

On Intuitionistic Q -Fuzzy Semiprime Ideals in Semigroups

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Abstract

In this paper, we introduce the notion of intuitionistic Q -fuzzy semiprimality in a semigroup, which is an extension of Q -fuzzy semiprimality and investigate some properties of intuitionistic Q -fuzzification of the concept of several ideals.

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1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh [8], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K. T. Atanassov [1, 2], as a generalization of the notion of fuzzy set. In [3], N. Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. In this paper, we introduce the notion of intuitionistic Q -fuzzy semiprimality in a semigroup, which is an extension of Q -fuzzy semiprimality and investigate some properties of intuitionistic Q -fuzzification of the concept of several ideals.

2. Preliminaries

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a *left (right) ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of S which is both a left and a right ideal of S . A subsemigroup A of a semigroup

S is called a *bi-ideal* of S if $ASA \subseteq A$. A semigroup G is said to be *right* (resp. *left*) *zero* if $xy = y$ (resp. $xy = x$) for all $x, y \in G$. A semigroup S is said to be *regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$. A semigroup G is said to be *left* (resp. *right*) *simple* if G itself is the only left (resp. right) ideal of G .

By a *fuzzy set* μ in a non-empty set S we mean a function $\mu : S \rightarrow [0, 1]$, and the complement of μ , denoted by $\bar{\mu}$, is the fuzzy set in S given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in S$.

An *intuitionistic Q -fuzzy set* (IQFS for short) A is an object having the form

$$A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\}$$

where the functions $\mu_A : X \times Q \rightarrow [0, 1]$ and $\gamma_A : X \times Q \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x, q)$) and the degree of nonmembership (namely $\gamma_A(x, q)$) of each element $(x, q) \in X \times Q$ to the set A , respectively, and $0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1$ for all $x \in X$ and $q \in Q$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IQFS $A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\}$.

3. Intuitionistic Q -Fuzzy Semiprime Ideals

In what follows, let S denote a semigroup unless otherwise specified.

Definition 3.1. An IQFS $A = (\mu_A, \gamma_A)$ in S is called an *intuitionistic Q -fuzzy subsemigroup* of S if

$$(IF1) \quad \mu_A(xy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\},$$

$$(IF2) \quad \gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\},$$

for all $x, y \in S$ and $q \in Q$.

Definition 3.2. An IQFS $A = (\mu_A, \gamma_A)$ in S is called an *intuitionistic Q -fuzzy left ideal* of S if

$$(IF3) \quad \mu_A(xy, q) \geq \mu_A(y, q),$$

$$(IF4) \quad \gamma_A(xy, q) \leq \gamma_A(y, q), \text{ for all } x, y \in S \text{ and } q \in Q.$$

Definition 3.3. An intuitionistic Q -fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an *intuitionistic Q -fuzzy bi-ideal* of S if

$$(IF5) \quad \mu_A(xwy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\},$$

$$(IF6) \quad \gamma_A(xwy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\}$$

for all $w, x, y \in S$ and $q \in Q$.

Example 3.4. Let $S := \{a, b, c, d, e\}$ be a semigroup with the following Cayley table:

\cdot	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Define an IFS $A = (\mu_A, \gamma_A)$ in S by $\mu_A(a, q) = 0.6, \mu_A(b, q) = 0.5, \mu_A(c, q) = 0.4, \mu_A(d, q) = \mu_A(e, q) = 0.3, \gamma_A(a, q) = \gamma_A(b, q) = 0.3, \gamma_A(c, q) = 0.4$ and $\gamma_A(d, q) = 0.5, \gamma_A(e, q) = 0.6$ for all $q \in Q$. By routine calculation, we can check that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q -fuzzy bi-ideal of S .

Lemma 3.5. If X is a bi-ideal of S , then the IQFS $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ is an intuitionistic Q -fuzzy bi-ideal of S .

Proof. Let $w, x, y \in S$ and $q \in Q$. If $x, y \in X$, then $xwy \in XSX \subseteq X$, and so

$$\chi_{X \times Q}(xwy, q) = 1 = \min\{\chi_{X \times Q}(x, q), \chi_{X \times Q}(y, q)\}$$

and

$$\bar{\chi}_{X \times Q}(xwy, q) = 1 - \chi_{X \times Q}(xwy, q) = 0 = \max\{\bar{\chi}_{X \times Q}(x, q), \bar{\chi}_{X \times Q}(y, q)\}.$$

If $x \notin U$ or $y \notin U$, then $\chi_{X \times Q}(x, q) = 0$ or $\chi_{X \times Q}(y, q) = 0$. Hence

$$\chi_{X \times Q}(xwy, q) \geq 0 = \min\{\chi_{X \times Q}(x, q), \chi_{X \times Q}(y, q)\}$$

and

$$\bar{\chi}_{X \times Q}(xwy, q) \leq 1 = \max\{\bar{\chi}_{X \times Q}(x, q), \bar{\chi}_{X \times Q}(y, q)\}.$$

Therefore $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ is an intuitionistic Q -fuzzy bi-ideal of S .

Definition 3.6. A semigroup S is said to be *intuitionistic Q -fuzzy left simple* if every intuitionistic Q -fuzzy left ideal of S is constant.

Theorem 3.7. If S is left simple, then S is intuitionistic Q -fuzzy left simple.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q -fuzzy left ideal of S and let $a, b \in S$. Since S is left simple, it follows from [3. p. 6] that there exist elements x and y in G such that $b = xa$ and $a = yb$. Using (I1) and (I2), we have

$$\mu_A(a, q) = \mu_A(yb, q) \geq \mu_A(b, q) = \mu_A(xa, q) = \mu_A(a, q)$$

and

$$\gamma_A(a, q) = \gamma_A(yb, q) \leq \gamma_A(b, q) = \gamma_A(xa, q) = \gamma_A(a, q)$$

for all $q \in Q$, and so $\mu_A(a, q) = \mu_A(b, q)$ and $\gamma_A(a, q) = \gamma_A(b, q)$. Thus $A(a, q) = A(b, q)$, which means A is a constant function because a and b are any elements of S . Therefore, S is intuitionistic Q -fuzzy left simple. ■

Theorem 3.8. If S is simple, then every intuitionistic Q -fuzzy interior ideal of S is constant.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q -fuzzy interior ideal of S , and a and b any elements of S . Since S is simple, it follows from [7. Lemma I.3.9] that there exist elements x and y in S such that $a = xby$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic Q -fuzzy interior ideal of S , we have $\mu_A(a, q) = \mu_A(xby, q) \geq \mu_A(b, q)$ and $\gamma_A(a, q) = \gamma_A(xby, q) \leq \gamma_A(b, q)$ for all $q \in Q$. It can be seen in a similar way that $\mu_A(b, q) \geq \mu_A(a, q)$ and $\gamma_A(b, q) \leq \gamma_A(a, q)$. Since a and b are arbitrary elements, this means that $A = (\mu_A, \gamma_A)$ is a constant function. ■

Theorem 3.9. If S is left simple, then every intuitionistic Q -fuzzy bi-ideal of S is an intuitionistic Q -fuzzy right ideal of S .

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q -fuzzy bi-ideal of S , and let $a, b \in S$ and $q \in Q$. Since G is left simple, there exists $x \in S$ such that $b = xa$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic Q -fuzzy bi-ideal of S , it follows that

$$\mu_A(ab, q) = \mu_A(axa, q) \geq \min\{\mu_A(a, q), \mu_A(a, q)\} = \mu_A(a, q)$$

and

$$\gamma_A(ab, q) = \gamma_A(axa, q) \leq \max\{\gamma_A(a, q), \gamma_A(a, q)\} = \gamma_A(a, q)$$

so that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q -fuzzy right ideal of S . ■

A subset A of a semigroup S is called *semiprime* if $a^2 \in A$ imply $a \in A$ for all $a \in S$.

Definition 3.10. An IFS $A = (\mu_A, \gamma_A)$ is called *intuitionistic Q -fuzzy semiprime* if

$$(IF7) \quad \mu_A(x, q) \geq \mu_A(x^2, q),$$

$$(IF8) \quad \gamma_A(x, q) \leq \gamma_A(x^2, q),$$

for all $x \in S$ and $q \in Q$.

Example 3.11. Let $S = \{0, e, f, a, b\}$ be a set with the following Cayley table:

\cdot	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	a	0	0	e
b	0	0	b	f	0

Then S is a semigroup. Define an IQFS $A = (\mu_A, \gamma_A)$ in S by $\mu_A(e, q) = \mu_A(f, q) = 1$, $\mu_A(a, q) = \mu_A(b, q) = \mu_A(0, q) = 0$, $\gamma_A(e, q) = \gamma_A(f, q) = 0$, and $\gamma_A(a, q) =$

$\gamma_A(b, q) = \gamma_A(0, q) = 1$. By routine calculations we know that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q -fuzzy semiprime of S .

Theorem 3.12. If X is semiprime, then, an IQFS $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ is intuitionistic Q -fuzzy semiprime.

Proof. Let $a \in S$ and $q \in Q$. If $a^2 \in X$, then since X is semiprime, we have $a \in X$. Thus

$$\chi_{X \times Q}(a, q) = 1 \geq \chi_{X \times Q}(a^2, q)$$

and

$$\bar{\chi}_{X \times Q}(a, q) = 1 - \chi_{X \times Q}(a, q) \leq 1 - \chi_{X \times Q}(a^2, q) = \bar{\chi}_{X \times Q}(a^2, q).$$

If $a^2 \notin X$, then we have $\chi_{X \times Q}(a^2, q) = 0$. Therefore,

$$\chi_{X \times Q}(a, q) \geq 0 = \chi_{X \times Q}(a^2, q)$$

and

$$\bar{\chi}_{X \times Q}(a^2, q) = 1 - \chi_{X \times Q}(a^2, q) \geq 1 - \chi_{X \times Q}(a, q) = \bar{\chi}_{X \times Q}(a, q).$$

This proves the theorem. ■

Theorem 3.13. Let X be a nonempty subset of S . If an IQFS $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ satisfy (IF7) or (IF8), then X is semiprime.

Proof. Suppose that $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ satisfy (IF7). Let $a^2 \in X$. Then,

$$\chi_{X \times Q}(a, q) \geq \chi_{X \times Q}(a^2, q) = 1.$$

So, $a \in X$. Hence X is semiprime. Now suppose that $\tilde{A} = (\chi_{X \times Q}, \bar{\chi}_{X \times Q})$ satisfy (IF8). Let $a^2 \in X$. Then

$$\bar{\chi}_{X \times Q}(a, q) \leq \bar{\chi}_{X \times Q}(a^2, q) = 1 - \chi_{X \times Q}(a^2, q) = 1 - 1 = 0,$$

i.e, $\chi_{X \times Q}(a, q) = 1$. This show that $a \in X$. This proves the theorem. ■

Theorem 3.14. For any intuitionistic Q -fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S , if $A = (\mu_A, \gamma_A)$ is intuitionistic Q -fuzzy semiprime, then $A(a, q) = A(a^2, q)$ holds for all $q \in Q$.

Proof. Let $a \in S$ and $q \in Q$. Then, since μ_A is a fuzzy subsemigroup of S , we have

$$\mu_A(a, q) \geq \mu_A(a^2, q) = \min\{\mu_A(a, q), \mu_A(a, q)\} = \mu_A(a, q),$$

and so we have $\mu_A(a, q) = \mu_A(a^2, q)$. Also, we have

$$\gamma_A(a, q) \leq \gamma_A(a^2, q) = \max\{\gamma_A(a, q), \gamma_A(a, q)\} = \gamma_A(a, q).$$

Thus $\gamma_A(a, q) = \gamma_A(a^2, q)$. This proves the theorem. ■

A semigroup S is called *left* (resp. *right*) regular if, for each element a of S , there exists an element x in S such that $a = xa^2$ (resp. $a = a^2x$).

Theorem 3.15. Let S be left regular. Then, every intuitionistic Q -fuzzy left ideal of S is intuitionistic Q -fuzzy semiprime.

Proof. Let IQFS $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S and let $a \in S$. Then, there exists an element x in S such that $a = xa^2$ since S is left regular. So, we have $\mu_A(a, q) = \mu_A(xa^2, q) \geq \mu_A(a^2, q)$, and $\gamma_A(a, q) = \gamma_A(xa^2, q) \leq \gamma_A(a^2, q)$ for all $q \in Q$. This proves the theorem. ■

A semigroup S is called *intra-regular* if, for each element a of S , there exist elements x and y in S such that $a = xa^2y$.

Definition 3.16. An intuitionistic Q -fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of S is called an *intuitionistic Q -fuzzy interior ideal* of S if

$$(IF9) \quad \mu_A(xay, q) \geq \mu_A(a, q),$$

$$(IF10) \quad \gamma_A(xay, q) \leq \gamma_A(a, q),$$

for all $x, y, a \in S$ and $q \in Q$.

Theorem 3.17. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q -fuzzy ideal of S . If S is intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy semiprime.

Proof. Let a be any element of S . Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(a, q) = \mu_A(xa^2y, q) \geq \mu_A(a^2y, q) \geq \mu_A(a^2, q),$$

and

$$\gamma_A(a, q) = \gamma_A(xa^2y, q) \leq \gamma_A(a^2y, q) \leq \gamma_A(a^2, q)$$

for all $q \in Q$. This proves the theorem. ■

Theorem 3.18. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q -fuzzy interior ideal of S . If S is an intra-regular, then $A = (\mu_A, \gamma_A)$ is intuitionistic Q -fuzzy semiprime.

Proof. Let $a \in S$ and $q \in Q$. Then since S is intra-regular, there exist x and y in S such that $a = xa^2y$. So, we have

$$\mu_A(a, q) = \mu_A(xa^2y, q) \geq \mu_A(a^2, q),$$

and

$$\gamma_A(a, q) = \gamma_A(xa^2y, q) \leq \gamma_A(a^2, q).$$

This proves the theorem. ■

A semigroup S is called *archimedean* if, for any elements a, b , there exists a positive integer n such that $a^n \in SbS$.

Theorem 3.19. Let S be an archimedean semigroup. Then, every intuitionistic Q -fuzzy semiprime fuzzy ideal of S is a constant function.

Proof. Let $A = (\mu_A, \gamma_A)$ be any intuitionistic Q -fuzzy semiprime fuzzy ideal of S and $a, b \in S$. Then since S is archimedean, there exist x and y in S such that $a^n = xby$ for some integer n . Then, we have for all $q \in Q$,

$$\mu_A(a, q) = \mu_A(a^n, q) = \mu_A(xby, q) \geq \mu_A(b, q),$$

and

$$\mu_A(b, q) = \mu_A(b^n, q) = \mu_A(xay, q) \geq \mu_A(a, q).$$

Thus, we obtain $\mu_A(a, q) = \mu_A(b, q)$. Also, we have

$$\gamma_A(a, q) = \gamma_A(a^n, q) = \gamma_A(xby, q) \leq \gamma_A(b, q),$$

and

$$\gamma_A(b, q) = \gamma_A(b^n, q) = \gamma_A(xay, q) \leq \gamma_A(a, q).$$

Therefore, we have $A(a, q) = A(b, q)$ for all $a, b \in S$ and $q \in Q$. This proves the theorem. ■

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