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Intuitionistic Fuzzy Topological *BCC*-Algebras

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Abstract

We further study some important properties of intuitionistic fuzzy topological BCC-algebras introduced by Dudek, Jun and Hong in 2000. A characterization theorem of the intuitionistic fuzzy Hausdorff spaces is given. Some properties of intuitionistic fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras) are also investigated.

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1. Introduction

Imai and Iséki [17] defined a kind of type (2,0) algebras called BCK-algebras which generalizes the notion of algebra of sets with set subtraction as its only fundamental non-nullary operation and on the other hand, such BCK-algebras also generalizes the notion of implication algebras (see Iséki and Tanaka [19]). It has been proved that the class of all BCK-algebras forms a quasivariety,however, Wroński [24]) shown that the class of BCK-algebras does not always form a variety. In this connection, Komori [21]

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introduced the notion of BCC-algebras, and Dudek [9] redefined the BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. Later on, Dudek and Zhang [13] introduced a new notion of ideals in BCC-algebras and described the connections between such ideals and congruences. The fuzzification of BCC-ideals in BCC-algebras was then considered by Dudek and Jun [11]. They shown that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and also pointed out that the converse is not true by giving a counter example. It was also proved by them that in a BCC-algebra, every fuzzy BCK-ideal is a fuzzy BCC-subalgebra and in a BCK-algebra, the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. Thus, the studying of BCK-ideals of a BCK-algebra is a special case of studying the BCC-ideals in a BCC-algebra.

After Zadeh [25], various notions of higher-order fuzzy sets have been proposed. Among them, the intuitionistic fuzzy sets introduced by Atanassov [5,6] have drawn the attention of many researchers in the last decades. This is mainly due to the fact that intuitionistic fuzzy sets are consistent with human behavior, by reflecting and modeling the hesitancy present in real-life situations. In fact, the fuzzy sets only describe the degree of membership of an element in a given set, while the intuitionistic fuzzy sets describe both the degree of membership and the degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. By using the notion of intuitionistic fuzzy set, Dudek, Jun and Hong [15] have recently introduced the fuzzy topological space endowed with a BCC-algebra and called it the intuitionistic fuzzy topological *BCC*-algebras. In this paper, we investigate some topological properties of such algebras such as the C_5-connectedness, connectedness, strongly connectedness and Hausdorff spaces. We also discuss the properties of the homomorphic images and inverse images of the intuitionistic fuzzy topological *BCC*-ideals (BCK-ideals) of *BCC*-algebras (BCK-algebras).

2. Preliminaries

In this section, we first review some definitions and properties which will be used in the sequel.

A nonempty set G with a constant 0 and a binary operation $*$ is called a *BCC-algebra* if it satisfies the following conditions:

(a) $((x * y) * (z * y)) * (x * z) = 0$,

(b) $x * x = 0$,

(c) $0 * x = 0$,

(d) $x * 0 = x$,

(e) $x * y = 0, y * x = 0$ implies that $x = y$

for all $x, y, z \in G$. In a BCC-algebra, the following equality holds

$$(x * y) * x = 0.$$

Obviously, any BCK-algebra is a BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebras [10]. We note that a BCC-algebra is a BCK-algebra if and only if it satisfies the equality

$$(x * y) * z = (x * z) * y.$$

A nonempty subset S of a BCC-algebra G is called a *subalgebra* of G if it is closed under the BCC-operation. Such subalgebra contains the constant 0 and it is clearly a BCC-algebra, but some subalgebras may be also BCK-algebras. Moreover, there exist BCC-algebras in which all subalgebras are BCK-algebras [9]. A mapping $f : G_1 \to G_2$ of *BCC*-algebras is called *a homomorphism* if $f(x * y) = f(x) * f(y)$ holds, for all x, $y \in G_1$.

For a nonempty given set G,let I be the closed unit interval [0, 1]. Then,an intuitionistic fuzzy set (IFS) is an object of the form $A = \{< x, \mu_A(x), \lambda_A(x) > |x \in G\}$, where the mappings $\mu_A : G \to I$ and $\lambda_A : G \to I$ denote the *degree of membership* (namely, $\mu_A(x)$) and the *degree of non-membership* (namely, $\lambda_A(x)$) of each element $x \in G$ to the object A respectively satisfying $0 \leq \mu_A(x)+\lambda_A(x) \leq 1$, for all $x \in G$. The complement of the IFSs A is $\overline{A} = \{< x, \lambda_A(x), \mu_A(x) > |x \in G\}$. Obviously, every fuzzy A on a nonempty G is an IFS of the form $A = \{(x, \mu_A(x), 1 - \lambda_A(x))|x \in G\}$.

For the sake of simplicity, we just write $A =< \mu_A, \lambda_A >$ instead of $A = \{< x, \mu_A(x), \lambda_A(x) > \mid x \in G\}$. The IFSs $0_\sim$ and $1_\sim$ in G are defined by $0_\sim = \{< x, 0, 1 >: x \in G\}$ and $1_\sim = \{< x, 1, 0 >: x \in G\}$, respectively. If f is a mapping which maps a set G_1 into another set G_2, then the following statements hold:

(a) If $B = \{< y, \mu_B(y), \lambda_B(y) >: y \in G_2\}$ is an IFS in G_2, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is still an IFS in G_1. We now write

$$f^{-1}(B) = \{< x, f^{-1}(\mu_B)(x), f^{-1}(\lambda_B)(x) >: x \in G_1\}.$$

(b) If $A = \{< x, \mu_A(x), \lambda_A(x) >: x \in G_1\}$ is an IFS in G_1, then the *image* of A under f, denoted by $f(A)$, is also an IFS in G_2 which is defined by

$$f(A) = \{< y, f_{\sup}(\mu_A)(y), f_{\inf}(\lambda_A)(y) >: y \in G_2\},$$

where

$$f_{\sup}(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset\,, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_{\inf}(\lambda_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \lambda_A(x), & \text{if } f^{-1}(y) \neq \emptyset\,, \\ 1, & \text{otherwise}\,, \end{cases} \qquad \text{for each } y \in G_2.$$

Proposition 2.1. Let A, $A_i(i \in I)$ be IFSs in G_1 and B an IFS in G_2. If $f : G_1 \to G_2$ is a function then the following properties hold for the function f:

(a) If f is surjective, then $f(f^{-1}(B)) = B$.

(b) $f^{-1}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} f^{-1}(A_i)$.

(c) $f^{-1}(1_\sim)= 1_\sim$.

(d) $f^{-1}(0_\sim)= 0_\sim$.

(e) $f(1_\sim)= 1_\sim$ if f is surjective.

(f) $f(0_\sim)= 0_\sim$.

Definition 2.2. [8] An *intuitionistic fuzzy topology* (in short, IFT) on a nonempty set G is a family τ of IFSs in G which satisfies the following conditions:

(i) $0_\sim, 1_\sim \in \tau$.

(ii) If $G_1, G_2 \in \tau$ then $G_1 \cap G_2 \in \tau$.

(iii) If $G_j \in \tau$ for all $j \in J$, then $\cup_{i \in I} G_i \in \tau$.

The pair (G, τ) is called an *intuitionistic fuzzy topological space* (briefly, IFTS) and any IFS in τ is called an *intuitionistic fuzzy open sets* (briefly, IFOS) in G. The topology τ on a IFTS is said to be *an indiscrete intuitionistic fuzzy topology* if its only elements are $(0_\sim)$ and $(1_\sim)$. On the other hand,the IFT τ on a space G is said to be a *discrete intuitionistic fuzzy topology* if the topology IFT τ contains all intuitionistic fuzzy subsets of G.

If A is an IFS in an IFTS (G, τ), then the *induced intuitionistic fuzzy topology* (IFT) on A is the family of IFSs in A which are the intersection with A of IFOSs in G. The induced intuitionistic fuzzy topology is denoted by τ_A, and the pair (A, τ_A) is called an *intuitionistic fuzzy subspace* of (G, τ). Let (G_1, τ_1) and (G_2, τ_2) be two IFTSs and $f : (G_1, \tau_1) \to (G_2, \tau_2)$ a function. Then f is said to be an *intuitionistic fuzzy continuous* function if and only if the preimage of each IFS in τ_2 is an IFS in τ_1. Let (G_1, τ_1) and (G_2, τ_2) be two IFTSs and let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ be a function. Then f is said to be an *intuitionistic fuzzy open* if and only if the image of each IFS in τ_1 is an IFS in τ_2.

3. Intuitionistic Fuzzy Topological Subalgebras

Definition 3.1. An intuitionistic fuzzy set $A =< \mu_A, \lambda_A >$ in G is called *an intuitionistic fuzzy subalgebra* of G if it satisfies the following conditions:

(IFS1) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$, (IFS2) $\lambda_A(x * y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$

for all $x, y \in G$.

Example 3.2. Let $G = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Let $A =< \mu_A, \lambda_A >$ be an intuitionistic fuzzy set in G defined by $\mu_A(d) = 0.06$, $\mu_A(x) = 0.7$, $\lambda_A(x) = 0.5$ and $\lambda_A(x)$= 0.06 for all $x \neq d$. Then $A =< \mu_A, \lambda_A >$ is an intuitionistic fuzzy subalgebra of G.

Definition 3.3. Let τ_1 and τ_2 be the intuitionistic fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $f : (G_1, \tau_1) \to (G_2, \tau_2)$ is called an *intuitionistic fuzzy continuous function* which maps (G_1, τ_1) into (G_2, τ_2) if f satisfies the following conditions:

(i) For every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$,

(ii) For every intuitionistic fuzzy subalgebras A (of G_2) in τ_2, $f^{-1}(A)$ is an intuitionistic fuzzy subalgebra (of G_1) in τ_1.

Proposition 3.4. If τ_1 is an intuitionistic fuzzy topology on a BCC-algebra G_1 and τ_2 is an indiscrete intuitionistic fuzzy topology on a BCC-algebras G_2, then every function $f : (G_1, \tau_1) \to (G_2, \tau_2)$ is an intuitionistic fuzzy continuous function.

Proof. Since τ_2 is an indiscrete intuitionistic fuzzy topology, τ_2= $\{0_\sim, 1_\sim\}$. Let $f : G_1 \to G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is an intuitionistic fuzzy subalgebra of BCC-algebra Y. We now show that f is an intuitionistic fuzzy continuous function, we only need to prove that for every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$. For this purpose, we let $0_\sim \in \tau_2$. Then for any $x \in G_1$, we have

$$\begin{aligned} f^{-1}(0_\sim)(x) &= 0_\sim(f(x)) \\ &= 0 \\ &= 0_\sim(x). \end{aligned}$$

This shows that $(f^{-1}(0_\sim)) = 0_\sim \in \tau_1$.

On the other hand, if $1_\sim \in \tau_2$ and $x \in G_1$,then

$$\begin{aligned} (f^{-1}(1_\sim))(x) &= 1_\sim(f(x)) \\ &= 1 \\ &= 1_\sim(x). \end{aligned}$$

Thus $(f^{-1}(1_\sim)) = 1_\sim \in \tau_1$. This shows that f is indeed an intuitionistic fuzzy continuous function of G_1 to G_2. ■

Theorem 3.5. Let τ_1 and τ_2 be any two discrete intuitionistic fuzzy topologies defined on the *BCC*-algebras G_1 and G_2 respectively. Then every homomorphism $f : (G_1, \tau_1) \to (G_2, \tau_2)$ is an intuitionistic fuzzy continuous function.

Proof. Since τ_1 and τ_2 are discrete IFTs on the *BCC*-algebras G_1 and G_2 respectively, we have $f^{-1}(A) \in \tau_1$, for every $A \in \tau_2$. We note here that f is not the usual inverse homomorphism from G_2 to G_1).

Let $A =< \mu_A, \lambda_A >$ be an intuitionistic fuzzy subalgebra (of G_2) in τ_2. Then for x, $y \in G_1$, we have

$$\begin{aligned}(f^{-1}(\mu_A))(x * y) &= \mu_A(f(x * y))\\ &= \mu_A(f(x) * f(y))\\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\}\\ &= \min\{(f^{-1}(\mu_A))(x), (f^{-1}(\mu_A)(y)\}\end{aligned}$$

and

$$\begin{aligned}(f^{-1}(\lambda_A))(x * y) &= \lambda_A(f(x * y))\\ &= \lambda_A(f(x) * f(y))\\ &\leq \max\{\lambda_A(f(x)), \lambda_A(f(y))\}\\ &= \max\{(f^{-1}(\lambda_A))(x), (f^{-1}(\lambda_A)(y)\}.\end{aligned}$$

Hence $f^{-1}(A)$ is an intuitionistic fuzzy subalgebra (of G_1) in τ_1 and consequently, f is an intuitionistic fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2). ■

Definition 3.6. Let (G_1, τ_1) and (G_2, τ_2) be IFTSs. A function $f : (G_1, \tau_1) \to (G_2, \tau_2)$ is said to be an *intuitionistic fuzzy homomorphism* if it satisfies the following conditions:

- f is and injective and surjective function;
- f is fuzzy continuous function which maps G_1 to G_2;
- f^{-1} is a fuzzy continuous function which maps G_2 to G_1.

Definition 3.7. Let τ be an IFT on *BCC*-algebra G. An IFTS (G, τ) is an *intuitionistic fuzzy Hausdorff space* if and only if for any distinct intuitionistic fuzzy points $x_1, x_2 \in G$, there exist IFOs $F_1 =< \mu_{F_1}, \lambda_{F_1} >$ and $F_2 =< \mu_{F_2}, \lambda_{F_2} >$ such that

$$\begin{aligned}\mu_{F_1}(x_1) &= 1, &\qquad \lambda_{F_1}(x_1) &= 0\\ \mu_{F_2}(x_2) &= 1, &\qquad \lambda_{F_2}(x_2) &= 0\end{aligned}$$

and $F_1 \cap F_2 = 0_\sim$.

Theorem 3.8. Let τ_1 and τ_2 be IFTs on BCC-algebras G_1 and G_2 respectively and let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ be an intuitionistic fuzzy homeomorphism. Then G_1 is an intuitionistic fuzzy Hausdorff space if and only if G_2 is an intuitionistic fuzzy Hausdorff space.

Proof. Suppose that G_1 is an intuitionistic fuzzy Hausdorff space. Let x_1 and x_2 be the intuitionistic fuzzy points in τ_2 with $x \neq y$ $(x, y \in G_1)$.Then $f^{-1}(x) \neq f^{-1}(y)$ because f is a injective function. For $z \in G_1$, we consider

$$\begin{aligned}(f^{-1}(x_1))(z) = x_1(f(z)) &= \begin{cases} t \in (0,1], & \text{if f(z)=x;} \\ 0, & \text{if f(z)}\neq \text{ x,} \end{cases} \\ &= \begin{cases} t \in (0,1], & \text{if } z = f^{-1}(x); \\ 0, & \text{if } z \neq f^{-1}(x), \end{cases} \\ &= (f^{-1}(x))_1(z).\end{aligned} \tag{3.1}$$

That is, $(f^{-1}(x_1))(z) = (f^{-1}(x))_1(z)$, for all $z \in G_1$. Hence $f^{-1}(x_1) = (f^{-1}(x))_1$. Similarly we can also prove that $f^{-1}(x_2) = (f^{-1}(x))_2$. Now, by the definition of an intuitionistic fuzzy Hausdorff space, there exist IFOs F_1 and F_2 of $f^{-1}(x_1)$ and $f^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0_\sim$. Since f is an intuitionistic fuzzy continuous map from G_1 to G_2 and f^{-1} is an intuitionistic fuzzy continuous map from G_2 to G_1, there exist IFOs $f(F_1)$ and $f(F_2)$ of x_1 and x_2 respectively such that $f(F_1) \cap f(F_2) = f(F_1 \cap F_2) = f(0_\sim) = 0_\sim$. This shows that G_2 is an intuitionistic fuzzy Hausdorff space.

Conversely, if (G_2, τ_2) is an intuitionistic fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both f and f^{-1} are intuitionistic fuzzy continuous functions, we can easily prove that (G_1, τ_1) is an intuitionistic fuzzy Hausdorff space. The proof is hence completed. ■

Definition 3.9. Let τ be an IFT on a BCC-algebra G. Then (G, τ) is called an *intuitionistic fuzzy C_5- disconnected* space if there exists an intuitionistic fuzzy open and closed set F such that $F \neq 1_\sim$ and $F \neq 0_\sim$.

Theorem 3.10. Let τ_1 and τ_2 be the IFTSs on BCC-algebras G_1 and G_2 respectively and let $f : G_1 \to G_2$ be an intuitionistic fuzzy continuous surjective function. If (G_1, τ_1) is an intuitionistic fuzzy C_5-connected space then (G_2, τ_2) is an intuitionistic fuzzy C_5-connected space.

Proof. Assume that (G_2, τ_2) is an intuitionistic fuzzy C_5-disconnected. Then there exist an intuitionistic fuzzy open and a closed set F such that $F \neq 1_\sim$ and $F \neq 0_\sim$. Since f is an intuitionistic fuzzy continuous function, $f^{-1}(F)$ is both IFOs (that is, intuitionistic fuzzy open set) and IFCs (that is, intuitionistic fuzzy closed set). In this case, $f^{-1}(F)$= $1_\sim$ or $f^{-1}(F)$= $0_\sim$. Since $F = f(f^{-1}(F)) = f(1_\sim) = 1_\sim$ and $F = f(f^{-1}(F)) = f(0_\sim) = 0_\sim$, we see that these results contradict to our assumption. Hence the space (G_2, τ_2) must be intuitionistic fuzzy C_5-connected.

Definition 3.11. Let τ be an intuitionistic fuzzy topology on a BCC-algebra G. An IFTS (G, τ) is called an *intuitionistic fuzzy disconnected* space if there exist intuitionistic fuzzy open sets $A \neq 0_\sim$ and $B \neq 0_\sim$ such that $A \cup B = 1_\sim$ and $A \cap B = 0_\sim$. Naturally,we call the set (G, τ) an *intuitionistic fuzzy connected* if (G, τ) is not intuitionistic fuzzy disconnected.

Theorem 3.12. Let τ_1 and τ_2 be IFTs on BCC-algebras G_1 and G_2 respectively and let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ be an intuitionistic fuzzy continuous and surjective function. If G_1 is an intuitionistic fuzzy connected space, then so is G_2.

Proof. Suppose that G_2 is an intuitionistic fuzzy disconnected, then there exists intuitionistic fuzzy open sets $C \neq 0_\sim$, $D \neq 0_\sim$ in G_2 such that $C \cup D = 1_\sim$ and $C \cap D = 0_\sim$. Since f is an intuitionistic fuzzy continuous function, $A = f^{-1}(C)$ and $B = f^{-1}(D)$ are intuitionistic fuzzy open sets in G_1. Clearly,$C \neq 0_\sim$ implies that $A = f^{-1}(C) \neq 0_\sim$, and $D \neq 0_\sim$ implies that $B = f^{-1}(D) \neq 0_\sim$.

$$\begin{aligned} \text{Now, we have } C \cup D &= 1_\sim \\ \Rightarrow f^{-1}(C \cup D) &= f^{-1}(1_\sim) \\ \Rightarrow f^{-1}(C) \cup f^{-1}(D) &= 1_\sim \\ \Rightarrow A \cup B &= 1_\sim, \end{aligned}$$

$$\begin{aligned} \text{and } C \cap D &= 0_\sim \\ \Rightarrow f^{-1}(C \cap D) &= f^{-1}(0_\sim) \\ \Rightarrow f^{-1}(C) \cap f^{-1}(D) &= 0_\sim \\ \Rightarrow A \cap B &= 0_\sim. \end{aligned}$$

This clearly contradicts our hypothesis. Hence G_2 is an intuitionistic fuzzy connected space. ■

Definition 3.13. An IFTS (G, τ) is said to be an *intuitionistic fuzzy strongly connected*, if there exists no nonzero intuitionistic fuzzy closed sets A and B in G such that $\mu_A + \mu_B \leq 1$ and $\lambda_A + \lambda_B \geq 1$.

The following fact follows immediately from our definition.

Proposition 3.14. *G is an intuitionistic fuzzy strongly connected if and only if there exist no intuitionistic fuzzy open sets A and B in G such that $A \neq 1_\sim \neq B$ and $\mu_A + \mu_B \geq 1$, $\lambda_A + \lambda_B \leq 1$.*

We now formulate the following theorem

Theorem 3.15. Let τ_1 and τ_2 be IFTs on BCC-algebras G_1 and G_2 respectively and let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ an intuitionistic fuzzy continuous and surjective mapping. If G_1 is an intuitionistic fuzzy strongly connected, then so is G_2.

Proof. Suppose that G_2 is not an intuitionistic fuzzy strongly connected. Then there exist intuitionistic fuzzy sets C and D in G_2 with $C \neq 0_\sim \neq D$ so that $\mu_C + \mu_D \leq 1$ and $\lambda_C + \lambda_D \geq 1$. Since f is an intuitionistic fuzzy continuous function, $f^{-1}(C)$ and $f^{-1}(D)$ are intuitionistic fuzzy closed sets in G_1. Now, we can deduce the following equalities:

$$\begin{aligned}\mu_{f^{-1}(C)} + \mu_{f^{-1}(D)} &= f^{-1}(\mu_C) + f^{-1}(\mu_D)\\ &= \mu_C \circ f + \mu_D \circ f\\ &\leq 1[since\ \mu_C + \mu_D \leq 1],\end{aligned}$$

$$\begin{aligned}\lambda_{f^{-1}(C)} + \lambda_{f^{-1}(D)} &= f^{-1}(\lambda_C) + f^{-1}(\lambda_D)\\ &= \lambda_C \circ f + \lambda_D \circ f\\ &\geq 1[\text{since } \lambda_C + \lambda_D \geq 1],\end{aligned}$$

$f^{-1}(C) \neq 0_\sim$ and $f^{-1}(D) \neq 0_\sim$. This contradicts our hypothesis. Hence G_2 is an intuitionistic fuzzy strongly connected space. ■

Definition 3.16. [16] Let τ be an IFT on a *BCC*-algebra G and A be an intuitionistic fuzzy *BCC*-algebra with IIFT τ_A. Then A is called an *intuitionistic fuzzy topological BCC-subalgebra* if the self mapping $r_a : (A, \tau_A) \to (A, \tau_A)$ defined by $r_a(x) = x * a$ for all $a \in G$, is a relatively intuitionistic fuzzy continuous function.

Dudek and Jun gave the following interesting properties of the *intuitionistic fuzzy topological BCC-algebras* in [15].

Theorem 3.17. Let $f : G_1 \to G_2$ be a homomorphism of $BCC-$algebras and let τ and τ^* be intuitionistic fuzzy topologies on G_1 and G_2 respectively such that $\tau = f^{-1}(\tau^*)$. If B is an intuitionistic fuzzy topological *BCC*-algebra in G_2 then $f^{-1}(B)$ is an intuitionistic fuzzy topological *BCC*-algebra in G_1.

Theorem 3.18. Let $f : G_1 \to G_2$ be an isomorphism of *BCC*-algebras. Let τ and τ^* be the respectively IFTs on the spaces G_1 and G_2 such that $f(\tau) = \tau^*$. If A is an intuitionistic fuzzy topological *BCC*-algebra in G_1, then $f(A)$ is also an intuitionistic fuzzy topological *BCC*-algebra in G_2.

4. Intuitionistic Fuzzy Topological BCC-ideals

Definition 4.1. An IFS $A = \{< \mu_A(x), \lambda_A(x) >\}$ in a BCK-algebra G is called an *intuitionistic fuzzy BCK-ideal* of G if the following conditions are satisfied:

(i) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$,

(ii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$,

(iii) $\lambda_A(x) \leq \max\{\lambda_A(x * y), \lambda_A(y)\}$

for all $x, y \in G$.

Definition 4.2. An intuitionistic fuzzy set $A =< \mu_A, \lambda_A >$ in G is called an *intuitionistic fuzzy BCC-ideal* of G if it satisfies the following inequalities:

(IF1) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$,

(IF2) $\mu_A(x * z) \geq \min\{\mu_A((x * y) * z), \mu_A(y)\}$,

(IF3) $\lambda_A(x * z) \leq \max\{\lambda_A((x * y) * z), \lambda_A(y)\}$

for all $x, y, z \in G$.

Putting $z = 0$ in (IF2-IF3). Then we can easily see that an intuitionistic fuzzy *BCC*-ideal is an intuitionistic fuzzy *BCK*-ideal. However, the converse does not hold, in general (see [11]).

Example 4.3. Let $G := \{0, a, b, c, d, e\}$ be a *BCC*-algebra with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Let $A =< \mu_A, \lambda_A >$ be an intuitionistic fuzzy set in G defined by $\mu_A(e) = 0.03$, $\mu_A(x)= 0.5$, $\lambda_A(e) = 0.3$ and $\lambda_A(x) = 0.05$ for all $x \neq e$. Then $A =< \mu_A, \lambda_A >$ is an intuitionistic fuzzy BCC-ideal of a BCC-algebra G.

Theorem 4.4. Let f be a homomorphism of a *BCC*-algebra G_1 into a *BCC*-algebra G_2 and B be an intuitionistic fuzzy *BCC*-ideal of G_2. Then $f^{-1}(B)$ is an intuitionistic fuzzy *BCC*-ideal of G_1.

Proof. It can be easily seen that $\mu_{f^{-1}(B)}(0) \geq \mu_{f^{-1}(B)}(x)$ and $\lambda_{f^{-1}(B)}(0) \leq \lambda_{f^{-1}(B)}(x)$, for all $x \in G_1$.
For any $x, y, z \in G$, we can deduce the following

$$\begin{aligned} \mu_{f^{-1}(B)}(x * z) &= \mu_B(f(x * z)) \\ &\geq \min\{\mu_B(f((x * y) * z)), \mu_B(f(y))\} \\ &= \min\{\mu_B((f(x) * f(y)) * f(z)), \mu_B(f(y))\} \\ &= \min\{\mu_{f^{-1}(B)}((x * y) * z), \mu_{f^{-1}(B)}(y)\} \end{aligned}$$

and

$$\begin{aligned}\lambda_{f^{-1}(B)}(x*z) &= \lambda_B(f(x*z))\\ &\le \max\{\lambda_B(f((x*y)*z)), \lambda_B(f(y))\}\\ &= \max\{\lambda_B((f(x)*f(y))*f(z)), \lambda_B(f(y))\}\\ &= \max\{\lambda_{f^{-1}(B)}((x*y)*z), \lambda_{f^{-1}(B)}(y)\}.\end{aligned}$$

Hence $f^{-1}(B)$ is an intuitionistic fuzzy BCC-ideal of G_1. ■

Putting $z = 0$ in the above proof, we can obtain the following corollary:

Corollary 4.5. Let f be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and let B an intuitionistic fuzzy BCK-ideal of G_2. Then $f^{-1}(B)$ is an intuitionistic fuzzy BCK-ideal of G_1.

Since an intuitionistic fuzzy BCC-ideal(BCK-ideal) is an intuitionistic fuzzy subalgebra (see [11]), as a consequence of the the above results and Theorem 3.17, we obtain the following corollary:

Corollary 4.6. Let $f : (G_1, \tau_1) \to (G_2, \tau_2)$ be a homomorphism of the $BCC-$algebras. Let τ_1 and τ_2 be the intuitionistic fuzzy topologies on G_2 and G_1 respectively such that $\tau_2 = f^{-1}(\tau_1)$. If B is an intuitionistic fuzzy topological BCC-ideal (BCK-ideal) of G_2 with membership function μ_B, then $f^{-1}(B)$ is an intuitionistic fuzzy topological BCC-ideal(BCK-ideal) of G_1 with membership function $\mu_{f^{-1}(B)}$.

Theorem 4.7. Let f be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is an intuitionistic fuzzy BCC-ideal of G_1. Then the homomorphic image $f(A)$ of A is still an intuitionistic fuzzy BCC-ideal of G_2.

Proof. Let $A =< \mu_A, \lambda_A >$ be an intuitionistic fuzzy topological BCC-ideal of G_1. Then, it is trivial that $\mu_{f(A)}(0) \ge \mu_{f(A)}(x)$ and $\lambda_{f(A)}(0) \le \lambda_{f(A)}(x)$, for all $x \in G_2$. Take $x, y, z \in G_2$, and let $x_0 \in f^{-1}(x)$, $y_0 \in f^{-1}(y)$, $z_0 \in f^{-1}(z)$ such that $\mu_A(x_0) = \sup\limits_{t\in f^{-1}(x)}(t)$, $\mu_A(y_0) = \sup\limits_{t\in f^{-1}(y)}(t)$ and $\mu_A(z_0) = \sup\limits_{t\in f^{-1}(z)}(t)$. Then, we can deduce the following:

$$\begin{aligned}\mu_{f(A)}(x*z) &= \sup_{t\in f^{-1}(x*z)} \mu_A(t)\\ &\ge \mu_A(x_0*z_0)\\ &\ge \min\{\mu_A((x_0*y_0)*z_0), \mu_A(y_0)\}\\ &= \min\{\sup_{t\in f^{-1}((x*y)*z)} \mu_A(t), \sup_{t\in f^{-1}(y)} \mu_A(t)\}\\ &= \min\{\mu_{f(A)}((x*y)*z), \mu_{f(A)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{f(A)}(x * z) &= \inf_{t \in f^{-1}(x*z)} \lambda_A(t) \\ &\leq \lambda_A(x_0 * z_0) \\ &\leq \max\{\lambda_A((x_0 * y_0) * z_0), \lambda_A(y_0)\} \\ &= \max\{\inf_{t \in f^{-1}((x*y)*z)} \lambda_A(t), \inf_{t \in f^{-1}(y)} \lambda_A(t)\} \\ &= \max\{\lambda_{f(A)}((x * y) * z), \lambda_{f(A)}(y)\}.\end{aligned}$$

Hence $f(A) =< f_{\sup}(\mu_A), f_{\inf}(\lambda_A) >$ is indeed an intuitionistic fuzzy BCC-ideal of G_2. ■

Putting $z = 0$ in the above proof, we obtain:

Corollary 4.8. Let f be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is an intuitionistic fuzzy BCK-ideal of G_1, then the homomorphic image $f(A)$ of A is an intuitionistic fuzzy BCK-ideal of G_2.

Summing up Theorem 3.18, Theorem 4.7 and Corollary 4.8, we conclude the following theorem.

Theorem 4.9. Let $f : G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively IFTs on the spaces G_1 and G_2 such that $f(\tau) = \tau^*$. If A is an intuitionistic fuzzy topological BCC-ideal(BCK-ideal) in G_1, then $f(A)$ is also an intuitionistic fuzzy topological BCC-ideal(BCK-ideal) in G_2.

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