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Nonresonance Conditions for a Nonlinear Hyperbolic Problem

Aomar Anane, Omar Chakrone and Abdellah Zerouali
Université Mohamed I, Faculté des Sciences,
Département de Mathématiques et Informatique, Oujda, Maroc
E-mail: anane@sciences.univ-oujda.ac.ma, chakrone@yahoo.fr,
abdellahzerouali@hotmail.com

Abstract

In this paper we study the existence of periodic weak solutions of semilinear wave equations in the case of nonresonance.

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1. Introduction

In this paper we will consider the existence of periodic solutions of wave equations of the form

$$\begin{cases} \Box u = g(x,t,u) + h(x,t) & \text{in } Q, \\ u(x,t+2\pi) = u(x,t) & \text{in } \Omega \times \mathbb{R}, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with boundary $\partial\Omega$, $Q = \Omega \times (0, 2\pi)$, $\Box = \dfrac{\partial^2}{\partial t^2} - \triangle$ is the D'Alembertian, h is a given function in $L^2(Q)$ and $g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and 2π-periodic in t. We are interested in the nonresonance for the problem (1.1), i.e., in the condition on the function g such that there exists a solution $u \in L^2(Q)$ for any given $h \in L^2(Q)$. Throughout this paper, we will suppose that $\sigma(\Box) = \{\lambda_k, k \in \mathbb{N}\}$ is closed in $\mathbb{R}$ and all points of this set are isolated (for example if $N = 1$ and $\Omega = (0, r\pi)$ with $r \in \mathbb{Q}$). We will assume that g satisfies three conditions:

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(C_1) $g(x,t,s)$ is nondecreasing in s;
(C_2) for all $R > 0$, there exists $\phi_R \in L^2(Q)$ such that a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, we have

$$\max_{|s|\leq R} |g(x,t,s)| \leq \phi_R(x,t);$$

(C_3) a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, we have

$$\lambda_k < l(x,t) := \liminf_{|s|\to+\infty} \frac{g(x,t,s)}{s} \leq \limsup_{|s|\to+\infty} \frac{g(x,t,s)}{s} := k(x,t) < \lambda_{k+1},$$

where λ_k and $\lambda_{k+1} \in \sigma(\Box)$ are two consecutive eigenvalues of the D'Alembertian and $\sigma(\Box)$ is the spectrum of the D'Alembertian. An example of g is $g(x,t,s) = \dfrac{cs^3}{s^2+1} + \phi(x,t)$, where $\phi \in L^2(Q)$ is an arbitrary function, $c \in (\lambda_k, \lambda_{k+1})$ is a real constant, λ_k and λ_{k+1} are two positive consecutive eigenvalues of the D'Alembertian.

The problem (1.1) has been studied with conditions of resonances by several authors, in particular: In the case $N = 1$, Benaoum in [3,4], Mustonen and Berkovits in [5–10], Brezis and Nirenberg in [12],.... In the general case, see Anane, Chakrone and Ghanim [2]. In the case of nonresonance, the problem (1.1) has been studied by Mustonen and Berkovits in [6] and [11], and by Brezis and Nirenberg in [12] but only for $N = 1$ and when $\dfrac{g(x,t,s)}{s}$ is located strictly between two consecutive eigenvalues of the D'Alembertian, more precisely when $\lambda_k < \dfrac{g(x,t,s)}{s} < \lambda_{k+1}$ for all t, s and a.e. x.

In our work, if $N \geq 1$, we show (see Corollary 2.3) while using a homotopy argument given by Mustonen and Berkovits in [6], and with analogous techniques developed by Anane and Chakrone in [1] for the Laplacian $(\triangle)$, that the problem (1.1) has at least one solution for all $h \in L^2(Q)$.

Note that the same result (see Theorem 2.13) is given for the problem

$$\begin{cases} \Box u = \mu u + \delta \nabla u - \gamma u_t + g(x,t,u) + h(x,t) & \text{in } Q, \\ u(x,t+2\pi) = u(x,t) & \text{in } \Omega \times \mathbb{R}, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{1.2}$$

where $(\mu, \delta, \gamma) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. This situation is basically characterized by the presence of the first partial derivatives to the level of the second members. This constitutes an extension of the case studied in Corollary 2.3.

2. Main Results

Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$. We introduce the following general hypothesis:
$(C_{\alpha,\beta})$ a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, we have

$$\alpha \leq\neq l(x,t) := \liminf_{|s|\to+\infty} \frac{g(x,t,s)}{s} \leq \limsup_{|s|\to+\infty} \frac{g(x,t,s)}{s} := k(x,t) \leq\neq \beta,$$

i.e., for all $\varepsilon > 0$, there exists $a_\varepsilon \in L^2(Q)$ such that a.e. $x \in \Omega$, for all $(t, s) \in \mathbb{R} \times \mathbb{R}$, we have

$$(l(x,t) - \varepsilon)s^2 - a_\varepsilon(x,t)|s| \leq sg(x,t,s) \leq (k(x,t) + \varepsilon)s^2 + a_\varepsilon(x,t)|s|.$$

The notation $\leq\neq$ means that one has a large inequality on Ω and strict on a set of measure different from zero.

Remark 2.1.

1. As $\sigma(\Box) = \{\lambda_k, k \in \mathbb{N}\}$ is closed in $\mathbb{R}$ and all points of this set are isolated, $\Box_0$ has a compact inverse, where $\Box_0$ is the restriction of the D'Alembertian operator on $\mathrm{Im}(\Box) = \Box(D(\Box))$ and $D(\Box)$ is the domain of the D'Alembertian operator.

2. Let
$$N : L^2(Q) \to L^2(Q) : N(u) = g(x,t,u)$$
be the Nemytskii operator generated by the function g. For $r \in [0, 1]$, consider the operator
$$T_r : D(\Box) \subset L^2(Q) \to L^2(Q) : T_r(u) = \Box u - r(N(u) + h) - (1 - r)\lambda u,$$
where $\alpha < \lambda < \beta$.

3. Under hypothesis (C_2) and $(C_{\alpha,\beta})$, it is easy to see that there exist $\theta > 0$ and $\eta \in L^2(Q)$ such that a.e. $x \in \Omega$, for all $(t, s) \in \mathbb{R} \times \mathbb{R}$, we have
$$|g(x,t,s)| \leq \theta|s| + \eta(x,t). \tag{2.1}$$

Consider the estimate:

There exists $R > 0$, such that

$$T_r(u) \neq 0 \text{ for all } r \in [0,1] \text{ and } u \in D(\Box), \text{ with } \|u\| = \left(\int_Q |u|^2\right)^{\frac{1}{2}} = R. \tag{2.2}$$

Theorem 2.2. Assume (C_1), (C_2) and $(C_{\alpha,\beta})$. If T_r does not verify the estimate (2.2), then there exist $m(x,t) \in L^\infty(Q)$, $v \in L^2(Q) \setminus \{0\}$ and $(u_n) \subset L^2(Q)$ such that v is a nontrivial solution of the problem

$$\begin{cases} \Box u = m(x,t)u & \text{in } Q, \\ u(x, t + 2\pi) = u(x,t) & \text{in } \Omega \times \mathbb{R}, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \|u_n\| \to +\infty, \dfrac{u_n}{\|u_n\|} \to v \text{ in } L^2(Q), \\ \alpha \leq\neq m(x,t) \leq\neq \beta \text{ a.e. in } Q. \end{cases}$$

Corollary 2.3. Assume (C_1), (C_2) and $(C_{\alpha,\beta})$. If there exist two positive consecutive eigenvalues of the D'Alembertian λ_k and λ_{k+1} such that $\lambda_k < \alpha < \beta < \lambda_{k+1}$, then the problem (1.1) has at least one solution for all $h \in L^2(Q)$.

Proof of Theorem 2.2. As the proof is relatively long, we organize it in several lemmas. Suppose that T_r does not verify the estimate (2.2). Then for all $n \in \mathbb{N}$, there exist $r_n \in [0, 1]$ and $u_n \in D(\Box)$ with $\|u_n\| = n$ such that

$$\Box u_n - r_n(N(u_n) + h) - (1 - r_n)\lambda u_n = 0. \tag{2.4}$$

Let

$$v_n = \frac{u_n}{\|u_n\|} \text{ and } g_n(x, t) = \frac{g(x, t, u_n)}{\|u_n\|} \text{ a.e. in } Q.$$

The sequence (v_n) is bounded in $L^2(Q)$. Then there exists a subsequence of v_n also denoted by v_n such that $v_n \to v$ weakly in $L^2(Q)$.

Lemma 2.4.

1) There exists a subsequence of g_n also denoted by g_n such that $g_n \to f$ weakly in $L^2(Q)$.
2) $v_n \to v$ strongly in $L^2(Q)$, in particular, $\|v\| = 1$, thus $v \neq 0$.

Proof.

1) Dividing (2.1) by $\|u_n\|$, we have

$$|g_n(x, t)| \le \theta|v_n| + \frac{\eta(x, t)}{n},$$

thus

$$\|g_n\| \le \theta\|v_n\| + \frac{\|\eta\|}{n} \le \theta + \frac{\|\eta\|}{n},$$

hence g_n is bounded in $L^2(Q)$. One deduces that there exists a subsequence of g_n also denoted by g_n such that $g_n \to f$ weakly in $L^2(Q)$.

2) Dividing by $\|u_n\|$ in (2.4) we have

$$\Box v_n = r_n g_n + (1 - r_n)\lambda v_n + r_n\frac{h}{n}.$$

This implies

$$v_n = (\Box_0^{-1})\left(r_n g_n + (1 - r_n)\lambda v_n + r_n\frac{h}{n}\right).$$

Since $g_n \to f$ weakly in $L^2(Q)$ and $v_n \to v$ weakly in $L^2(Q)$, we have

$$r_n g_n + (1 - r_n)\lambda v_n + r_n\frac{h}{n} \to rf + (1 - r)\lambda v \text{ weakly in } L^2(Q).$$

The operator $\Box_0^{-1}$ is compact. Thus

$$v_n = (\Box_0^{-1})\left(r_n g_n + (1-r_n)\lambda v_n + r_n\frac{h}{n}\right)$$
$$\to (\Box_0^{-1})(rf + (1-r)\lambda v) \text{ strongly in } L^2(Q).$$

Therefore, $v_n \to (\Box_0^{-1})(rf + (1-r)\lambda v) = v$ strongly in $L^2(Q)$. ■

Lemma 2.5. Suppose (2.1). Then $f(x,t) = 0$ a.e. in $A = \{(x,t) \in Q : v(x,t) = 0$ a.e. in $Q\}$.

Proof. Let ψ be the function defined by

$$\psi(x,t) = \text{sign}(f(x,t))\chi_A(x,t) \text{ a.e. in } Q,$$

where χ_A is the indicator function and sign : $\mathbb{R} \to \{-1, 1\}$: sign$(x) = 1$ if $x \geq 0$ and sign$(x) = -1$ if $x < 0$. Since $g_n \to f$ weakly in $L^2(Q)$, we have $\int_Q g_n\psi \to \int_Q f\psi = \int_A |f(x,t)|$. On the other hand, as $v_n \to v$, and using (2.1), we have

$$\left|\int_Q g_n\psi\right| \leq \int_Q |g_n\psi| \leq \theta\int_Q |v_n\chi_A| + \int_Q \frac{\eta(x,t)\chi_A}{n} \to \theta\int_Q v\chi_A = 0,$$

thus $\int_A |f(x,t)| = 0$ and $f = 0$ a.e. in A. ■

Choose $\lambda \in \mathbb{R}$ such that $\alpha < \lambda < \beta$ and define the function d by

$$d(x,t) = \begin{cases} \dfrac{f(x,t)}{v(x,t)} & \text{a.e. in } Q \setminus A, \\ \lambda & \text{a.e. in } A. \end{cases}$$

Lemma 2.6. Suppose $(C_{\alpha,\beta})$. Then $\alpha \leq d(x,t) \leq \beta$ a.e. in Q.

Proof. We prove $\alpha \leq d(x,t)$ a.e. in Q. We denote $B = \{(x,t) \in Q : \alpha(v(x,t))^2 > v(x,t)f(x,t)$ a.e.$\}$. It is sufficient to prove that meas$B = 0$. Under $(C_{\alpha,\beta})$, we have

$$(\alpha - \varepsilon)u_n^2 - a_\varepsilon(x,t)|u_n| \leq u_n g(x,t,u_n).$$

Dividing by $\|u_n\|^2$, we get

$$(\alpha - \varepsilon)v_n^2 - a_\varepsilon(x,t)\frac{|v_n|}{n} \leq v_n g_n(x,t).$$

Multiplying by χ_B and integrating, we get

$$(\alpha - \varepsilon)\int_Q v_n^2\chi_B - \int_Q \frac{a_\varepsilon(x,t)}{n}|v_n|\chi_B \leq \int_Q v_n\chi_B g_n(x,t).$$

Passing to the limit, we have

$$(\alpha - \varepsilon) \int_Q |v(x,t)|^2 \chi_B \leq \int_Q v(x,t) f(x,t) \chi_B.$$

As ε is arbitrary, one concludes that

$$\int_Q [v(x,t) f(x,t) - \alpha |v(x,t)|^2] \chi_B \geq 0.$$

Therefore, by the definition of B, meas$B = 0$. By analogous method, we prove that $d(x,t) \leq \beta$ a.e. in Q. ■

Lemma 2.7. Supposes (2.4) and $(C_{\alpha,\beta})$. Then

$$\begin{cases} \Box v = m(x,t) v & \text{in } Q, \\ v(x, t+2\pi) = v(x,t) & \text{in } \Omega \times \mathbb{R}, \\ v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases}$$

where $m(x,t) = rd(x,t) + (1-r)\lambda$ and $r = \lim_n r_n$.

Remark 2.8. It is easy to see that $m(x,t)$ is t 2π-periodic and $\alpha \leq m(x,t) \leq \beta$ a.e. in Q.

Proof of Lemma 2.7. In the proof of Lemma 2.4, we have

$$rf + (1-r)\lambda v = \Box v.$$

By the definition of the function m, we have $\Box v = mv$. ■

Finally, it remains to prove the following lemma:

Lemma 2.9. Suppose (2.4) and $(C_{\alpha,\beta})$. Then

$$\alpha \leq\neq m(x,t) \leq\neq \beta \text{ a.e. in } Q.$$

Proof. We prove $m(x,t) \leq\neq \beta$ a.e. in Q. (By analogous method, we prove that $\alpha \leq\neq m(x,t)$ a.e. in Q.) Suppose by contradiction that $m(x,t) = \beta$ a.e. in Q. By Lemma 2.7, we have

$$\int_Q \nabla v^2 - \int_Q \dot{v}^2 = \beta \int_Q v^2. \tag{2.5}$$

Under $(C_{\alpha,\beta})$, we have

$$v_n g_n \leq (k(x,t) + \varepsilon) v_n^2 + \frac{a_\varepsilon |v_n|}{n}, \tag{2.6}$$

where $k(x,t) \in L^\infty(Q)$ such that $k(x,t) \leq\neq \beta$. Dividing (2.4) by $\|u_n\|$, multiplying by v_n and integrating, we obtain

$$\int_Q \nabla v_n^2 - \int_Q \dot{v_n}^2 = r_n \int_Q g_n v_n + (1-r_n)\lambda \int_Q v_n^2 + r_n \int_Q \frac{h v_n}{n}. \tag{2.7}$$

By (2.6) and (2.7), we deduce

$$\int_Q \nabla v_n^2 - \int_Q \dot{v_n}^2 \le r_n \int_Q (k(x,t)+\varepsilon)v_n^2 + (1-r_n)\lambda \int_Q v_n^2 + r_n \int_Q \left(h\frac{v_n}{n} + a_\varepsilon \frac{|v_n|}{n} \right).$$

Passing to the limit, we get

$$\int_Q \nabla v^2 - \int_Q \dot{v}^2 \le r \int_Q (k(x,t)+\varepsilon)v^2 + (1-r)\lambda \int_Q v^2.$$

As ε is arbitrary, we have

$$\int_Q \nabla v^2 - \int_Q \dot{v}^2 \le \int_Q [rk(x,t) + (1-r)\lambda]v^2. \tag{2.8}$$

Finally, by (2.5) and (2.8), we obtain

$$\int_Q [\beta - rk(x,t) - (1-r)\lambda]v^2 \le 0.$$

Since $k(x,t) \le \beta$ a.e. in Q and $\lambda < \beta$, we have

$$\beta - rk(x,t) - (1-r)\lambda \ge 0 \text{ and } \int_Q [\beta - rk(x,t) - (1-r)\lambda]v^2 = 0.$$

Therefore $[\beta - rk(x,t) - (1-r)\lambda]v^2 = 0$ a.e. in Q. As $m(x,t) = \beta$ a.e. in Q, by the definition of the function d ($d(x,t) \ne \lambda$), we have meas$A = 0$ (i.e., $v(x,t) \ne 0$ a.e. in Q). Thus, $\beta = rk(x,t) + (1-r)\lambda$ a.e. in Q. This contradiction concludes the proof. ■

For the proof of Corollary 2.3 we will need the following two lemmas.

Lemma 2.10. [6] Assume (C_1), (2.2), $\lambda \in \sigma(\Box)$ and $\lambda \ge 0$. Let $h \in L^2(Q)$. If there exists $R > 0$ such that $\Box u - r(N(u)+h) - (1-r)\lambda u \ne 0$, for all $u \in D(\Box)$, $\|u\| = R$, $0 \le r \le 1$, then the problem (1.1) admits at least one solution $u \in D(\Box)$ with $\|u\| < R$.

Proof. By (2.2) and (C_1), N is continuous and monotone. Therefore the result follows by the homotopy studied in [6]. ■

Lemma 2.11. [5] If there exist two reals α and β such that

$$\alpha \le m(x,t) \le \beta \text{ a.e. in } Q \text{ with } [\alpha,\beta] \cap \sigma(\Box) = \emptyset, \tag{2.9}$$

then the problem (2.3) has only the trivial solution.

Proof. Let $c \in [\alpha, \beta]$ be arbitrary with

$$\frac{\max(|\beta - c|, |\alpha - c|)}{\operatorname{dist}(c, \sigma(\Box))} < 1$$

(for example, $c = \dfrac{\alpha+\beta}{2}$), where $\operatorname{dist}(c, \sigma(\Box)) := \inf_{\lambda \in \sigma(\Box)} |c - \lambda|$ is the distance function. Then the operator $\Box - cI$ is invertible and

$$\|(\Box - cI)^{-1}\| = \frac{1}{\operatorname{dist}(c, \sigma(\Box))}.$$

Hence for all $u \in D(\Box)$,

$$\|\Box u - cu\| \geq \operatorname{dist}(c, \sigma(\Box))\|u\|.$$

Assume now that $u \in D(\Box)$ is a solution of the problem (2.3). Then

$$\|\Box u - cu\| = \|mu - cu\|$$

and therefore

$$\|u\| \leq \frac{\|mu - cu\|}{\operatorname{dist}(c, \sigma(\Box))}.$$

On the other hand, by the condition (2.9),

$$|mu - cu| = |m - c||u| \leq \max(|\beta - c|, |c - \alpha|)|u|.$$

Therefore

$$\|mu - cu\| \leq \max(|\beta - c|, |c - \alpha|)\|u\|.$$

Thus

$$\|u\| \leq \frac{\max(|\beta - c|, |c - \alpha|)}{\operatorname{dist}(c, \sigma(\Box))}\|u\|.$$

Since

$$\frac{\max(|\beta - c|, |\alpha - c|)}{\operatorname{dist}(c, \sigma(\Box))} < 1,$$

we have $u = 0$. ■

Proof of Corollary 2.3. Suppose by contradiction that the problem (1.1) does not admit a solution. Thus by Lemma 2.10, the homotopy T_r does not verify the estimate (2.2). By Theorem 2.2, there exist $m(x,t) \in L^\infty(Q)$, $v \in L^2(Q) \setminus \{0\}$ such that v is a nontrivial solution of the problem (2.3) and $\alpha \leq\neq m(x,t) \leq\neq \beta$ a.e. in Q. Since $\lambda_k < \alpha < \beta < \lambda_{k+1}$, this is in contradiction with Lemma 2.11. The proof is thus complete. ■

Remark 2.12.

1. We have an analogous result, if in Corollary 2.3, λ_k and λ_{k+1} are two negative consecutive eigenvalues, while replacing the D'Alembertian ($\Box$) by the operator ($-\Box$).

2. For the existence of periodic solutions of the problem with conditions of nonresonances

$$\begin{cases} \Box u &= \mu u + \delta \nabla u - \gamma u_t + g(x,t,u) + h(x,t) & \text{in} \quad Q, \\ u(x, t+2\pi) &= u(x,t) & \text{in} \quad \Omega \times \mathbb{R}, \\ u(x,t) &= 0 & \text{on} \quad \partial\Omega \times \mathbb{R}, \end{cases} \tag{2.10}$$

where $(\mu, \delta, \gamma) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, we have the following theorem.

Theorem 2.13. If the nonlinearity g satisfies

(C_1) $g(x,t,s)$ is nondecreasing in s

(C_1')

$$e^{\delta \frac{x}{2}} \left(\frac{g(x,t,r) - g(x,t,s)}{r-s} \right) \geq -\lambda,$$

where $\lambda = \mu - \dfrac{\delta^2}{4}$

(C_2) for all $R > 0$, there exists $\phi_R \in L^2(Q)$ such that a.e. $x \in \Omega$, for all $t \in \mathbb{R}$,

$$\max_{|s| \leq R} |g(x,t,s)| \leq \phi_R(x,t)$$

(C_3') a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, we have

$$\lambda_k - \frac{\gamma^2}{4} - \lambda < l(x,t) := \liminf_{|s| \to +\infty} \frac{g(x,t,s)}{s} \leq \limsup_{|s| \to +\infty} \frac{g(x,t,s)}{s}$$
$$:= k(x,t) < \lambda_{k+1} - \frac{\gamma^2}{4} - \lambda,$$

where λ_k and λ_{k+1} are two consecutive eigenvalues of the D'Alembertian,

then the problem (2.10) has at least one solution for all $h \in L^2(Q)$.

Remark 2.14. Note that if $\mu = 0$, $\delta = 0$ and $\gamma = 0$, we recover Corollary 2.3.

Proof of Theorem 2.13. It is easy to see that (2.10) is equivalent to the problem

$$\begin{cases} T_\gamma v &= \hat{g}(x,t,v) + \hat{h}(x,t) & \text{in} \quad Q, \\ v(x, t+2\pi) &= v(x,t) & \text{in} \quad \Omega \times \mathbb{R}, \\ v(x,t) &= 0 & \text{on} \quad \partial\Omega \times \mathbb{R}, \end{cases} \tag{2.11}$$

where $T_\gamma v = \Box v + \gamma v$, $\hat{g}(x,t,s) = \lambda s + e^{\frac{\delta}{2}x} g(x,t,e^{-\frac{\delta}{2}x}s)$ and $\hat{h}(x,t) = e^{\frac{\delta}{2}x} h(x,t)$. For the solution of the problem (2.11), we follow the same method used to solve the problem (1.1), while replacing the D'Alembertian ($\Box$) by the operator T_γ. ■

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