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On Domination in Fuzzy Graphs

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Abstract

Given two fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$. We consider some parameters of domination such as dominating sets, connected dominating sets, total dominating sets, independent dominating sets and dominating cliques in Cartesian product and composition of G_1 and G_2. Then we introduce the concepts of categorical and strong product of G_1 and G_2 and study above parameters for them.

AMS Subject Classification:
Keywords: Dominating set, cartesian product, categorical product, strong product, composition.

1. Introduction

The first definition of fuzzy graphs was proposed by Kafmann [3], from the fuzzy relations introduced by Zadeh [10]. Although Rosenfeld [6] introduced another elaborated definition , including fuzzy vertex and fuzzy edges, and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness and etc. Then after it many concepts such as fuzzy outomorphic groups, fuzzy intersection graphs, fuzzy line graphs, was introduced, (see [1], [4], [5]). The concept of domination in fuzzy graphs was investigated by A. Somasundaram, S. Somasundaram [7] and A. Somasundaram present the concepts of independent domination, total domination, connected domination and domination in Cartesian product and composition of fuzzy graphs ([8], [9]). In this paper we develop the domination in Cartesian and composition of fuzzy graphs to another domination parameters and investigate the concepts of domination in categorical and strong product of fuzzy graphs.

2. Preliminaries

We review some definitions in fuzzy graphs. Let V be a finite nonempty set. Let E be the collection of all two-element subsets of V. A *fuzzy graph* $G = (\sigma, \mu)$ is a set with two functions $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ such that $\mu(\{x, y\}) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. We write $\mu(xy)$ for $\mu(\{x, y\})$. An edge $e = xy$ of a fuzzy graph is called an *effective edge* if $\mu(xy) = \sigma(x) \wedge \sigma(y)$. $N(x) = \{y \in V \mid \mu(xy) = \sigma(x) \wedge \sigma(y)\}$ is called the *neighborhood* of x.

Let $G = (\sigma, \mu)$ be a fuzzy graph on V. Let $u, v \in V$. We say that u *dominates* v in G if $\mu(uv) = \sigma(u) \wedge \sigma(v)$. A subset S of V is called a *dominating set* in G if for every $v \in V - S$, there exists $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a dominating set in G is called the *domination number* of G and denote by $\gamma(G)$ or simply γ.

A vertex u of a fuzzy graph G is said to be an *isolated vertex* if $\mu(uv) < \sigma(u) \wedge \sigma(v)$ for all $v \in V - \{u\}$.

Let $G = (\sigma, \mu)$ be a fuzzy graph on V. A subset S of V is said to be an *independent set* if $\mu(uv) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in S$. The minimum fuzzy cardinality of an independent dominating set of G is called the *independent domination number* of G and is denoted by $\gamma_i(G)$ or simply γ_i.

Let G be a fuzzy graph without isolated vertices. A subset S of V is called a *total dominating set* of G if every vertex in V is dominated by a vertex in S or equivalently S is a dominating set of G and the induced subgraph $G[S]$ has no isolated vertices. The minimum fuzzy cardinality of a total dominating set of G is called the *total domination number* of G and denoted by $\gamma_t(G)$ or simply γ_t.

A *path* p in fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices $(u_0, u_1, u_2, \ldots, u_n)$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, \ldots, n$ and n is called the *length* of p. The path p is called $u_0 - u_n$ path.

The strength of a path p is defined to be $\wedge_{i=1}^{n} \mu(u_{i-1} u_i)$. In other word, the strength of the path p is the weight of the weakest edge of p. The fuzzy graph G is said to be *connected* if there exists a $u - v$ path in G for any two vertices u and v.

Let G be connected fuzzy graph of V. A subset S of V is called a *connected dominating set* of G if S is a dominating set and $G[S]$ is connected subgraph of G. The minimum fuzzy cardinality of a connected dominating set is called the *connected domination number* of G and is denoted by $\gamma_c(G)$ or simply γ_c.

Let $G = (\sigma, \mu)$ be fuzzy graph on V and $D \subseteq V$, then the *scaler cardinality* of D is defined to be $\Sigma_{v \in D}\, \sigma(v)$. We denote the scaler cardinality of D by $|D|_s$.

A. Somasundaram introduced the following definitions [9].

Definition 2.1. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively. Then the *composition* of G_1 and G_2, denoted by $G_1 \circ G_2$, is the fuzzy graph on $V_1 \times V_2$ defined as follows:

$$G_1 \circ G_2 = (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$$

where

$$\sigma_1 \circ \sigma_2(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$$

and

$$\mu_1 \circ \mu_2((u_1, u_2), (v_1, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2) & \text{if } u_1 = v_1,\ u_2 \neq v_2 \\ \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1v_1) & \text{otherwise.} \end{cases}$$

Definition 2.2. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively. Then the *Cartesian product* of G_1 and G_2, denoted by $G_1 \Box G_2$, is the fuzzy graph on $V_1 \times V_2$ defined as follows:

$$G_1 \Box G_2 = (\sigma_1 \Box \sigma_2, \mu_1 \Box \mu_2)$$

where

$$\sigma_1 \Box \sigma_2(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$$

and

$$\mu_1 \Box \mu_2((u_1, u_2), (v_1, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2) & \text{if } u_1 = v_1 \\ \sigma_2(u_2) \wedge \mu_1(u_1v_1) & \text{if } u_2 = v_2 \\ 0 & \text{otherwise.} \end{cases}$$

The followings are useful.

Theorem 2.3. [9] Let D_1 and D_2 be dominating sets of the fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ respectively. Then $D_1 \times D_2$ is a dominating set of $G_1 \circ G_2$.

Theorem 2.4. [9] Let D_1 and D_2 be minimum dominating sets of $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ respectively. Then

$$\gamma(G_1 \Box G_2) \leq \min\{|D_1 \times V_2|_s, |V_1 \times D_2|_s\}.$$

3. Main Results

Theorem 3.1. Let G_1, G_2 be connected and D_1, D_2 be dominating sets of G_1 and G_2, respectively. Then $G_1 \Box G_2$ is connected and we have:

(i) If D_1 is connected then $D_1 \times V_2$ is a connected dominating set of $G_1 \Box G_2$.
(ii) If D_2 is connected then $V_1 \times D_2$ is a connected dominating set of $G_1 \Box G_2$.

Proof. For connectivity of $G_1 \Box G_2$ we consider two arbitrary distinct vertices of $V_1 \times V_2$ such as (x_i, y_j) and (x_l, y_k). We show that there exists a path between this two vertices in each three following cases:

Case (i) $x_i = x_l$. There exists a path $p : y_j, y_{i_1}, y_{i_2}, \ldots, y_k$ because of connectivity of G_2 such that $\mu_2(uv) > 0$ for each two vertices u, v of path p. But this implies that $\mu((x_i, u), (x_i, v)) = \sigma_1(x_i) \wedge \mu_2(uv) > 0$ and hence $p' : (x_i, y_j), (x_i, y_{i_1}), (x_i, y_{i_2}), \ldots, (x_i, y_k)$ is the path between (x_i, y_j) and (x_l, y_k) in $G_1 \Box G_2$.

Case (ii) $y_j = y_k$. Consider the path $q : x_i, x_{j_1}, x_{j_2}, \ldots, x_l$ and therefore $q' : (x_i, y_j), (x_{j_1}, y_j), (x_{j_2}, y_j), \ldots, (x_l, y_j)$ is the needed path.

Case (iii) $x_i \neq x_l, y_j \neq y_k$. Find a path between $(x_i, y_j), (x_i, y_k)$ by case (i) and a path from (x_i, y_k) to (x_l, y_k) by case (ii). The union of this two disjoint paths is a path between $(x_i, y_j), (x_l, y_k)$.

$D_1 \times V_2$ and $V_1 \times D_2$ are dominating sets (Theorem 2.4) and the connectivity of them is similarly proved as above. ■

Theorem 3.2. Suppose that G_1 has no isolated vertex and D_1 is a total dominating set of G_1. Then $G_1 \Box G_2$ has no isolated vertex and $D_1 \times V_2$ is a total dominating set of $G_1 \Box G_2$.

Proof. Let (u, v) be an arbitrary vertex in $V_1 \times V_2$. There exists a vertex x in D_1 such that $u \in N(x)$. Now

$$\begin{aligned} \mu((u, v), (x, v)) &= \mu_1(ux) \wedge \sigma_2(v) \\ &= \sigma_1(u) \wedge \sigma_1(x) \wedge \sigma_2(v) \\ &= \sigma(u, v) \wedge \sigma(x, v) \end{aligned}$$

implies that $(u, v) \in N((x, v))$. So (u, v) is not an isolated vertex and since $(x, v) \in D_1 \times V_2$ the proof is complete. ■

Remark 3.3. Under the similar conditions $V_1 \times D_2$ is a total dominating set of $G_1 \Box G_2$.

Theorem 3.4. Let D_1 and D_2 be dominating sets of G_1 and G_2 respectively.

(a) $D_1 \times V_2$ is an independent dominating set of $G_1 \Box G_2$ if and only if D_1 is independent and

$$\mu_1(uv) < \sigma_2(w) \qquad u, v \in D_1, w \in V_2. \tag{3.1}$$

$$\begin{cases} (i)\ \ \mu_2(wz) < \sigma_1(u) & u \in D_1, w, z \in V_2; \\ (ii)\ \ \mu_2(wz) < \sigma_2(w) \wedge \sigma_2(z) & w, z \in V_2. \end{cases} \tag{3.2}$$

(b) $V_1 \times D_2$ is an independent dominating set of $G_1 \Box G_2$ if and only if D_2 is independent and

$$\begin{cases} (i)\ \ \mu_1(uv) < \sigma_2(w) & u, v \in V_1, w \in D_2; \\ (ii)\ \ \mu_1(uv) < \sigma_1(u) \wedge \sigma_1(v) & u, v \in V_1. \end{cases} \tag{3.3}$$

$$\mu_2(wz) < \sigma_1(u) \qquad u \in V_1, w, z \in D_2. \tag{3.4}$$

Proof. (a) Sufficiency: We show that every two distinct vertices of $D_1 \times V_2$ such as $(x_1, y_1), (x_2, y_2)$ are not adjacent. If $x_1 = x_2$, then by (2)(i) and (2) (ii)

$$\mu((x_1, y_1), (x_1, y_2)) = \sigma_1(x_1) \wedge \mu_2(y_1 y_2)$$

$$
\begin{aligned}
&= \mu_2(y_1 y_2) \\
&< \sigma_2(y_1) \wedge \sigma_2(y_2) \wedge \sigma_1(x_1) \\
&= \sigma(x_1, y_1) \wedge \sigma(x_1, y_2).
\end{aligned}
$$

If $y_1 = y_2$ the request result will be obtained by independence of D_1 and inequality (1) and we omit the details. If $x_1 \neq x_2, y_1 \neq y_2$ then by Definition B we have $\mu((x_1, y_1), (x_2, y_2)) = 0$, so (x_1, y_1) and (x_2, y_2) con not be adjacent.

Necessity: We use contrapositive law. That is we show if D_1 is not independent or $D_1 \times V_2$ dos not satisfy in (1) or (2)(i) or (2)(ii) then $D_1 \times V_2$ is not an independent dominating set of $G_1 \Box G_2$. Here we have proved the necessity of (2)(ii) and left the easy proof of others to the reader. Suppose that (2)(ii) is false, i.e., there exists $w, z \in V_2$ such that $\mu_2(wz) = \sigma_2(w) \wedge \sigma_2(z)$. Then if u is any vertex of D_1, we will have

$$
\begin{aligned}
\mu((u, w), (u, z)) &= \sigma_1(u) \wedge \mu_2(wz) \\
&= \sigma_1(u) \wedge \sigma_2(w) \wedge \sigma_2(z) \\
&= \sigma(u, w) \wedge \sigma(u, z).
\end{aligned}
$$

But this means $D_1 \times V_2$ is not independent. The proof of (a) is complete. The same proof is right for (b). ■

Definition 3.5. Let D be a dominating set of a fuzzy graph G. We say D is a *dominating clique* if $G[D]$ is a complete fuzzy graph. The minimum fuzzy cardinality among all dominating cliques is the *clique domination number* of G and denote by $\gamma_{cl}(G)$.

Lemma 3.6. If $S_1 \times S_2$ is a dominating set of $G_1 \Box G_2$ where $S_i \subseteq V_i,\ i = 1, 2$ then at least one of the subsets equals to the its hyper set.

Proof. Let $S_i \subseteq V_i, i = 1, 2$ and $S_1 \times S_2$ be any dominating set of $G_1 \Box G_2$ and moreover on the contrary suppose that $S_1 \neq V_1, S_2 \neq V_2$. Therefore if $x \in V_1 - S_1, y \in V_2 - S_2$ then for each member of $S_1 \times S_2$ such as (s_1, s_2) we have $\mu((s_1, s_2), (x, y)) = 0$. But this means that (x, y) is not dominated by $S_1 \times S_2$. This contradiction to establish the lemma. ■

By a similar argument we have the following generalization of Lemma 3.6.

Proposition 3.7. If $A = \{(x_1, y_1), \ldots, (x_k, y_l)\}$ be any dominating set of $G_1 \Box G_2$, then $\cup_{i=1}^{k}\{x_i\} = V_1$ or $\cup_{j=1}^{l}\{y_j\} = V_2$.

Theorem 3.8. Let G_1 and G_2 be two fuzzy graphs. If $|V_1|, |V_2| \geq 2$ then any dominating clique of $G_1 \Box G_2$ has one of the following forms:

(1) $\{x\} \times V_2$ for a vertex $x \in V_1$

(2) $V_1 \times \{y\}$ for a vertex $y \in V_2$.

Proof. If $D = \{(x_1, y_1), \ldots, (x_k, y_l)\}$ is a dominating clique of $G_1 \Box G_2$ then by Proposition 6, $\bigcup_{i=1}^{k} x_i = V_1$ or $\bigcup_{j=1}^{l} y_j = V_2$. Suppose that $\bigcup_{i=1}^{k} x_i = V_1$. Since $|V_1| \geq 2$, we can choose two vertices $x, x' \in V_1$. Let $(x, y), (x', y') \in D$. If $y \neq y'$ then $\mu((x, y), (x', y')) = 0$ which is a contradiction because of $G_1 \Box G_2[D]$ is a complete subgraph. So $y = y'$ and therefore $D = V_1 \times \{y\}$. Similarly if $\bigcup_{j=1}^{l} y_j = V_2$ then we will have $D = \{x\} \times V_2$. ■

Connectivity of $G_1 \circ G_2$ obtained very easier than the connectivity of $G_1 \Box G_2$.

Theorem 3.9. Let G_1 be a connected fuzzy graph, D_1 be a connected dominating set of G_1 and D_2 be a dominating set of G_2 . Then $G_1 \circ G_2$ is connected and $D_1 \times D_2$ is a connected dominating set of it.

Proof. Suppose that $(u, v), (w, z) \in V_1 \times V_2$. Sine G_1 is connected, there exists a path $p : u, x_1, x_2, \ldots, x_{n-1}, w$ in G_1. But this implies that $p' : (u, v), (x_1, v), (x_2, v), \ldots, (x_{n-1}, v), (w, z)$ is a $(u, v) - (w, z)$ path in $G_1 \circ G_2$. A same argument prove that $D_1 \times D_2$ is connected too. ■

The proof of the next theorem is similar to the proof of the Theorem 3.9 and we omit it.

Theorem 3.10. Let D_1 and D_2 be dominating sets of G_1 and G_2 respectively. Then $D_1 \times D_2$ is an independent dominating set of $G_1 \circ G_2$ if and only if D_1 and D_2 are independent and

(1) $\mu_1(uv) < \sigma_2(w) \quad u, v \in D_1,\ w \in D_2$;

(2) $\mu_2(wz) < \sigma_1(u) \quad u \in D_1,\ w, z \in D_2$.

Dominating cliques in $G_1 \circ G_2$ have more variation in comparison with $G_1 \Box G_2$. We give two types of them in the next two propositions, which the proofs are straightforward.

Proposition 3.11. If D_i is a dominating clique of G_i $(i = 1, 2)$ then $D_1 \times D_2$ is a dominating clique of $G_1 \circ G_2$.

Proposition 3.12. Let D_1 and D_2 be dominating sets of G_1 and G_2 respectively. Then under the following conditions $D_1 \times D_2$ is a dominating clique of $G_1 \circ G_2$.

(1) $\mu_1(u, v) \geq \sigma_2(w) \quad u, v \in D_1,\ w \in D_2$;

(2) $\mu_2(w, z) \geq \sigma_1(u) \quad u \in D_1,\ w, z \in D_2$.

Definition 3.13. Let $G_1 = (V_1, \sigma_1, \mu_1)$, $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs on V_1, V_2 respectively. Then the *categorical product* of G_1 and G_2 denoted by $G_1 \times G_2$ is the fuzzy graph on $V_1 \times V_2$ defined as follows:

$$G_1 \times G_2 = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$$

where

$$\sigma_1 \times \sigma_2(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$$

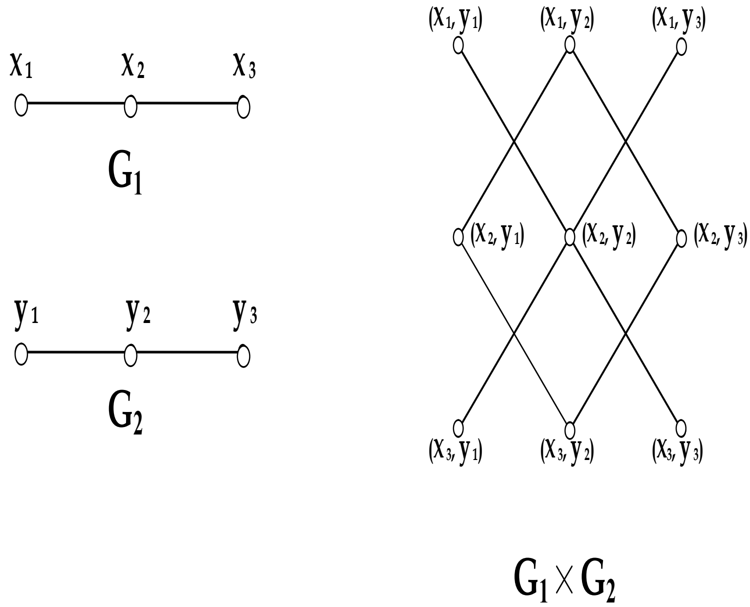
and

$$\mu_1 \times \mu_2((u_1, u_2), (v_1, v_2)) = \begin{cases} \mu_1(u_1v_1) \wedge \mu_2(u_2v_2) & if\ u_1 \neq v_1, u_2 \neq v_2 \\ 0 & \text{otherwise.} \end{cases}$$

By the definition we have

Corollary 3.14. If D_i is a total dominating set of G_i for $i = 1, 2$ such that D_1, D_2 both contain at least two vertices, then $D_1 \times D_2$ is a total dominating set of $G_1 \times G_2$.

Remark 3.15. If some of D_i in the last corollary is just a dominating set or don't have the mentioned property then $D_1 \times D_2$ is not necessary a dominating set of $G_1 \times G_2$. Consider the fuzzy graph $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ below and the categorical product of them whit $D_1 = \{x_1, x_3\}$ and $D_2 = \{y_1, y_3\}$ or $D_1 = \{x_2\}$ and $D_2 = \{y_2\}$.



$\sigma_1 = (x_1|0.3, x_2|0.4, x_3|0.5)$, $\sigma_2 = (y_1|0.4, y_2|0.3, y_3|0.5)$, $\mu_1 = (x_1x_2|0.3, x_2x_3|0.4)$, $\mu_2 = (y_1y_2|0.3, y_2y_3|0.3)$ and $\sigma_1 \times \sigma_2 = ((x_1, y_1)|0.3, (x_1, y_2)|0.3,$

$(x_1, y_3)|0.3$, $(x_2, y_1)|0.4$, $(x_2, y_2)|0.3$,
$(x_2, y_3)|0.4$, $(x_3, y_1)|0.4$, $(x_3, y_2)|0.3$, $(x_3, y_3)|0.5$) and $\mu_1 \times \mu_2((s,t),(s',t')) = 0.3$ for every $s, s' \in V_1$ and $t, t' \in V_2$.

Corollary 3.16. If we give up the trivial case that both of V_1 and V_2 contain just one vertex, then $G_1 \times G_2$ don't have a dominating clique.

Proof. First note that a vertex can not be a dominating set of $G_1 \times G_2$. Assume that $|V_1|, |V_2| \geq 2$. If $S = \{(u_1, v_1), \ldots, (u_k, v_k)\}$ be a complete subgraph of $G_1 \times G_2$ where $k = |S| \geq 2$, then it is clear that $u_i \neq u_j$, $v_i \neq v_j$ for $i, j = 1, \ldots, k$. So if $v \neq v_i$ for each $i = 1, \ldots, k$ then neither $(u_1, v) \in S$ nor (u_1, v) is dominated by a vertex of S. Therefore S can not be a dominating set of $G_1 \times G_2$. If $|V_1| = 1$ or $|V_2| = 1$ then the proof is very easy and we skip the proof. ■

Theorem 3.17. Let G_1 and G_1 be fuzzy graphs on V_1 and V_2 with total connected dominating sets D_1 and D_2 respectively where both contain at least two vertices. If for each $x, x' \in V_1$ and $y, y' \in V_2$ there exists paths p and q between x, x' and y, y' respectively such that the length of p and q are both even or both odd then $G_1 \times G_2$ is connected and more over $D_1 \times D_2$ is a total connected dominating set of $G_1 \times G_2$.

Proof. Let $(x, x'), (y, y') \in V_1 \times V_2$, and suppose that $p : x, x_1, x_2, \ldots, x_{n-1}, x'$ and $q : y, y_1, y_2, \ldots, y_{m-1}, y'$ be the $x - x'$ and $y - y'$ paths and $n \leq m$. If n, m are both even then let the sequence R of vertices in $V_1 \times V_2$ to be $R = (x, y), (x_1, y_1), (x_2, y_2), \ldots,$ $(x_{n-1}, y_{n-1}), (\boldsymbol{x}_{n-2}, \boldsymbol{y}_n), (\boldsymbol{x}_{n-1}, \boldsymbol{y}_{n+1}), (\boldsymbol{x}_{n-2}, \boldsymbol{y}_{n+2}), (\boldsymbol{x}_{n-1}, \boldsymbol{y}_{n+3}), \ldots, (\boldsymbol{x}_{n-2}, \boldsymbol{y}_{m-4}),$ $(\boldsymbol{x}_{n-1}, \boldsymbol{y}_{m-3}), (x_{n-2}, y_{m-2}), (x_{n-1}, y_{m-1}), (x', y')$. Note that the number of vertices in R equals to $m+1$ and the number of bold vertices (which equals to m-n-2) is even. More over for each two consecutive vertices (s, t) and (s', t') since s, s' and t, t' are consecutive in p and q respectively, then we have $\mu((s, t), (s', t')) = \mu_1(s, s') \wedge \mu_2(t, t')) > 0$. Therefore R is a path between $(x, y), (x', y')$ in $G_1 \times G_2$. If n, m are both odd then let $R = (x, y), (x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (\boldsymbol{x}_{n-2}, \boldsymbol{y}_n),$
$(\boldsymbol{x}_{n-1}, \boldsymbol{y}_{n+1}), \ldots, (\boldsymbol{x}_{n-2}, \boldsymbol{y}_{m-2}), (\boldsymbol{x}_{n-1}, \boldsymbol{y}_{m-1}), (x', y')$. ■

Definition 3.18. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively. Then the *strong product* of G_1 and G_2, denoted by $G_1 \Box times G_2$, is the fuzzy graph on $V_1 \times V_2$ defined as follows:

$$G_1 \Box times G_2 = (\sigma_1 \Box times \sigma_2, \mu_1 \Box times \mu_2)$$

where

$$\sigma_1 \Box times \sigma_2(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$$

and

$$\mu_1 \Box times \mu_2((u_1, u_2), (v_1, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2 v_2) & \text{if } u_1 = v_1 \\ \sigma_2(u_2) \wedge \mu_1(u_1 v_1) & \text{if } u_2 = v_2 \\ \mu_1(u_1 v_1) \wedge \mu_2(u_2 v_2) & \text{otherwise.} \end{cases}$$

$G_1 \Box times G_2$ treat very similar to $G_1 \circ G_2$ about dominating sets by its definition. So we give up the proofs of the following results except Theorem 21. Note that this assumptions that D_2 is connected added to Theorem 21, in comparison with Theorem 3.17. More over with the conditions of Proposition 3.11 the similar result is not correct.

Corollary 3.19. If D_1 and D_2 be minimum dominating sets of G_1 and G_2 respectively, then $D_1 \times D_2$ is a minimum dominating set of $G_1 \Box times G_2$.

Corollary 3.20. Suppose that G_1 and G_2 be two fuzzy graphs having no isolated vertices. If D_1 and D_2 are minimum total dominating sets of G_1 and G_2 respectively then $\gamma_t(G_1 \Box times G_2) \leq |D_1 \times D_2|_s$.

Corollary 3.21. Let D_1 and D_2 be dominating sets of G_1 and G_2 respectively. Then $D_1 \times D_2$ is an independent dominating set of $G_1 \Box times G_2$ if and only if D_1 and D_2 are independent and

(1) $\mu_1(uv) < \sigma_2(w) \;\; u, v \in D_1, \;\; w \in D_2$

(2) $\mu_2(wz) < \sigma_1(u) \;\; u \in D_1, \;\; w, z \in D_2$.

Corollary 3.22. Let D_1 and D_2 be minimum dominating cliques of G_1 and G_2 respectively. Then $\gamma_{cl}(G_1 \Box times G_2) \leq |D_1 \times D_2|_s$.

Theorem 3.23. Let D_1, D_2 be connected dominating sets of connected fuzzy graphs G_1 and G_2 respectively. Then $G_1 \Box times G_2$ is connected and $\gamma_c(G_1 \Box times G_2) \leq |D_1 \times D_2|_s$.

Proof. Given any two vertices $(u, v), (w, z)$ of $V_1 \times V_2$, there exists paths $p = u, x_1, x_2, \ldots, x_{n-1}, w$ and $q = v, y_1, y_2, \ldots, y_{m-1}, z$. If $n \leq m$ then let $R = (u, v), (x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_{n-2}, y_n), (x_{n-1}, y_{n+1}), \ldots, (x_{n-1}, y_{m-1}), (x_{n-2}, y_{m-1}), (w, z)$. If $m < n$ then let $R = (u, v), (x_1, y_1), (x_2, y_2), \ldots, (x_{m-1}, y_{m-1}), (x_m, y_{m-1}), (x_{m+1}, y_{m-1}), \ldots, (x_{n-1}, y_{m-1}), (w, z)$. In each case R is the needed path by definition of $G_1 \Box times G_2$. ■

References

[1] Bhutani, K.R., 1989, On automorphism of fuzzy graphs, Pattern Recognition Letters, 9, pp. 159–162.

[2] Haynes, T.W., Hedetniemi S.T., and Slater P.T., 1997, Fundamental of Domination in Graphs (Marcel Dekker, Inc.).

[3] Kaufmann, A., 1976, Introduction à la théorie des sous-ensembles flous, 1o Eléments théoriques de base. Paris: Masson et Cie.

[4] McAlister, K.L.N., 1988, Fuzzy intersection graphs, Comp. Math. Appl., 15, pp. 871–886.

[5] Mordeson, J.N., 1993, Fuzzy line graphs, Pattern Prcognition Letters, 14, pp. 381–384.

[6] Rosenfeld, A., Fuzzy graphs In: Zadeh, L.A., Fu, K.S., Shimura, M. (Eds), Fuzzy Sets and their Applications (Academic Press, New York).

[7] Somasundaram, A., Somasundaram, S., 1998, Domination in Fuzzy Graphs-I, Pattern Recognition Letters, 19, pp. 787–791.

[8] Somasundaram, A., 2004, Domination in Fuzzy Graph-II, Journal of Fuzzy Mathematics, 20.

[9] Somasundaram, A., 2005, Domination in Product of Fuzzy Graphs, Intenational journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 13(2), pp. 195–204.

[10] Zadeh, L.A. 1971, Similarity relations and fuzzy ordering, Information Siences, 3(2), pp. 177–200.