Advances in Fuzzy Mathematics.
ISSN 0973-533X Vol. 1 No. 1 (2006), pp. 55–65

http://www.ripublication.com/afm.htm

Fixed Point Theorem in $\mathcal{M}$-Fuzzy Metric Spaces with property (E)

S. Sedghi[1] **and N. Shobe**[2]
[1]*Department of Mathematics, Islamic Azad University-Ghaemshahr Branch, Iran*
E-mail: sedghi_gh@yahoo.com
[2]*Department of Mathematics, Islamic Azad University-Babol Branch, Iran*
E-mail: nabi_shobe@yahoo.com
The corresponding author: sedghi_gh@yahoo.com (Shaban Sedghi)

Abstract

In this paper, a common fixed point theorem in $\mathcal{M}$-fuzzy metric spaces for property (E) is proved.

AMS Subject Classification: 54E40, 54E35, 54H25.
Keywords: $\mathcal{M}$-Fuzzy contractive mapping; Complete $\mathcal{M}$-fuzzy metric space; Common fixed point theorem.

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [15] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [6] and Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [2–5, 14]. Many authors [7, 10, 11, 13] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.One should there exists a space between spaces. And one such generalization is generalized metric space or D-metric space initiated by Dhage [1] in 1992.He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [9] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [12] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition.

In what follows (X, D) will denote a D-metric space, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D-metric) on X is a function: $D : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D(x, y, z) \geq 0$,

(2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D(x, y, z) = D(p\{x, y, z\})$,(symmetry) where p is a permutation function,

(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called a generalized metric (or D-metric) space.

Immediate examples of such a function are

(a) $D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}^+$ then we define

$$D(x, y, z, t) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise }, \end{cases}$$

Remark 1.2. In a D-metric space, we prove that $D(x, x, y) = D(x, y, y)$. For

(i) $D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)$ and similarly

(ii) $D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)$.

Hence by (i),(ii) we get $D(x, x, y) = D(x, y, y)$.

Let (X, D) be a D-metric space. For $r > 0$ define

$$B_D(x, r) = \{y \in X : D(x, y, y) < r\}$$

Example 1.3. Let $X = \mathbb{R}$. Denote $D(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_D(1, 2) &= \{y \in \mathbb{R} : D(1, y, y) < 2\} = \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2) \end{aligned}$$

Definition 1.4. Let (X, D) be a D-metric space and $A \subset X$.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_D(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D-bounded if there exists $r > 0$ such that $D(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D(x_n, x_n, x) = D(x, x, x_n) \to 0$ as $n \to \infty$.That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow D(x, x, x_n) < \epsilon \quad (*)$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \Longrightarrow D(x, x_n, x_m) < \epsilon \quad (**)$$

Indeed, if have $(*)$,then

$$D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon$$

Conversely, set $m = n$ in $(**)$ we have $D(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D-metric space (X, D) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $r > 0$ such that $B_D(x, r) \subset A$. Then τ is a topology on X (induced by the D-metric D).

Lemma 1.5. Let (X, D) be a D-metric space. If $r > 0$, then ball $B_D(x, r)$ with center $x \in X$ and radius r is open ball.

Proof. Let $z \in B_D(x, r)$, hence $D(x, z, z) < r$. If set $D(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_D(z, r') \subseteq B_D(x, r)$. Let $y \in B_D(z, r')$, by triangular inequality we have $D(x, y, y) = D(y, y, x) \leq D(y, y, z) + D(z, x, x) < r' + \delta = r$. Hence $B_D(z, r') \subseteq B_D(x, r)$. That is ball $B_D(x, r)$ is open ball. ■

Lemma 1.6. Let (X, D) be a D-metric space. If sequence $\{x_n\}$ in X converges to x,then x is unique.

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D(x, x, x_n) < \frac{\epsilon}{2}$ and $n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D(y, y, x_n) < \frac{\epsilon}{2}$.

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D(x, x, y) = 0$ is a contradiction. So, $x = y$. ■

Lemma 1.7. Let (X, D) be a D-metric space. If sequence $\{x_n\}$ in X is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Since $x_n \longrightarrow x$ for each $\epsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D(x_n, x_n, x) < \frac{\epsilon}{2}$ and $n_2 \in \mathbb{N}$ such that for every $m \geq n_2 \Longrightarrow D(x, x_m, x_m) < \frac{\epsilon}{2}$.

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by triangular inequality we have

$$D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. ■

Definition 1.8. A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.9. A 3-tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}$-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

(1) $\mathcal{M}(x, y, z, t) > 0$,

(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,

(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(5) $\mathcal{M}(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Remark 1.10. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. We prove that for every $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$. Because for each $\epsilon > 0$ by triangular inequality we have

(i) $\mathcal{M}(x, x, y, \epsilon + t) \geq \mathcal{M}(x, x, x, \epsilon) * \mathcal{M}(x, y, y, t) = \mathcal{M}(x, y, y, t)$

(ii) $\mathcal{M}(y, y, x, \epsilon + t) \geq \mathcal{M}(y, y, y, \epsilon) * \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.

By taking limits of (i) and (ii) when $\epsilon \longrightarrow 0$, we obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space . For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ as $n \longrightarrow \infty$,for each $t > 0$. It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$. The $\mathcal{M}$-fuzzy metric $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.11. Let X is a nonempty set and D is the D-metric on X . Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in]0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}$$

for all $x, y, z \in X$. It is easy to see that $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Lemma 1.12. Let $(X, M, *)$ is a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \longrightarrow [0, 1]$ by

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$$

for every x, y, z in X, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Proof.

(1) It is easy to see that for every $x, y, z \in X$, $\mathcal{M}(x, y, z, t) > 0 \quad \forall t > 0$.

(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $M(x, y, t) = M(y, z, t) = M(z, x, t) = 1$ if and only if $x = y = z$.

(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function.

(4)

$$\begin{aligned}\mathcal{M}(x, y, z, t + s) &= M(x, y, t + s) * M(y, z, t + s) * M(z, x, t + s)\\ &\geq M(x, y, t) * M(y, a, t) * M(a, z, s) * M(z, a, s) * M(a, x, t)\\ &= \mathcal{M}(x, y, a, t) * M(a, z, s) * M(z, a, s) * M(z, z, s)\\ &= \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s)\end{aligned}$$

for every $s > 0$. ■

Lemma 1.13. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t, for all x, y, z in X.

Proof. By Definition 1.9 (4) for each $x, y, z, a \in X$ and $t, s > 0$ we have $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$. If set $a = z$ we get $\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,that is $\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}(x, y, z, t)$. ■

Definition 1.14. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t)$$

Whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$ i.e.

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z \text{ and } \lim_{n\to\infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t)$$

Lemma 1.15. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.

Proof. Let $x, y, z \in X$ and $t > 0$, and let $(x'_n, y'_n, z'_n, t'_n)_n$ be a sequence in $X^3 \times (0, \infty)$ that converges to (x, y, z, t).

Since $(\mathcal{M}(x'_n, y'_n, z'_n, t'_n))_n$ is a sequence in $(0, 1]$, there is a subsequence $(x_n, y_n, z_n, t_n)_n$ of sequence $(x'_n, y'_n, z'_n, t'_n)_n$ such that sequence $(\mathcal{M}(x_n, y_n, z_n, t_n))_n$ converges to some point of $[0, 1]$.

Fix $\delta > 0$ such that $\delta < \frac{t}{2}$. Then, there is $n_0 \in \mathbb{N}$ such that $|t - t_n| < \delta$ for every $n \geq n_0$. Hence,

$$\begin{aligned}
\mathcal{M}(x_n, y_n, z_n, t_n) &\geq \mathcal{M}(x_n, y_n, z_n, t - \delta)\\
&\geq \mathcal{M}\left(x_n, y_n, z, t - \frac{4\delta}{3}\right) * \mathcal{M}\left(z, z_n, z_n, \frac{\delta}{3}\right)\\
&\geq \mathcal{M}\left(x_n, z, y, t - \frac{5\delta}{3}\right) * \mathcal{M}\left(y, y_n, y_n, \frac{\delta}{3}\right) * \mathcal{M}\left(z, z_n, z_n, \frac{\delta}{3}\right)\\
&\geq \mathcal{M}(z, y, x, t - 2\delta) * \mathcal{M}\left(x, x_n, x_n, \frac{\delta}{3}\right) * \mathcal{M}\left(y, y_n, y_n, \frac{\delta}{3}\right)\\
&\quad * \mathcal{M}\left(z, z_n, z_n, \frac{\delta}{3}\right)
\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(x, y, z, t + 2\delta) &\geq \mathcal{M}(x, y, z, t_n + \delta)\\ &\geq \mathcal{M}\left(x, y, z_n, t_n + \frac{2\delta}{3}\right) * \mathcal{M}\left(z_n, z, z, \frac{\delta}{3}\right)\\ &\geq \mathcal{M}\left(x, z_n, y_n, t_n + \frac{\delta}{3}\right) * \mathcal{M}\left(y_n, y, y, \frac{\delta}{3}\right) * \mathcal{M}\left(z_n, z, z, \frac{\delta}{3}\right)\\ &\geq \mathcal{M}(z_n, y_n, x_n, t_n) * \mathcal{M}\left(x_n, x, x, \frac{\delta}{3}\right) * \mathcal{M}\left(y_n, y, y, \frac{\delta}{3}\right)\\ &\quad * \mathcal{M}\left(z_n, z, z, \frac{\delta}{3}\right),\end{aligned}$$

for all $n \geq n_0$. By taking limits when $n \longrightarrow \infty$, we obtain

$$lim_{n\to\infty}\mathcal{M}(x_n, y_n, z_n, t_n) \geq \mathcal{M}(x, y, z, t - 2\delta) * 1 * 1 * 1 = \mathcal{M}(x, y, z, t - 2\delta)$$

and

$$\mathcal{M}(x, y, z, t + 2\delta) \geq \lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n) 1 * 1 * 1 = \lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n),$$

respectively.
So,by continuity of the function $t \longmapsto \mathcal{M}(x, y, z, t)$, we immediately deduce that

$$lim_{n\to\infty}\mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t).$$

Therefore $\mathcal{M}$ is continuous on $X^3 \times (0, \infty)$. ■

Definition 1.16. Let A and B be two self-mappings of a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$. We say that A and B satisfy the property (E), if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n\to\infty} \mathcal{M}(Bx_n, u, u, t) = 1$$

for some $u \in X$ and $t > 0$.

Example 1.17. Let $X = \mathbb{R}$ and $\mathcal{M}(x, y, z, t) = \dfrac{t}{t + |x - y| + |y - z| + |x - z|}$ for every $x, y, z \in X$ and $t > 0$. Let A and B defined

$$Ax = 2x + 1, \quad Bx = x + 2.$$

Consider the sequence $x_n = \dfrac{1}{n} + 1,\ n = 1, 2, \cdots$. Thus we have

$$\lim_{n\to\infty} \mathcal{M}(Ax_n, 3, 3, t) = \lim_{n\to\infty} \mathcal{M}(Bx_n, 3, 3, t) = 1$$

for every $t > 0$. Then A and B satisfying in the property (E).

In the next example we show that there are some mappings that have not property (E).

Example 1.18. Let $X = \mathbb{R}$ and $\mathcal{M}(x, y, z, t) = \dfrac{t}{t + |x - y| + |y - z| + |x - z|}$ for every $x, y, z \in X$ and $t > 0$. Let $Ax = x + 1$ and $Bx = x + 2$, if sequence $\{x_n\}$ there exist such that

$$\lim_{n\to\infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n\to\infty} \mathcal{M}(Bx_n, u, u, t) = 1$$

for some $u \in X$. Therefore

$$\lim_{n\to\infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n\to\infty} \mathcal{M}(x_n + 1, u, u, t) = \lim_{n\to\infty} \mathcal{M}(x_n, u - 1, u - 1, t) = 1,$$

and

$$\lim_{n\to\infty} \mathcal{M}(Bx_n, u, u, t) = \lim_{n\to\infty} \mathcal{M}(x_n + 2, u, u, t) = \lim_{n\to\infty} \mathcal{M}(x_n, u - 2, u - 2, t) = 1.$$

We conclude that, $x_n \to u - 1$ and $x_n \to u - 2$.Which is a contradiction. Hence A and B do not satisfy the property (E).

Definition 1.19. Let A and S be mappings from a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.20. Let A and S be mappings from a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

2. The Main Results

A class of implicit relation

Let Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : [0, 1]^{12} \longrightarrow [0, 1]$, and ϕ is continuous and increasing in each co-ordinate variable. Also $\phi(s, s, \cdots, s) > s$ for every $s \in [0, 1)$.

Example 2.1. Let $\phi : [0, 1]^{12} \longrightarrow [0, 1]$ is define by

$$\phi(x_1, x_2, \cdots, x_{12}) = (min\, \{x_i\})^h$$

for some $0 < h < 1$.

We begin by recalling some basic concepts of the main theory of this paper.

Theorem 2.2. Let A, B, S and T be self-mappings of a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ satisfying:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ *or* $S(X)$ is a complete fuzzy metric subspace of X,

(ii) The pair (A, S) and (B, T) are weakly compatible and (A, S) or (B, T) satisfy the property (E),

(iii)

$$\begin{aligned}&\mathcal{M}(Ax, By, Bz, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Sx, Ty, Tz, kt), \mathcal{M}(Sx, By, Tz, kt), \mathcal{M}(Sx, Ty, Bz, kt), \mathcal{M}(Sx, By, By, kt)\\ \mathcal{M}(Ty, By, Bz, kt), \mathcal{M}(Ty, Ty, Bz, kt), \mathcal{M}(Ty, By, By, kt), \mathcal{M}(Ty, Bz, Bz, kt)\\ \mathcal{M}(By, Ty, Tz, kt), \mathcal{M}(By, By, Tz, kt), \mathcal{M}(By, Tz, Tz, kt), \mathcal{M}(Tz, Bz, Bz, kt)\end{pmatrix}.\end{aligned}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let the pair (B, T) satisfy in property (E), hence there exist a sequence $\{x_n\}$ such that,

$$\lim_{n\to\infty} \mathcal{M}(Bx_n, u, u, t) = \lim_{n\to\infty} \mathcal{M}(Tx_n, u, u, t) = 1$$

for some $u \in X$ and every $t > 0$. As $BX \subseteq SX$, there exist a sequence $\{y_n\}$ such that,$Bx_n = Sy_n$, hence $\lim\limits_{n\to\infty} \mathcal{M}(Sy_n, u, u, t) = 1$.

We prove that $\lim\limits_{n\to\infty} \mathcal{M}(Ay_n, u, u, t) = 1$. Since

$$\begin{aligned}&\mathcal{M}(Ay_n, Bx_n, Bx_{n+1}, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Sy_n, Tx_n, Tx_{n+1}, kt), \mathcal{M}(Sy_n, Bx_n, Tx_{n+1}z, kt), \mathcal{M}(Sy_n, Tx_n, Bx_{n+1}, kt),\\ \mathcal{M}(Sy_n, Bx_n, Bx_n, kt), \mathcal{M}(Tx_n, Bx_n, Bx_{n+1}, kt), \mathcal{M}(Tx_n, Tx_n, Bx_{n+1}, kt),\\ \mathcal{M}(Tx_n, Bx_n, Bx_n, kt), \mathcal{M}(Tx_n, Bx_{n+1}, Bx_{n+1}, kt), \mathcal{M}(Bx_n, Tx_n, Tx_{n+1}, kt),\\ \mathcal{M}(Bx_n, Bx_n, Tx_{n+1}, kt), \mathcal{M}(Bx_n, Tx_{n+1}, Tx_{n+1}, kt), \mathcal{M}(Tx_{n+1}z, Bx_{n+1}, Bx_{n+1}, kt)\end{pmatrix}.\end{aligned}$$

On making $n \to \infty$ the above inequality, we get

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{M}(Ay_n, Bx_n, Bx_{n+1}, t)\\ &\qquad\geq \phi(\mathcal{M}(u, u, u, kt), \mathcal{M}(u, u, u, kt), \cdots, \mathcal{M}(u, u, u, kt)) = 1.\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} M(Ay_n, u, u, t) = 1$, hence

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = u.$$

Let $S(X)$ be complete $\mathcal{M}$-fuzzy metric space, then there exist $x \in X$ such that $Sx = u$. If $Ax \neq u$, then we have

$$\begin{aligned}&M(Ax, Bx_n, Bx_{n+1}, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Sx, Tx_n, Tx_{n+1}, kt), \mathcal{M}(Sx, Bx_n, Tx_{n+1}z, kt), \mathcal{M}(Sx, Tx_n, Bx_{n+1}, kt),\\ \mathcal{M}(Sx, Bx_n, Bx_n, kt), \mathcal{M}(Tx_n, Bx_n, Bx_{n+1}, kt), \mathcal{M}(Tx_n, Tx_n, Bx_{n+1}, kt),\\ \mathcal{M}(Tx_n, Bx_n, Bx_n, kt), \mathcal{M}(Tx_n, Bx_{n+1}, Bx_{n+1}, kt), \mathcal{M}(Bx_n, Tx_n, Tx_{n+1}, kt),\\ \mathcal{M}(Bx_n, Bx_n, Tx_{n+1}, kt), \mathcal{M}(Bx_n, Tx_{n+1}, Tx_{n+1}, kt), \mathcal{M}(Tx_{n+1}z, Bx_{n+1}, Bx_{n+1}, kt)\end{pmatrix}.\end{aligned}$$

On making $n \longrightarrow \infty$ we get $\mathcal{M}(Ax, u, u, t) = 1$, hence $Ax = u = Sx$. By (A, S) be weakly compatible , we have $ASx = SAx$, so

$$AAx = ASx = SAx = SSX.$$

As $AX \subset TX$, there exist $v \in X$ such that $Ax = Tv$. We prove that $Tv = Bv$. If $Tv \neq Bv$ then

$$\begin{aligned}&\mathcal{M}(Ax, Bv, Bv, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Sx, Tv, Tv, kt), \mathcal{M}(Sx, Bv, Tv, kt), \mathcal{M}(Sx, Tv, Bv, kt), \mathcal{M}(Sx, Bv, Bv, kt)\\ \mathcal{M}(Tv, Bv, Bv, kt), \mathcal{M}(Tv, Tv, Bv, kt), \mathcal{M}(Tv, Bv, Bv, kt), \mathcal{M}(Tv, Bv, Bv, kt)\\ \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Bv, Bv, Tv, kt), \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Tv, Bv, Bv, kt)\end{pmatrix}.\end{aligned}$$

If $Bv \neq u$ then

$$\mathcal{M}(Ax, Bv, Bv, t) > \mathcal{M}(Ax, Bv, Bv, kt),$$

is a contradiction. Thus $Tv = Bv = u$. By B and T be weakly compatible, we get $TTv = TBv = BTv = BBv$, so $Tu = Bu$. We prove $Au = u$, for

$$\begin{aligned}&\mathcal{M}(Au, u, u, t) = \mathcal{M}(Au, Bv, Bv, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Su, Tv, Tv, kt), \mathcal{M}(Su, Bv, Tv, kt), \mathcal{M}(Su, Tv, Bv, kt), \mathcal{M}(Su, Bv, Bv, kt)\\ \mathcal{M}(Tv, Bv, Bv, kt), \mathcal{M}(Tv, Tv, Bv, kt), \mathcal{M}(Tv, Bv, Bv, kt), \mathcal{M}(Tv, Bv, Bv, kt)\\ \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Bv, Bv, Tv, kt), \mathcal{M}(Bv, Tv, Tv, kt), \mathcal{M}(Tv, Bv, Bv, kt)\end{pmatrix}.\end{aligned}$$

If $Au \neq u$, then

$$\mathcal{M}(Au, u, u, t) > \mathcal{M}(Au, u, u, kt),$$

is a contradiction. Thus $Au = Su = u$. Now, we prove $Bu = u$. For

$$\begin{aligned}&\mathcal{M}(u, Bu, Bu, t) = \mathcal{M}(Au, Bu, Bu, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Su, Tu, Tu, kt), \mathcal{M}(Su, Bu, Tu, kt), \mathcal{M}(Su, Tu, Bu, kt), \mathcal{M}(Su, Bu, Buv, kt)\\ \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Tu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt)\\ \mathcal{M}(Bu, Tu, Tu, kt), \mathcal{M}(Bu, Bu, Tu, kt), \mathcal{M}(Bu, Tu, Tu, kt), \mathcal{M}(Tu, Bu, Bu, kt)\end{pmatrix}.\end{aligned}$$

If $Bu \neq u$, then

$$\mathcal{M}(u, Bu, Bu, t) > \mathcal{M}(u, Bu, Bu, kt),$$

is a contradiction. Thus $Au = Bu = Su = Tu = u$. So, A, B, S and T have a fixed common point u.
Now to prove uniqueness, if possible $v \neq u$ be another common fixed point of A, B, S and T. Then

$$\begin{aligned}&\mathcal{M}(v, u, u, t) = \mathcal{M}(Av, Bu, Bu, t)\\ &\quad\geq \phi\begin{pmatrix}\mathcal{M}(Sv, Tu, Tu, kt), \mathcal{M}(Sv, Bu, Tu, kt), \mathcal{M}(Sv, Tu, Bu, kt), \mathcal{M}(Sv, Bu, Bu, kt)\\ \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Tu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt), \mathcal{M}(Tu, Bu, Bu, kt)\\ \mathcal{M}(Bu, Tu, Tu, kt), \mathcal{M}(Bu, Bu, Tu, kt), \mathcal{M}(Bu, Tu, Tu, kt), \mathcal{M}(Tu, Bu, Bu, kt)\end{pmatrix}\\ &\quad> \mathcal{M}(v, u, u, kt),\end{aligned}$$

is contradiction. ■

Acknowledgment

The authors thank the referee for his careful reading of the original manuscript and helpful suggestions.

References

[1] Dhage B.C., 1992, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc., 84(4), pp. 329–336.

[2] El Naschie MS, 1998, On the uncertainty of Cantorian geometry and two-slit experiment. Chaos, Solitons and Fractals, 9, pp. 517–29.

[3] El Naschie MS, 2004, A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons and Fractals, 19, pp. 209–36.

[4] El Naschie MS, 2005, On a fuzzy Kahler-like Manifold which is consistent with two-slit experiment. Int J of Nonlinear Science and Numerical Simulation, 6, pp. 95–8.

[5] El Naschie MS, 2006, The idealized quantum two-slit gedanken experiment revisited-Criticism and reinterpretation. Chaos, Solitons and Fractals, 27, pp. 9–13.

[6] George A., and Veeramani P., 1994, On some result in fuzzy metric space. Fuzzy Sets Syst, 64, pp. 395–9.

[7] Gregori V. and Sapena A., 2002, On fixed-point theorem in fuzzy metric spaces. Fuzzy Sets and Sys, 125, pp. 245–52.

[8] Kramosil I. and Michalek J., 1975, Fuzzy metric and statistical metric spaces. Kybernetica, 11, pp. 326–34.

[9] Rhoades B.E., 1996, A fixed point theorem for generalized metric spaces, Int. J. Math. Math. Sci., 19(1), pp. 145–153.

[10] Miheţ D., 2004, A Banach contraction theorem in fuzzy metric spaces. Fuzzy Sets Sys, 144, pp. 431–9.

[11] Saadati R. and Park JH., 2006, On the intuitionistic fuzzy topological spaces. Chaos, Solitons and Fractals, 27, pp. 331–44.

[12] Singh B. and Sharma R.K., 2002, Common fixed points via compatible maps in D-metric spaces, Rad. Mat., 11(1), pp. 145–153.

[13] Schweizer B., Sherwood H., and Tardiff RM., 1988, Contractions on PM-space examples and counterexamples. Stochastica, 1, pp. 5–17.

[14] Tanaka Y., Mizno Y., and Kado T., 2005, Chaotic dynamics in Friedmann equation. Chaos, Solitons and Fractals, 24, pp. 407–22.

[15] Zadeh LA., 1965, Fuzzy sets. Inform and Control, 8, pp. 338–53.