lambda_1), \forall \lambda_1, \lambda_2 \in \tilde{P}(X)$.

Remark 2. 13. By Lemma 2. 14 of [13], for $\lambda_i, \gamma_i \in \tilde{P}(X)$, $i \in I$, We have

$$\bigwedge_{i\in I} \nu(\lambda_i \equiv \gamma_i) \leq \nu(\bigcup_{i\in I} \lambda_i \equiv \bigcup_{i\in I} \gamma_i) \wedge \nu(\bigcap_{i\in I} \lambda_i \equiv \bigcap_{i\in I} \gamma_i).$$

Theorem 2. 14. Let $\tilde{F}^*$ be a max-min general fuzzy automaton. $\eta \in \tilde{P}(\tilde{P}(Q_{act}(t_0)))$ is defined as follows:

$$\eta(\lambda) = \nu(\overline{D}(\lambda) \equiv \lambda).$$

Then η is a bifuzzy topology on $Q_{act}(t_0)$.

Proof. Our purpose is to demonstrate that η fulfils the three conditions in Definition 1.4. (In fact, the conclusions we will verify are stronger than those in Definition 1.4.)

(i) For any $p \in Q_{act}(t_0)$, we have

$$\overline{D}(1)(p) = \vee\{1(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} = 1(p) \wedge r^{\tilde{F}^*}(\Lambda, p) = 1(p) \wedge 1 = 1(p).$$

So we get that

$$\eta(1) = \nu(\overline{D}(1) \equiv 1) = \nu(\overline{D}(1) \subseteq 1) \wedge \nu(1 \subseteq \overline{D}(1))$$

$$= (\bigwedge_{p\in Q_{act}(t_0)} (1 \wedge 1 - \overline{D}(1)(p) + 1(p))) \wedge (\bigwedge_{p\in Q_{act}(t_0)} (1 \wedge 1 - 1(p) + \overline{D}(1)(p)))$$

$$= (\bigwedge_{p\in Q_{act}(t_0)} (1 \wedge 1 - 1(p) + 1(p))) \wedge (\bigwedge_{p\in Q_{act}(t_0)} (1 \wedge 1 - 1(p) + 1(p))) = 1.$$

Similarly, since $\overline{D}(0)(p) = \vee\{0(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} = 0(p)$, therefore $\eta(0) = 1$.

(ii) For any $i \in I$, $\lambda_i \in \tilde{P}(Q_{act}(t_0))$,

$$\bigwedge_{i\in I} \eta(\lambda_i) \leq \eta(\bigcap_{i\in I} \lambda_i).$$

Utilizing Remarks 2.11, 2.13 and Theorem 2.12, we have

$$\bigwedge_{i\in I} \eta(\lambda_i) = \bigwedge_{i\in I} \nu(\overline{D}(\lambda_i) \equiv \lambda_i)$$

$$\leq \nu(\bigcap_{i\in I} \overline{D}(\lambda_i) \equiv \bigcap_{i\in I} \lambda_i) \wedge \nu(\overline{D}(\bigcap_{i\in I} \lambda_i) \equiv \bigcap_{i\in I} \lambda_i)$$

$$\leq \nu(\overline{D}(\bigcap_{i\in I} \lambda_i) \subseteq \bigcap_{i\in I} \lambda_i)$$

$$= \nu(\overline{D}(\bigcap_{i\in I} \lambda_i) \equiv \bigcap_{i\in I} \lambda_i) = \eta(\bigcap_{i\in I} \lambda_i).$$

(iii) For any $i \in I$, $\lambda_i \in \tilde{P}(Q_{act}(t_0))$,

$$\bigwedge_{i\in I} \eta(\lambda_i) \leq \eta(\bigcup_{i\in I} \lambda_i).$$

Utilizing Remark 2.13 and Theorem 2.12, we have

$$\bigwedge_{i\in I} \eta(\lambda_i) = \bigwedge_{i\in I} \nu(\overline{D}(\lambda_i) \equiv \lambda_i)$$

$$\leq \nu(\bigcup_{i\in I} \overline{D}(\lambda_i) \equiv \bigcup_{i\in I} \lambda_i) \wedge \nu(\bigcup_{i\in I} \overline{D}(\lambda_i) \equiv \overline{D}(\bigcup_{i\in I} \lambda_i))$$

$$\leq \nu(\overline{D}(\bigcup_{i\in I} \lambda_i) \equiv \bigcup_{i\in I} \lambda_i) = \eta(\bigcup_{i\in I} \lambda_i).$$

Theorem 2. 15. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $p \in Q$. Then

$\overline{D}(\lambda)(p) > c$ if and only if p is an accessible state of $\tilde{F}^*$ with threshold c and $\lambda(p) > c$.

Proof. Let p be an accessible state of $\tilde{F}^*$ with threshold c and $\lambda(p) > c$. Then there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, p) > c$. Thus $\lambda(p) \wedge r^{\tilde{F}^*}(x, p) > c$, which implies that $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} > c$. Conversely, let $\overline{D}(\lambda)(p) > c$. Then $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} > c$. Thus, there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, p) > c$ and $\lambda(p) > c$. Consequently, p is an accessible state of $\tilde{F}^*$ with threshold c and $\lambda(p) > c$. □

Definition 2. 16. Let $\tilde{F}^*$ be a max-min general fuzzy automaton. $\overline{B}$ is called bifuzzy operator on Σ^* from $\tilde{P}(\Sigma^*)$ to $\tilde{P}(\Sigma^*)$ and it defined as follows:

$$\overline{B}(\lambda)(x) = \vee\{\lambda(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}, \forall \lambda \in \tilde{P}(\Sigma^*), \ \forall x \in \Sigma^*.$$

Remark 2. 17. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and $\Sigma' = \{x \in \Sigma^* : r^{\tilde{F}^*}(x, p) = 1, \ \forall p \in \ Q_{act}(t_0)\}$. Then we have $\lambda(x) \le \overline{B}(\lambda)(x), \forall x \in \Sigma'$. In fact, we have $\lambda(x) = \lambda(x) \wedge 1 = \lambda(x) \wedge r^{\tilde{F}^*}(x, \ , p) \le \overline{B}(\lambda)(x)$, where $p \in Q_{act}(t_0)$ and $x \in \Sigma'$.

Theorem 2. 18. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $\lambda_i \in \tilde{P}(\Sigma^*), \forall i \in I$. Then

$$\text{(i)}\ \overline{B}(\bigcap_{i \in I} \lambda_i)(x) \le \bigwedge_{i \in I} \overline{B}(\lambda_i)(x), \ \forall x \in \Sigma^*,$$

$$\text{(ii)}\ \overline{B}(\bigcup_{i \in I} \lambda_i)(x) = \bigvee_{i \in I} \overline{B}(\lambda_i)(x), \ \forall x \in \Sigma^*.$$

Proof. (i) $\overline{B}(\bigcap_{i \in I} \lambda_i)(x) = \vee\{(\bigwedge_{i \in I} \lambda_i(x)) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}$

$$\le \bigwedge_{i \in I} (\vee\{\lambda_i(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\})$$

$$= \bigwedge_{i \in I} \overline{B}(\lambda_i)(x).$$

(ii) In the process of the above proof, using $\bigcup_{i \in I}$ and $\bigvee_{i \in I}$ to replace $\bigcap_{i \in I}$ and $\bigwedge_{i \in I}$, respectively, we can obtain (ii) and therefore complete the proof of theorem. □

Theorem 2. 19. Let $\tilde{F}^*$ be a max-min general fuzzy automaton. $\Gamma \in \tilde{P}(\tilde{P}(\Sigma'))$ is defined as follows:

$$\Gamma(\lambda) = \nu(\overline{B}(\lambda) \equiv \lambda).$$

Then Γ is a bifuzzy topology on $\Sigma' = \{x \in \Sigma^* : r^{\tilde{F}^*}(x, p) = 1, \ \forall p \in Q_{act}(t_0)\}$.

Proof. Our purpose is to demonstrate that Γ fulfils the three conditions in Definition 1.4. (In fact, the conclusions we will verify are stronger than those in Definition 1.4.)

(i) For any $x \in \Sigma'$, we have

$$\overline{B}(1)(x) = \vee\{1(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\} = 1(x) \wedge r^{\tilde{F}^*}(x, p_0) = 1(x) \wedge 1 = 1(x),$$

where $p_0 \in Q_{act}(t_0)$. So we get that

$$\Gamma(1) = \nu(\overline{B}(1) \equiv 1) = \nu(\overline{B}(1) \subseteq 1) \wedge \nu(1 \subseteq \overline{B}(1))$$

$$= (\bigwedge_{x \in \Sigma'} (1 \wedge 1 - \overline{B}(1)(x) + 1(x))) \wedge (\bigwedge_{x \in \Sigma'} (1 \wedge 1 - 1(x) + \overline{B}(1)(x)))$$

$$= (\bigwedge_{x \in \Sigma'} (1 \wedge 1 - 1(x) + 1(x))) \wedge (\bigwedge_{x \in \Sigma'} (1 \wedge 1 - 1(x) + 1(x))) = 1.$$

Similarly, since $\overline{B}(0)(x) = \vee\{0(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\} = 0(p)$, therefore $\Gamma(0) = 1$.

(ii) For any $i \in I$, $\lambda_i \in \tilde{P}(\Sigma')$,

$$\mathop{\wedge}_{i\in I} \Gamma(\lambda_i) \leq \Gamma(\bigcap_{i\in I} \lambda_i).$$

Utilizing Remarks 2.13, 2.17 and Theorem 2.18, we have

$$\mathop{\wedge}_{i\in I} \Gamma(\lambda_i) = \mathop{\wedge}_{i\in I} \nu(\overline{B}(\lambda_i) \equiv \lambda_i)$$

$$\leq \nu(\bigcap_{i\in I} \overline{B}(\lambda_i) \equiv \bigcap_{i\in I} \lambda_i) \wedge \nu(\overline{B}(\bigcap_{i\in I} \lambda_i) \equiv \bigcap_{i\in I} \lambda_i)$$

$$\leq \nu(\overline{B}(\bigcap_{i\in I} \lambda_i) \subseteq \bigcap_{i\in I} \lambda_i)$$

$$= \nu(\overline{B}(\bigcap_{i\in I} \lambda_i) \equiv \bigcap_{i\in I} \lambda_i) = \Gamma(\bigcap_{i\in I} \lambda_i).$$

(iii) For any $i \in I$, $\lambda_i \in \tilde{P}(\Sigma')$,

$$\mathop{\wedge}_{i\in I} \Gamma(\lambda_i) \leq \Gamma(\bigcup_{i\in I} \lambda_i).$$

Utilizing Remark 2.13 and Theorem 2.18, we have

$$\mathop{\wedge}_{i\in I} \Gamma(\lambda_i) = \mathop{\wedge}_{i\in I} \nu(\overline{B}(\lambda_i) \equiv \lambda_i)$$

$$\leq \nu(\bigcup_{i\in I} \overline{B}(\lambda_i) \equiv \bigcup_{i\in I} \lambda_i) \wedge \nu(\bigcup_{i\in I} \overline{B}(\lambda_i) \equiv \overline{B}(\bigcup_{i\in I} \lambda_i))$$

$$\leq \nu(\overline{B}(\bigcup_{i\in I} \lambda_i) \equiv \bigcup_{i\in I} \lambda_i) = \Gamma(\bigcup_{i\in I} \lambda_i).$$

3. A General Topology on Max-Min General Fuzzy Automata

Definition 3. 1. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q, p and q belong to Q and $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x,p) : x \in \Sigma^*\}$. Also, let $\rho' : q_0 = p, q_1, \ldots, q_n = q$ be a path from p to q and $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\}$. Then the degree of connectedness states p and q respect to λ is defined by $\deg_\lambda(p,q) = \vee\{\ S_\lambda(\rho) : \rho$ is a path from p to $q\ \}$.

If T is a subset of Q, then the degree of connectedness T respect to λ is defined by

$\deg_\lambda(T) = \wedge\{\deg_\lambda(p,q) : p,q \in Q\}$.

Theorem 3. 2. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then

(i) $\deg_\lambda(q,q) = \overline{D}(\lambda)(q)$, $\forall q \in Q$,

(ii) $\deg_\lambda(p,q) = \deg_\lambda(q,p)$, $\forall p,q \in Q$.

Proof. (i) Since q lies on every path ρ from q to q and for this type of paths we have $S_\lambda(\rho) = \wedge\{\overline{D}(\lambda)\ (q_i) : 0 \leq i \leq n\} \leq \overline{D}(\lambda)(q)$. Consequently, $\deg_\lambda(q,q) \leq \overline{D}(\lambda)(q)$. On the other hand, q is a path ρ' of length zero from q to q. Then $S_\lambda(\rho') = \overline{D}(\lambda)(q)$. Hence, $\deg_\lambda(q,q) = \vee\{S_\lambda(\rho) : \rho$ is a path from q to $q\} \geq \overline{D}(\lambda)(q)$.

(ii) Let $\rho' : q_0 = p, q_1, \ldots, q_n = q$ be a path from p to q and ρ'' be the converse of path ρ', i.e. , $\rho'' : p_0 = q_n = q, p_1, \ldots, p_n = q_0 = p$ is a path from q to p. Thus we have :

$$S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\} = \overline{D}(\lambda)(q_0) \wedge \ldots \wedge \overline{D}(\lambda)(q_n),$$

$$S_\lambda(\rho'') = \wedge\{\overline{D}(\lambda)(p_i) : 0 \leq i \leq n\} = \overline{D}(\lambda)(q_n) \wedge \ldots \wedge \overline{D}(\lambda)(q_0).$$

Since $S_\lambda(\rho') = S_\lambda(\rho'')$, then we have $\deg_\lambda(p,q) = \deg_\lambda(q,p)$. □

Theorem 3. 3. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then the fuzzy subset $\deg_\lambda(p,q)$ on $Q \times Q$ is a fuzzy relation of the fuzzy subset $\overline{D}(\lambda)$ to the fuzzy subset $\overline{D}(\lambda)$.

Proof. By Definition 1.5, we should prove $\deg_\lambda(p,q) \leq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$, $\forall p, q \in Q$. For every path $\rho : q_0 = p, q_1, \ldots, q_n = q$, we have

$$S_\lambda(\rho) = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\} \leq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q). \text{ So,}$$

$$\deg_\lambda(p,q) \leq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q). \quad \square$$

Definition 3. 4. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and T be a subset of Q. Then we say that

(i) p and q are connected with respect to λ if $\deg_\lambda(p,q) = \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$.

(ii) T is connected with respect to λ if p and q are connected with respect to λ, for any $p, q \in T$.

Theorem 3. 5. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then p and q are connected with respect to λ if and only if there exists a path $\rho' : q_0 = p, q_1, \ldots, q_n = q$ such that

$$\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q), \quad 0 \leq i \leq n.$$

Proof. Let p and q be connected with respect to λ. Then $\deg_\lambda(p,q) = \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$. Since , by

Definition 3.1, $\deg_\lambda(p,q) = \vee\{S_\lambda(\rho) : \rho$ is a path from p to $q\}$, so there exists a path $\rho' : q_0 = p, q_1, \ldots, q_n = q$ such that $S_\lambda(\rho') = \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$. By Definition 3.1, since $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\}$, then for every q_i which lies on the path ρ', we have

$$\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q), 0 \leq i \leq n.$$

Conversely, if there exists a path $\rho' : q_0 = p, q_1, \ldots, q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$, $0 \leq i \leq n$. Then $\deg_\lambda(p,q) \geq S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\} \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$. On the other hand, Theorem 3.3 implies that $\deg_\lambda(p,q) \leq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$. Hence, p and q are connected with respect to λ. □

Definition 3. 6. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and T be a subset of Q. Then T is called a component of $\tilde{F}^*$ with respect to λ if

(i) T is connected with respect to λ,

(ii) $\overline{D}(\lambda)(p) = \overline{D}(\lambda)(q)$, $\forall p,q \in T$,

(iii) $\overline{D}(\lambda)(q) > \overline{D}(\lambda)(p)$, $\forall q \in T$, $\forall p \in Q - T$.

Definition 3. 7. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and T, T' be two subsets of Q. Then we say that T and T' are connected with respect to λ if p and q are connected with respect to λ for any p and q in T and T', respectively.

Theorem 3. 8. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q and T be a component of $\tilde{F}^*$ with respect to λ. Also, let

$$C_T = \{\ p \in Q - T : \text{There exists a path } \rho : q_0 = p, q_1, \ldots, q_n = q \in T \text{ such that}$$
$$\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p), 0 \leq i \leq n\ \}.$$

Then C_T is in fact the set of those states of $Q - T$ which is connected with T, respect to λ.

Proof. Let $q \in T, p \in Q - T$ and p, q are connected with respect to λ. By Theorem 3.5, there exists a path $\rho : q_0 = p, q_1, \ldots, q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$, $0 \leq i \leq n$. Since $p \notin T$, by

Definition 3.6, we have $\overline{D}(\lambda)(q) > \overline{D}(\lambda)(p)$. Then $\overline{D}(\lambda)(q) \wedge \overline{D}(\lambda)(p) = \overline{D}(\lambda)(p)$. So $\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p), \forall i, 0 \leq i \leq n$. Thus, we have $p \in C_T$. Conversely, let $p \in C_T$ and $q \in T$. Then there exists a path $\rho : q_0 = p, q_1, \ldots, q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \overline{D}(\lambda)(p), \forall i, 0 \leq i \leq n$. Since $q \in T$ and $p \notin T$, by

Definition 3.6, we have $\overline{D}(\lambda)(q) > \overline{D}(\lambda)(p)$ and $\overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q) = \overline{D}(\lambda)(p)$. So, we have $\overline{D}(\lambda)\ (q_i) \geq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q), 0 \leq i \leq n$. Therefore, by Theorem 3.5, p and q are connected with respect to λ. □

Definition 3. 9. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $0 \leq c < 1$.

Then we say that p and q are concordant with threshold c and with respect to λ if $\deg_\lambda(p,q) > c$.

Theorem 3. 10. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $0 \leq c < 1$.

Then $\tilde{F}^*$ is connected with threshold c and $\lambda(p) > c$, $\forall p \in Q$ if and only if p and q are concordant with threshold c and with respect to λ, $\forall p,q \in Q$.

Proof. Let p, q be two arbitrary states of Q and $\tilde{F}^*$ be connected with threshold c. By Definitions 2.3, 2.7, there exists $x \in \sum^*$ such that $r^{\tilde{F}^*}(x,q) > c$. Since $\lambda(q) > c$, then $r^{\tilde{F}^*}(x,q) \wedge \lambda(q) > c$. So $\overline{D}(\lambda)(q) = \vee_{x\in\Sigma^*}(\lambda(q) \wedge r^{\tilde{F}^*}(x,q)) > c, \forall q \in Q$. Let $\rho' : q_0 = p, q_1, ..., q_n = q$ be a path from p to q. Then we have $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \le i \le n\} > c$. Thus $\deg_\lambda(p,q) = \vee\{\ S_\lambda(\rho) : \rho$ is a path from p to q $\} > c$.

Thus p and q are concordant with threshold c and with respect to λ.

Conversely, let p and q be two arbitrary elements of Q, also p and q are concordant with threshold c and with respect to λ. Then $\deg_\lambda(p,q) > c$. Thus there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that

$S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \le i \le n\} > c$. So $\overline{D}(\lambda)(q_i) > c, \forall i$, $0 \le i \le n$. Let $i = n$. Then $\overline{D}(\lambda)(q =$

$q_n) = \vee_{x\in\Sigma^*}(\lambda(q) \wedge r^{\tilde{F}^*}(x,q)) > c$. Thus, there exists $x \in \Sigma^*$ such that $\lambda(q) > c$ and $r^{\tilde{F}^*}(x,q) > c$.

Hence, $\tilde{F}^*$ is connected with threshold c and $\lambda(q) > c, \forall q \in Q$. □

Theorem 3. 11. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $0 \le c < 1$.

Then p and q are concordant with threshold c and with respect to λ if and only if there exists a path ρ': $q_0 = p, q_1, ..., q_n = q$ such that $\overline{D}(\lambda)(q_i) > c, \forall i$,$0 \le i \le n$.

Proof. Let p and q are concordant with threshold c and with respect to λ. Then $\deg_\lambda(p,q) > c$. Thus exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \le i \le n\} > c$. Hence, we have $\overline{D}(\lambda)(q_i) > c, \forall i$, $0 \le i \le n$. Conversely, Suppose that there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that $\overline{D}(\lambda)(q_i) > c$,$\forall i$,$0 \le i \le n$. Then we have $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \le i \le n\} > c$. So we get that $\deg_\lambda(p,q) = \vee\{\ S_\lambda(\rho) : \rho$ is a path from p to q $\} > c$. Consequently, p and q are concordant with threshold c and with respect to λ. □

Definition 3. 12. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then we say that p and q are semi-connected with respect to λ if $\deg_\lambda(p,q) = \lambda(p) \wedge \lambda(q)$.

Definition 3. 13. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then we say that λ is normal if $\lambda(p) < r^{\tilde{F}^*}(x,p)$, $\forall p \in Q$,$\forall x \in \Sigma^*$.

Theorem 3. 14. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a normal fuzzy subset on Q. Then p and q are semi-connected with respect to λ if and only if p and q are connected with respect to λ.

Proof. Since λ is normal, then we have :

$$\overline{D}(\lambda)(p) = \vee_{x\in\Sigma^*}(\lambda(p) \wedge r^{\tilde{F}^*}(x,p)) = \lambda(p) \ , \ \forall p \in Q .$$

Therefore, by Definitions 3.4 , 3.12, the proof is obvious. □

Theorem 3. 15. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Then

(i) If p and q are semi-connected with respect to λ, then there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \lambda(p) \wedge \lambda(q)$, $\forall i, 0 \leq i \leq n$.

(ii) If there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that $\overline{D}(\lambda)(q_i) \geq \lambda(p) \wedge \lambda(q)$, $\forall i, 0 \leq i \leq n$ and λ is normal, then p and q are semi-connected with respect to λ.

Proof. (i) Since p and q are semi-connected with respect to λ, then $\deg_\lambda(p,q) = \lambda(p) \wedge \lambda(q)$. Thus, there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that $S_\lambda(\rho') = \wedge\{\overline{D}(\lambda)(q_i) : 0 \leq i \leq n\} = \lambda(p) \wedge \lambda(q)$.

Consequently, $\overline{D}(\lambda)(q_i) \geq \lambda(p) \wedge \lambda(q), \forall i, 0 \leq i \leq n$.

(ii) Suppose that there exists a path $\rho' : q_0 = p, q_1, ..., q_n = q$ such that

$$\overline{D}(\lambda)(q_i) \geq \lambda(p) \wedge \lambda(q), \forall i, 0 \leq i \leq n.$$

Then $S_\lambda(\rho') \geq \lambda(p) \wedge \lambda(q)$. Thus, we have $\deg_\lambda(p,q) = \vee\{S_\lambda(\rho) : \rho$ is a path from p to $q\} \geq \lambda(p) \wedge \lambda(q)$. On the other hand, by Theorem 3.3, we get that $\deg_\lambda(p,q) \leq \overline{D}(\lambda)(p) \wedge \overline{D}(\lambda)(q)$. Since λ is normal, then $\overline{D}(\lambda)(p) = \lambda(p), \overline{D}(\lambda)(q) = \lambda(q)$. Thus $\deg_\lambda(p,q) \leq \lambda(p) \wedge \lambda(q)$. Hence, p and q are semi-connected with respect to λ. □

Theorem 3. 16. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q, $q \in Q_{act}(t_0)$, T be a subset of $Q_{act}(t_0)$, $0 \leq c < 1$ and

$A_c(q) = \{\ p \in Q_{act}(t_0) : p$ and q are concordant with threshold c and with respect to $\lambda\ \}$.

Also, let $A_c(T) = \bigcup\{A_c(q) : q \in T\}$. Then

(i) If $\lambda(q) > c$ for all $q \in T$, then $T \subseteq A_c(T)$,

(ii) If T_1 and T_2 are two subsets of $Q_{act}(t_0)$, then $A_c(T_1 \cup T_2) = A_c(T_1) \cup A_c(T_2)$.

Proof. (i) Let $q \in T$. Utilizing Theorem 3.2, we have $\deg_\lambda(q,q) = \overline{D}(\lambda)(q) = \underset{x \in \Sigma^*}{\vee}(\lambda(q) \wedge r^{\tilde{F}^*}(x,q)) \geq \lambda(q) \wedge r^{\tilde{F}^*}(\Lambda, q) = \lambda(q) \wedge 1 = \lambda(q) > c$.

So $q \in A_c(q) \subseteq A_c(T)$. Consequently, $T \subseteq A_c(T)$.

(ii) Let T_1 and T_2 be two subsets of $Q_{act}(t_0)$. Then we have

$$A_c(T_1 \cup T_2) = \bigcup_{q \in T_1 \cup T_2} A_c(q) = (\bigcup_{q \in T_1} A_c(q)) \cup (\bigcup_{q \in T_2} A_c(q)) = A_c(T_1) \cup A_c(T_2). \quad \square$$

Theorem 3. 17. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $\lambda(q) > c$, $\forall q \in Q_{act}(t_0)$, $0 \leq c < 1$. Then $\tau = \{\ T^C : A_c(T) = T$ and T is a subset of $Q_{act}(t_0)\ \}$ is a general topology on $Q_{act}(t_0)$.

Proof. (i) Since $A_c(\emptyset) = \emptyset$, then $\emptyset^C = Q_{act}(t_0) \in \tau$.

(ii) Since $\lambda(q) > c$, $\forall q \in Q_{act}(t_0)$, by Theorem 3.16, we get that $Q_{act}(t_0) \subseteq A_c(Q_{act}(t_0))$. On the other hand, we have $A_c(Q_{act}(t_0)) \subseteq Q_{act}(t_0)$.

Thus, $A_c(Q_{act}(t_0)) = Q_{act}(t_0)$. So $Q_{act}(t_0)^C = \emptyset \in \tau$.

(iii) Let $T_1^C, T_2^C \in \tau$. Then $A_c(T_1) = T_1$ and $A_c(T_2) = T_2$. By Theorem 3.16, we conclude that $A_c(T_1 \cup T_2) = A_c(T_1) \cup A_c(T_2) = T_1 \cup T_2$. That is $(T_1 \cup T_2)^C = T_1^C \cap T_2^C \in \tau$.

(iv) Let $T_i^C \in \tau, \forall i \in I$. Then $A_c(T_i) = T_i$. By Theorem 3.16, we have $\bigcap_{i\in I} T_i \subseteq A_c(\bigcap_{i\in I} T_i)$. Also, $T_i \cup (\bigcap_{i\in I} T_i) = T_i$ implies that $A_c(T_i) \cup A_c(\bigcap_{i\in I} T_i) = A_c(T_i)$. Thus, $A_c(\bigcap_{i\in I} T_i) \subseteq A_c(T_i) = T_i$. Consequently, $A_c(\bigcap_{i\in I} T_i) \subseteq \bigcap_{i\in I} T_i$. So we get that $A_c(\bigcap_{i\in I} T_i) = \bigcap_{i\in I} T_i$. Hence, $(\bigcap_{i\in I} T_i)^C = \bigcup_{i\in I} T_i^C \in \tau$.

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