

Linear k -Step Methods for Solving N th-order Fuzzy Differential Equations

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Abstract

In this paper a particular numerical algorithm of linear k -step methods for solving initial value problem of N th-order fuzzy differential equations is presented. First of all we will transform an initial value problem of N th-order fuzzy differential equations to an initial value problem of first-order systems of fuzzy differential equations, then we solve this system, numerically. After that we obtain the approximate solution of the initial value problem of N th-order fuzzy differential equation. Also we will prove that the algorithm converges to the exact solution as the stepsize goes to zero and at the end the validity of the algorithm is illustrated by solving some examples.

AMS Subject Classification: 65L05, 65L06.

Keywords: Fuzzy differential equations; Linear multi-step methods.

1. Preliminaries

A fuzzy subset of \mathcal{R} , the set of all real numbers, is a function $u : \mathcal{R} \rightarrow [0, 1]$. For each $r \in (0, 1]$ the r -level set of u is defined by

$$[u]_r = \{x \in \mathcal{R} : u(x) \geq r\},$$

and the support of u is defined as the closure of the union of all its level sets, that is,

$$[u]_0 = \overline{\bigcup_{r \in (0,1]} [u]_r}.$$

We denote by E the set of all fuzzy subsets of \mathcal{R} which are normal, convex and upper semicontinuous with bounded r -level sets [4,8]. It means that, if $u \in E$, then $[u]_r$ is a closed bounded interval which is denoted by

$$[u]_r = [\underline{u}(r), \bar{u}(r)], \quad r \in [0, 1].$$

The elements of E are called fuzzy numbers. Assume that I be a subinterval of \mathcal{R} . The mapping $y : I \rightarrow E$ is called a fuzzy process and for each $x \in I$ its r -level set is denoted by

$$[y(x)]_r = [\underline{y}(x; r), \bar{y}(x; r)], \quad r \in [0, 1].$$

The first derivative $y'(x)$ of $y(x)$ is defined by

$$[y'(x)]_r = [\underline{y}'(x; r), \bar{y}'(x; r)], \quad x \in I, \quad r \in [0, 1],$$

provided that this equation determines a fuzzy number, according to Seikkala [8]. Similarly the j th derivative, $y^{(j)}$, of y is defined, where j is a natural number.

In [6] and [7] the initial value problem (IVP) of first-order fuzzy differential equations (FDEs) has been solved, numerically, by Euler method and two-step predictor-corrector methods, respectively. In this paper we will solve the IVP of N th-order fuzzy differential equations by linear k -step methods.

Consider the IVP of N th-order fuzzy differential equations given by

$$\begin{cases} y^{(N)}(x) = f(x, y(x), y'(x), \dots, y^{(N-1)}(x)), & x \in [x_0, \bar{x}], \\ y(x_0) = \alpha_1, \dots, y^{(N-1)}(x_0) = \alpha_N, \end{cases} \quad (1.1)$$

where $y(x)$ is a fuzzy function of the crisp variable x and for $1 \leq j \leq N$, $y^{(j)}$ is the j th fuzzy derivative of y , α_j is a fuzzy number and $f(x, y, y', \dots, y^{(N-1)})$ is a fuzzy function of the crisp variable x and fuzzy variables $y, y', \dots, y^{(N-1)}$.

Since the FIVP (1.1) can be transformed to a first-order system of fuzzy differential equations [9], it is sufficient to solve the IVP of first-order systems of FDEs, in general; that is,

$$\begin{cases} u_1'(x) = f_1(x, u_1(x), \dots, u_N(x)), \\ \vdots \\ u_N'(x) = f_N(x, u_1(x), \dots, u_N(x)), \\ u_1(x_0) = \alpha_1, \dots, u_N(x_0) = \alpha_N, \end{cases} \quad (1.2)$$

where $x \in [x_0, \bar{x}]$ and for $j, 1 \leq j \leq N$, $u_j(x)$ is a fuzzy function of crisp variable x , $f_j(x, u_1, \dots, u_N)$ is a fuzzy function of the crisp variable x and fuzzy variables u_j , α_j is a fuzzy number and u'_j is the first derivative of fuzzy function u_j . Note that the FIVP (1.2) can be written as:

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \bar{x}], \\ u(x_0) = \alpha, \end{cases} \quad (1.3)$$

where $u = (u_1, \dots, u_N)^t$, $u' = (u'_1, \dots, u'_N)^t$, $f = (f_1, \dots, f_N)^t$, $\alpha = (\alpha_1, \dots, \alpha_N)^t$.

For each $r \in [0, 1]$, the lower and upper r -level sets of the fuzzy N -vector $a = (a_1, \dots, a_N)^t$, where $a_j, 1 \leq j \leq N$, is a fuzzy number, are defined as

$$\begin{cases} \underline{a}(r) = (\underline{a}_1(r), \dots, \underline{a}_N(r))^t, \\ \bar{a}(r) = (\bar{a}_1(r), \dots, \bar{a}_N(r))^t, \end{cases}$$

respectively. By using r -levels, (1.2) can be written as:

$$\begin{cases} \underline{u}'_1(x; r) = \underline{f}_1(x, u_1(x), \dots, u_N(x); r), \\ \quad \vdots \\ \underline{u}'_N(x; r) = \underline{f}_N(x, u_1(x), \dots, u_N(x); r), \\ \bar{u}'_1(x; r) = \bar{f}_1(x, u_1(x), \dots, u_N(x); r), \\ \quad \vdots \\ \bar{u}'_N(x; r) = \bar{f}_N(x, u_1(x), \dots, u_N(x); r), \\ \underline{u}_1(x_0; r) = \underline{\alpha}_1(r), \dots, \underline{u}_N(x_0; r) = \underline{\alpha}_N(r), \\ \bar{u}_1(x_0; r) = \bar{\alpha}_1(r), \dots, \bar{u}_N(x_0; r) = \bar{\alpha}_N(r). \end{cases}$$

By the extension principle [1], there exist functions $F_1, \dots, F_N, G_1, \dots, G_N$ such that

$$\begin{cases} \underline{u}'_1(x; r) = F_1(x, \underline{u}_1(x; r), \dots, \underline{u}_N(x; r), \bar{u}_1(x; r), \dots, \bar{u}_N(x; r)), \\ \quad \vdots \\ \underline{u}'_N(x; r) = F_N(x, \underline{u}_1(x; r), \dots, \underline{u}_N(x; r), \bar{u}_1(x; r), \dots, \bar{u}_N(x; r)), \\ \bar{u}'_1(x; r) = G_1(x, \underline{u}_1(x; r), \dots, \underline{u}_N(x; r), \bar{u}_1(x; r), \dots, \bar{u}_N(x; r)), \\ \quad \vdots \\ \bar{u}'_N(x; r) = G_N(x, \underline{u}_1(x; r), \dots, \underline{u}_N(x; r), \bar{u}_1(x; r), \dots, \bar{u}_N(x; r)). \end{cases}$$

Therefore, we have

$$\begin{cases} \underline{u}'(x; r) = \underline{f}(x, u(x); r) = F(x, \underline{u}(x; r), \bar{u}(x; r)), \\ \bar{u}'(x; r) = \bar{f}(x, u(x); r) = G(x, \underline{u}(x; r), \bar{u}(x; r)), \\ \underline{u}(x_0; r) = \underline{\alpha}(r), \quad \bar{u}(x_0; r) = \bar{\alpha}(r), \end{cases} \quad (1.4)$$

where $F = (F_1, \dots, F_N)^t$, $G = (G_1, \dots, G_N)^t$.

Remember that the IVP of first-order systems of ordinary differential equations given by

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \bar{x}], \\ u(x_0) = \alpha, \end{cases}$$

can be solved, numerically, by linear k -step methods[3, 9]. The linear k -step method given by

$$u_n = u_{n-k} + h \sum_{i=0}^k \beta_i f(x_{n-i}, u_{n-i}), \quad (1.5)$$

where $h = \frac{\bar{x} - x_0}{\tilde{N}}$ and $x_n = x_0 + ih$ for $i = 0, 1, \dots, \tilde{N}$, is of order q [3], if

$$\sum_{i=0}^k \beta_i = k, \quad j \sum_{i=1}^k i^{j-1} \beta_i = k^j, \quad 2 \leq j \leq q.$$

In this paper we will use the linear k -step methods given by (1.5) for solving FIVPs where $\beta_j \geq 0$ for $0 \leq j \leq k$.

At the end of this section we state some lemmas which are employed in section two.

Lemma 1.1. Assume that all of constants A_1, \dots, A_k are nonnegative and $A_k > 0$. Then the equation

$$x(A_1 + x)^{k-1} - A_2(A_1 + x)^{k-2} - \dots - A_{k-1}(A_1 + x) - A_k = 0, \quad (1.6)$$

has a positive root.

Proof. Denote the left-hand side of (1.6) by $f(x)$. Then f is a continuous mapping, $f(0) < 0$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. Therefore there exists a positive real number c such that $f(c) = 0$. ■

Lemma 1.2. Assume that $\{T_n\}_{n=k-1}^{\tilde{N}}$ be a sequence of nonnegative numbers such that

$$T_n \leq AT_{n-1} + B, \quad k \leq n \leq \tilde{N},$$

for some nonnegative constants A and B . If $A \neq 1$, then

$$T_n \leq A^{n-k+1}T_{k-1} + B \frac{A^{n-k+1} - 1}{A - 1}, \quad k \leq n \leq \tilde{N}.$$

Proof. The proof is straightforward by the mathematical induction. ■

Lemma 1.3. Suppose that $\{P_n\}_{n=0}^{\tilde{N}}$ be a sequence of nonnegative numbers such that

$$P_n \leq A_1 P_{n-1} + \dots + A_{k-1} P_{n-k+1} + A_k P_{n-k} + B, \quad k \leq n \leq \tilde{N}, \quad (1.7)$$

where all of A_1, \dots, A_k, B are nonnegative constants and $A_k > 0$. Also assume that θ_1 be a positive root (Lemma 1.1) of the equation given by

$$x(A_1 + x)^{k-1} - A_2(A_1 + x)^{k-2} - \dots - A_{k-1}(A_1 + x) - A_k = 0, \quad (1.8)$$

and

$$\theta_1 = \frac{A_2 + \theta_2}{A_1 + \theta_1}, \dots, \theta_{k-2} = \frac{A_{k-1} + \theta_{k-1}}{A_1 + \theta_1}, \quad \theta_{k-1} = \frac{A_k}{A_1 + \theta_1}. \quad (1.9)$$

If $A_1 + \theta_1 \neq 1$, then

$$P_n + \theta_1 P_{n-1} + \dots + \theta_{k-1} P_{n-k+1} \leq (A_1 + \theta_1)^{n-k+1} [P_{k-1} + \theta_1 P_{k-2} + \dots + \theta_{k-1} P_0] \\ + B \frac{(A_1 + \theta_1)^{n-k+1} - 1}{A_1 + \theta_1 - 1}, \quad k \leq n \leq \tilde{N}. \quad (1.10)$$

Proof. According to (1.9) it is clear that $\theta_i \geq 0$, $2 \leq i \leq k-1$. Now by adding $\theta_1 P_{n-1} + \dots + \theta_{k-1} P_{n-k+1}$ to both sides of (1.7) we obtain

$$P_n + \theta_1 P_{n-1} + \dots + \theta_{k-1} P_{n-k+1} \leq (A_1 + \theta_1) P_{n-1} + \dots + (A_{k-1} + \theta_{k-1}) P_{n-k+1} \\ + A_k P_{n-k} + B,$$

and then by (1.9) we have

$$P_n + \theta_1 P_{n-1} + \dots + \theta_{k-1} P_{n-k+1} \leq (A_1 + \theta_1) [P_{n-1} + \theta_1 P_{n-2} + \dots \\ + \theta_{k-1} P_{n-k}] + B.$$

Let $T_n = P_n + \theta_1 P_{n-1} + \dots + \theta_{k-1} P_{n-k+1}$, $k \leq n \leq \tilde{N}$. Now by Lemma 1.2, we obtain the inequality (1.10). ■

Lemma 1.4. If the vector function $u \in C^{q+1}[x_{n-k}, x_n]$ and the order of linear k -step method given by (1.5) is q , then

$$u(x_n) = u(x_{n-k}) + h \sum_{i=0}^k \beta_i u'(x_{n-i}) - \frac{u^{(q+1)}(\xi_k) p_{q+1}(x_{n-k})}{(q+1)!} \\ - h \sum_{i=1}^k \beta_i \frac{u^{(q+1)}(\eta_i) p'_{q+1}(x_{n-i})}{(q+1)!},$$

where $p_{q+1}(x) = (x - x_n)^{q+1}$, $x_{n-k} < \xi_k < x_n$ and η_i is between x_{n-i} and x_n for $i = 1, \dots, k$.

Proof. See [3]. ■

2. Linear k -step Methods for Solving the IVP of First-Order Systems of Fuzzy Differential Equations

Consider the IVP of first-order system of FDEs given by

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \bar{x}], \\ u(x_0) = \alpha. \end{cases} \quad (2.1)$$

First of all note that the equation (1.5) is fuzzified by equations

$$\begin{cases} \underline{u}_n(r) = \underline{u}_{n-k}(r) + h \sum_{i=0}^k \beta_i \underline{f}(x_{n-i}, u_{n-i}; r), \\ \bar{u}_n(r) = \bar{u}_{n-k}(r) + h \sum_{i=0}^k \beta_i \bar{f}(x_{n-i}, u_{n-i}; r), \end{cases} \quad (2.2)$$

where $n = k, \dots, \tilde{N}$. In order to start with the linear k -step methods we will use the Euler method from [6] to find $\underline{u}_i(r), \bar{u}_i(r)$ for $1 \leq i \leq k-1$. Now we prove the convergence of linear k -step methods for solving FIVPs.

Theorem 2.1. Let $K = \{(x, u_1, \dots, u_k, v_1, \dots, v_k) : x_0 \leq x \leq \bar{x}, -\infty < u_j \leq v_j, -\infty < v_j < \infty, 1 \leq j \leq k\}$ and suppose that $F(x, U, V)$ and $G(x, U, V)$ [see (1.4)] are in $C^m(K)$, where $U = (u_1, \dots, u_k)$, $V = (v_1, \dots, v_k)$ and m is the order of linear k -step method given by (1.5). Also assume that the partial derivatives of F and G are bounded on K and

$$\begin{cases} \|F(x, \underline{U}, \bar{U}) - F(x, \underline{V}, \bar{V})\| \leq L \max\{\|\underline{U} - \underline{V}\|, \|\bar{U} - \bar{V}\|\}, \\ \|G(x, \underline{U}, \bar{U}) - G(x, \underline{V}, \bar{V})\| \leq L \max\{\|\underline{U} - \underline{V}\|, \|\bar{U} - \bar{V}\|\}, \end{cases} \quad (2.3)$$

where L is a positive constant, $L \neq \frac{1}{2}$ and the norm in use is ℓ_∞ -norm. Then the approximate solution of FIVP (2.1) converges to its exact solution as the stepsize goes to zero; that is, for fixed r , $0 \leq r \leq 1$,

$$\lim_{h \rightarrow 0} \underline{u}_{\tilde{N}}(r) = \underline{u}(x_{\tilde{N}}; r), \quad \lim_{h \rightarrow 0} \bar{u}_{\tilde{N}}(r) = \bar{u}(x_{\tilde{N}}; r).$$

Proof. Assume that $w_n = \|\underline{u}_n(r) - \underline{u}(x_n; r)\|$, $v_n = \|\bar{u}_n(r) - \bar{u}(x_n; r)\|$ and $z_n = v_n + w_n$, where $0 \leq r \leq 1$ and $n = 0, 1, \dots, \tilde{N}$. Suppose that $u(x)$ is the exact solution of the FIVP (2.1), then by Lemma 1.4, (1.4) and (2.2) we have

$$\begin{aligned} \|\underline{u}_n(r) - \underline{u}(x_n; r)\| &\leq \|\underline{u}_{n-k}(r) - \underline{u}(x_{n-k}; r)\| + \sup_{x \in [x_{n-k}, x_n]} \|\underline{u}^{(m+1)}(x; r)\| . M . h^{m+1} \\ &\quad + h \sum_{i=0}^k \beta_i \|F(x_{n-i}, \underline{u}_{n-i}(r), \bar{u}_{n-i}(r)) - F(x_{n-i}, \underline{u}(x_{n-i}; r), \bar{u}(x_{n-i}; r))\|, \end{aligned}$$

where $M = \frac{1}{(m+1)!} k^{m+1} + \frac{1}{m!} \sum_{i=1}^k (\beta_i) i^m$. Then by (2.3) we obtain

$$\begin{aligned} \|\underline{u}_n(r) - \underline{u}(x_n; r)\| &\leq w_{n-k} + \underline{M}(r) . M . h^{m+1} \\ &\quad + hL \sum_{i=0}^k \beta_i \max(\|\underline{u}_{n-i}(r) - \underline{u}(x_{n-i}; r)\|, \|\bar{u}_{n-i}(r) - \bar{u}(x_{n-i}; r)\|), \end{aligned}$$

where $\underline{M}(r) = \sup_{x \in [x_{n-k}, x_n]} \|\underline{u}^{(m+1)}(x; r)\|$. Therefore we have

$$w_n \leq w_{n-k} + hL \sum_{i=0}^k \beta_i z_{n-i} + \underline{M}(r).M.h^{m+1}. \quad (2.4)$$

Similarly by Lemma 1.4, (1.4), (2.2) and (2.3) we obtain

$$v_n \leq v_{n-k} + hL \sum_{i=0}^k \beta_i z_{n-i} + \overline{M}(r).M.h^{m+1}, \quad (2.5)$$

where $\overline{M}(r) = \sup_{x \in [x_{n-k}, x_n]} \|\overline{u}^{(m+1)}(x; r)\|$. Now by (2.4)–(2.5) we have

$$z_n \leq \sum_{i=1}^{k-1} \frac{2hL\beta_i}{1-2hL\beta_0} z_{n-i} + \frac{1+2hL\beta_k}{1-2hL\beta_0} z_{n-k} + \frac{M(r).M.h^{m+1}}{1-2hL\beta_0}, \quad k \leq n \leq \tilde{N},$$

where $M(r) = \underline{M}(r) + \overline{M}(r)$ and $h > 0$ is such that $1 - 2Lh\beta_0 > 0$. Suppose that

$$A_i = \frac{2hL\beta_i}{1-2hL\beta_0}, \quad A_k = \frac{1+2hL\beta_k}{1-2hL\beta_0}, \quad B = \frac{M(r).M.h^{m+1}}{1-2hL\beta_0}, \quad 1 \leq i \leq k-1. \quad (2.6)$$

By Lemma 1.3, we get

$$z_n + \theta_1 z_{n-1} + \cdots + \theta_{k-1} z_{n-k+1} \leq (A_1 + \theta_1)^{n-k+1} [z_{k-1} + \theta_1 z_{k-2} + \cdots + \theta_{k-1} z_0] \\ + B \frac{(A_1 + \theta_1)^{n-k+1} - 1}{(A_1 + \theta_1) - 1}, \quad k \leq n \leq \tilde{N}.$$

If $n = \tilde{N}$, then

$$0 \leq z_{\tilde{N}} + \theta_1 z_{\tilde{N}-1} + \cdots + \theta_{k-1} z_{\tilde{N}-k+1} \\ \leq (A_1 + \theta_1)^{\tilde{N}-k+1} [z_{k-1} + \theta_1 z_{k-2} + \cdots + \theta_{k-1} z_0] + B \frac{(A_1 + \theta_1)^{\tilde{N}-k+1} - 1}{(A_1 + \theta_1) - 1}.$$

Let us denote the positive root of (1.8) which depends on h by $\theta_1(h)$. Since all coefficients of (1.8) are continuous, when they are considered as functions of h according to (2.6), by (1.8) we have $\lim_{h \rightarrow 0} \theta_1(h) = 1$. It is clear from (1.9) that $\lim_{h \rightarrow 0} \theta_i(h) = 1$, $2 \leq i \leq k-1$. Also $\lim_{h \rightarrow 0} z_j = 0$, $0 \leq j \leq k-1$ (see [6]) and $\lim_{h \rightarrow 0} (A_1(h) + \theta_1(h)) = 2L$. Thus we obtain

$$\lim_{h \rightarrow 0} \left\{ (A_1 + \theta_1)^{\tilde{N}-k+1} [z_{k-1} + \theta_1 z_{k-2} + \cdots + \theta_{k-1} z_0] + B \frac{(A_1 + \theta_1)^{\tilde{N}-k+1} - 1}{A_1 + \theta_1 - 1} \right\} = 0,$$

and then $\lim_{h \rightarrow 0} [z_{\tilde{N}} + \sum_{j=1}^{k-1} \theta_j z_{\tilde{N}-j}] = 0$. Therefore $\lim_{h \rightarrow 0} z_{\tilde{N}} = 0$ and the proof is completed. \blacksquare

Convergence theorem. The approximate solution of the FIVP given by (1.1) converges to its exact solution as the step size goes to zero.

3. Numerical Results

In this section we will solve some examples by the following explicit methods.

Linear 2-step method of order two:

$$\begin{cases} \underline{u}_n(r) = \underline{u}_{n-2}(r) + 2h \underline{f}(x_{n-1}, u_{n-1}; r), \\ \bar{u}_n(r) = \bar{u}_{n-2}(r) + 2h \bar{f}(x_{n-1}, u_{n-1}; r), \\ 0 \leq r \leq 1, \quad n = 2, \dots, \tilde{N}. \end{cases} \quad (3.1)$$

Linear 3-step method of order three:

$$\begin{cases} \underline{u}_n(r) = \underline{u}_{n-3}(r) + \frac{3}{4}h\{3\underline{f}(x_{n-1}, u_{n-1}; r) + \underline{f}(x_{n-3}, u_{n-3}; r)\}, \\ \bar{u}_n(r) = \bar{u}_{n-3}(r) + \frac{3}{4}h\{3\bar{f}(x_{n-1}, u_{n-1}; r) + \bar{f}(x_{n-3}, u_{n-3}; r)\}, \\ 0 \leq r \leq 1, \quad n = 3, \dots, \tilde{N}. \end{cases} \quad (3.2)$$

Example 3.1. Consider the FIVP given by

$$\begin{cases} y''(x) = 2xy'(x) + y(x), & x \in [0, 1], \\ \underline{y}(0; r) = 0.4 \exp(r) - 0.3, & \underline{y}'(0; r) = 0.5r^{0.5} - 0.3, \\ \bar{y}(0; r) = 0.4 \exp(2-r) - 0.3, & \bar{y}'(0; r) = 0.2(1-r)^{0.5} + 0.2, \end{cases} \quad (3.3)$$

where $0 \leq r \leq 1$. The exact solution of the FIVP (3.3) is

$$\begin{aligned} \underline{y}(x; r) &= \underline{y}(0; r) \left[1 + \sum_{j=0}^{\infty} \frac{(4j+1) \times \dots \times 9 \times 5 \times 1}{(2j+2)!} x^{2j+2} \right] \\ &\quad + \underline{y}'(0; r) \left[x + \sum_{j=1}^{\infty} \frac{(4j-1) \times \dots \times 7 \times 3}{(2j+1)!} x^{2j+1} \right], \\ \bar{y}(x; r) &= \bar{y}(0; r) \left[1 + \sum_{j=0}^{\infty} \frac{(4j+1) \times \dots \times 9 \times 5 \times 1}{(2j+2)!} x^{2j+2} \right] \\ &\quad + \bar{y}'(0; r) \left[x + \sum_{j=1}^{\infty} \frac{(4j-1) \times \dots \times 7 \times 3}{(2j+1)!} x^{2j+1} \right], \end{aligned}$$

where $x \in [0, 1]$, $0 \leq r \leq 1$. In order to obtain the approximate solution of FIVP (3.3), suppose that

$$\begin{cases} u_1(x) = y(x), \\ u_2(x) = y'(x). \end{cases} \quad (3.4)$$

Then the FDE of (3.3) is transformed to the first-order system of FDEs given by

$$\begin{cases} u_1'(x) = u_2(x), \\ u_2'(x) = 2xu_2(x) + u_1(x). \end{cases}$$

Now by using r -level sets we have

$$\begin{cases} \underline{u}_1'(x; r) = \underline{u}_2(x; r), & \underline{u}_1(0; r) = 0.4 \exp(r) - 0.3, \\ \bar{u}_1'(x; r) = \bar{u}_2(x; r), & \bar{u}_1(0; r) = 0.4 \exp(2 - r) - 0.3, \\ \underline{u}_2'(x; r) = 2x\underline{u}_2(x; r) + \underline{u}_1(x; r), & \underline{u}_2(0; r) = 0.5r^{0.5} - 0.3, \\ \bar{u}_2'(x; r) = 2x\bar{u}_2(x; r) + \bar{u}_1(x; r), & \bar{u}_2(0; r) = 0.2(1 - r)^{0.5} + 0.2. \end{cases} \quad (3.5)$$

After computing the approximate solution of FIVP (3.5) using method given by (3.1), the approximate solution of FIVP (3.3), that is, $[y_{\tilde{N}}]_r = [\underline{y}_{\tilde{N}}(r), \bar{y}_{\tilde{N}}(r)]$, $0 \leq r \leq 1$, according to (3.4) is obtained. For some values of r , $0 \leq r \leq 1$, the approximate and exact solutions of FIVP (3.3) at $x = 1$ with $\tilde{N} = 300$ have been given in table 1 and their graphs in figure 1.

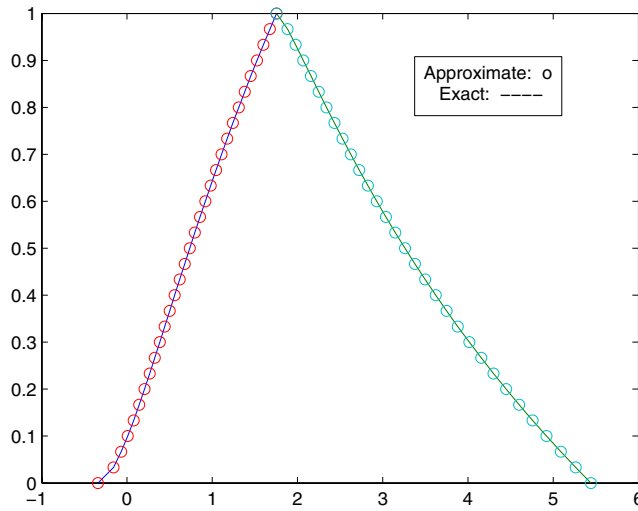


Figure 1: $h = \frac{1}{\tilde{N}}$

r	$\underline{y}_{\tilde{N}}(r)$	$\underline{y}(x_{\tilde{N}}; r)$	$\bar{y}_{\tilde{N}}(r)$	$\bar{y}(x_{\tilde{N}}; r)$
0	-0.3408197277	-0.3408342109	5.4426412521	5.4427351756
0.2	0.2049205492	0.2049212245	4.4478203772	4.4478987597
0.4	0.5588298024	0.5588386689	3.6217602825	3.6218255593
0.6	0.9182423072	0.9182587731	2.9301845005	2.9302385436
0.8	1.3106485495	1.3106727594	2.3401112668	2.3401553246
1	1.7546066087	1.7546391105	1.7546066087	1.7546391105

Table 1: Approximate and exact solutions.

Example 3.2. Consider the FIVP given by

$$\begin{cases} y^{(4)}(x) = y'''(x) + y''(x) + y'(x) + 2y(x), & x \in [0, 1], \\ \underline{y}(0; r) = 0.5r^{0.5} - 0.3, & \bar{y}(0; r) = 0.2(1 - r)^{0.5} + 0.2, \\ \underline{y}'(0; r) = 0.4 \exp(r) - 0.3, & \bar{y}'(0; r) = 0.4 \exp(2 - r) - 0.3, \\ \underline{y}''(0; r) = \exp(r), & \bar{y}''(0; r) = \exp(2 - r), \\ \underline{y}'''(0; r) = r + 2, & \bar{y}'''(0; r) = 4 - r, \end{cases} \quad (3.6)$$

where $0 \leq r \leq 1$. The exact solution of the FIVP (3.6) is

$$\begin{aligned} \underline{y}(x; r) &= ((10\underline{y}(0; r) - 5\underline{y}'(0; r) + 10\underline{y}''(0; r) - 5\underline{y}'''(0; r))/30) \exp(-x) \\ &\quad + ((\underline{y}(0; r) + \underline{y}'(0; r) + \underline{y}''(0; r) + \underline{y}'''(0; r))/15) \exp(2x) \\ &\quad + ((18\underline{y}(0; r) + 3\underline{y}'(0; r) - 12\underline{y}''(0; r) + 3\underline{y}'''(0; r))/30) \cos(x) \\ &\quad + ((6\underline{y}(0; r) + 21\underline{y}'(0; r) + 6\underline{y}''(0; r) - 9\underline{y}'''(0; r))/30) \sin(x); \\ \bar{y}(x; r) &= ((10\bar{y}(0; r) - 5\bar{y}'(0; r) + 10\bar{y}''(0; r) - 5\bar{y}'''(0; r))/30) \exp(-x) \\ &\quad + ((\bar{y}(0; r) + \bar{y}'(0; r) + \bar{y}''(0; r) + \bar{y}'''(0; r))/15) \exp(2x) \\ &\quad + ((18\bar{y}(0; r) + 3\bar{y}'(0; r) - 12\bar{y}''(0; r) + 3\bar{y}'''(0; r))/30) \cos(x) \\ &\quad + ((6\bar{y}(0; r) + 21\bar{y}'(0; r) + 6\bar{y}''(0; r) - 9\bar{y}'''(0; r))/30) \sin(x); \end{aligned}$$

where $x \in [0, 1]$, $0 \leq r \leq 1$. In order to obtain the approximate solution of FIVP (3.6) suppose that

$$\begin{cases} u_1(x) = y(x), \\ u_2(x) = y'(x), \\ u_3(x) = y''(x), \\ u_4(x) = y'''(x). \end{cases} \quad (3.7)$$

Then the FDE of (3.6) is transformed to the first-order system of FDEs given by

$$\begin{cases} u_1'(x) = u_2(x), \\ u_2'(x) = u_3(x), \\ u_3'(x) = u_4(x), \\ u_4'(x) = u_4(x) + u_3(x) + u_2(x) + 2u_1(x). \end{cases}$$

Now by using r -level sets we have

$$\left\{ \begin{array}{ll} \underline{u}'_1(x; r) = \underline{u}_2(x; r), & \underline{u}_1(0; r) = 0.5r^{0.5} - 0.3, \\ \overline{u}'_1(x; r) = \overline{u}_2(x; r), & \overline{u}_1(0; r) = 0.2(1 - r)^{0.5} + 0.2, \\ \underline{u}'_2(x; r) = \underline{u}_3(x; r), & \underline{u}_2(0; r) = 0.4 \exp(r) - 0.3, \\ \overline{u}'_2(x; r) = \overline{u}_3(x; r), & \overline{u}_2(0; r) = 0.4 \exp(2 - r) - 0.3, \\ \underline{u}'_3(x; r) = \underline{u}_4(x; r), & \underline{u}_3(0; r) = \exp(r), \\ \overline{u}'_3(x; r) = \overline{u}_4(x; r), & \overline{u}_3(0; r) = \exp(2 - r), \\ \underline{u}'_4(x; r) = \underline{u}_4(x; r) + \underline{u}_3(x; r) + \underline{u}_2(x; r) + 2\underline{u}_1(x; r), & \underline{u}_4(0; r) = r + 2, \\ \overline{u}'_4(x; r) = \overline{u}_4(x; r) + \overline{u}_3(x; r) + \overline{u}_2(x; r) + 2\overline{u}_1(x; r), & \overline{u}_4(0; r) = 4 - r. \end{array} \right. \quad (3.8)$$

After computing the approximate solution of FIVP (3.8) using method given by (3.2), the approximate solution of FIVP (3.6) according to (3.7) is obtained. For some values of r , $0 \leq r \leq 1$, the approximate and exact solutions of FIVP (3.6) at $x = 1$ with $\tilde{N} = 400$ have been given in table 2 and their graphs in figure 2.

r	$\underline{y}_{\tilde{N}}(r)$	$\underline{y}(x_{\tilde{N}}; r)$	$\overline{y}_{\tilde{N}}(r)$	$\overline{y}(x_{\tilde{N}}; r)$
0	0.8082702555	0.8082856664	8.4201587020	8.4202453971
0.2	1.3233027959	1.3233224575	7.0146244316	7.0146979557
0.4	1.7421210580	1.7421446516	5.8480294088	5.8480919382
0.6	2.1967685502	2.1967964911	4.8747046894	4.8747579935
0.8	2.7119777948	2.7120106786	4.0541144953	4.0541599939
1	3.3083532458	3.3083918378	3.3083532458	3.3083918378

Table 2: Approximate and exact solutions.

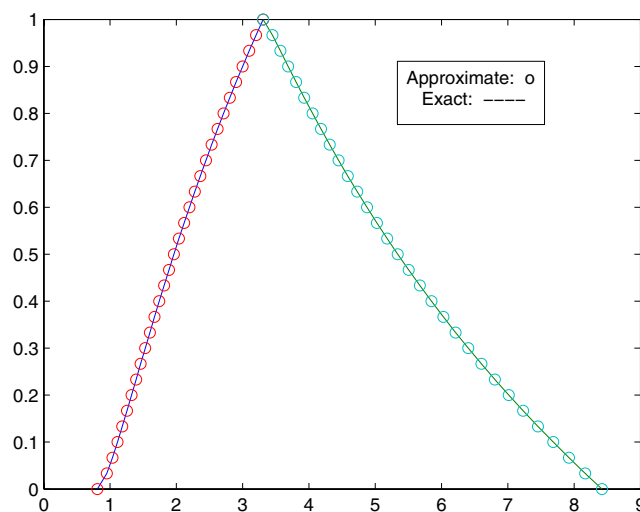


Figure 2: $h = \frac{1}{\tilde{N}}$

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