Advances in Dynamical Systems and Applications.
ISSN 0973-5321, Volume 17, Number 2 (2022) pp. 581-598

https://www.ripublication.com/adsa.htm

Numerical Methods for Solving the Home Heating System

***Abdulkafi Mohammed Saeed and Sarah Hasan Alotaibi**

Department of Mathematics, College of Science,
Qassim University, Buraydah, Kingdom of Saudi Arabia

Abstract

Many numerical approaches are developed in the last and in the present centuries to provide solutions of linear systems. Runge-Kutta (RK) method is a universal method that is widely used for solving Ordinary Differential Equations (ODEs). One of the most important physical applications of the linear systems is modeling the temperature distribution of a typical home with attic, basement, and main floor. In this paper, a system of ordinary differential equations that models the home heating system of a typical home with attic, basement, and main living area will be numerically solved by using different order of RK method. In addition, numerical experiments will be conducted to show the most superior numerical method for solving the home heating system.

Keywords: Ordinary differential equations, Initial Value Problem, Runge-kutta Method, Error Analysis.

1. Introduction

The Ordinary differential equations occur as mathematical models in many branches of science, engineering as well as economics, business, health care, aeronautics, elasticity, astronomy, medicine, environmental sciences, social sciences, banking etc ([2], [5], [7], [9], [12]). Differential equations can describe nearly all systems undergoing change. Often those systems described by differential equations are too complex, or the systems they describe are too large, to trace the purely analytical solution of the equations. In these complex systems, computer simulations and numerical methods are useful for finding the solution. Numerical methods provide a powerful alternative tool for solving of both first and higher order differential equations ([1], [13], [18]). The problems most frequently encountered are classified into initial value and boundary value problems,

depending on the conditions indicated at the end points of the domain. ([3],[4],[16]). Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using numerous methods, such as Taylor's method, Euler method, Runge-Kutta method, and also some other methods. A more robust and intricate numerical technique is the Runge Kutta method. This method is the most widely used one since it gives reliable starting values and is particularly suitable when the computation of higher derivatives is complicated ([6],[11],[18]).
The main purpose of this paper is to formulate and compare different orders of Runge-Kutta by obtaining the approximate solutions of a home heating system. Section 2, deals with a brief description of the Runge-Kutta of fourth order (RK4), Runge-Kutta of fifth order (RK5), Runge-Kutta of sixth order (RK6), Runge-Kutta of seventh order (RK7) methods. Section 3, presented the error analysis of the studied RK methods. The numerical results will be introduced in section 4. Section 5 includes the discussion of results and conclusion is given in section 6.

2. Numerical Methods

The Runge Kutta method is a technique for approximating the solution of ODEs. This method is most popular because it is quite accurate, stable and used in most computer programmes for differential equations. This technique was developed by two German Mathematicians, Karl Runge about 1894 and extended by Wilhelm Kutta a few years later. This method is distinguished by their order they agree with Taylor's series method ([1],[14],[17]). One of the most important physical applications of systems ODE is modeling the temperature distribution of a typical home with attic, basement, and main floor. The home system considered in this paper is an insulated main floor (living area), and un-insulated attic and basement. The functions $x(t), y(t)$ and $z(t)$ represent the attic temperature, the main floor temperature, and the basement temperature respectively at any time t According to Newton's Cooling Law, the following system of ODEs models the considered home heating system:

$$x'(t) = k_0(a - x(t)) + k_1(y(t) - x(t))$$

$$y'(t) = k_1(x(t) - y(t)) + k_1(b - y(t)) + k_3(z(t) - y(t)) + d$$

$$z'(t) = k_3(y(t) - z(t)) + k_4(c - z(t))$$

Where $a = x(0),\ b = y(0),\ c = z(0),\ d$ is the provided temperature to the main floor per unit time , and $k_i,\ i = 0, \cdots, 4$ are the cooling coefficients determined by the Newton's Cooling Law and depend on the system ([19]).
In this section we represent RK4, RK5, RK6 and RK7, for solving initial value problem (IVP) of ordinary differential equations (ODEs). Then we will apply these methods for third order of ODEs to solving system home heating.

2.1 Runge-Kutta of Fourth Order Method

The fourth order RK is most popular Runge-Kutta method and widely used in solving the first order DEs ([8]). Consider the general initial value problem in ODEs of the form:

$$y'(x) = f\left(x, y(x)\right), \qquad y(x_0) = y_0 \tag{1}$$

The general formula for fourth order RK approximation as the following:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4);$$

where:

$$x_{n+1} = x_n + h, \quad k_1 = hf\left(x_n, y_n\right), \quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2\right), \qquad k_4 = hf\left(x_n + h, y_n + k_3\right),\ n = 0,1,2,\cdots$$

Consider the following system of differential equations:

$$\frac{dx}{dt} = P\left(t, x, y, z\right), \quad \frac{dy}{dt} = Q\left(t, x, y, z\right), \quad \frac{dz}{dt} = R\left(t, x, y, z\right) \tag{2}$$

With the initial conditions $x\left(t_0\right) = x_0, y\left(t_0\right) = y_0, z\left(t_0\right) = z_0$.

The formulation of fourth order RK method for the system of three differential equations (2) can be formulated as:

$$\begin{aligned} x_{i+1} &= x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{i+1} &= y_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\ z_{i+1} &= z_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{aligned} \tag{3}$$

Where

$$t_{i+1} = t_i + h, \quad k_1 = hP\left(t_i, x_i, y_i, z_i\right), \quad l_1 = hQ\left(t_i, x_i, y_i, z_i\right), \quad m_1 = hR\left(t_i, x_i, y_i, z_i\right)$$

$$k_2 = hP\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_1, y_i + \frac{1}{2}l_1, z_i + \frac{1}{2}m_1\right), \quad l_2 = hQ\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_1, y_i + \frac{1}{2}l_1, z_i + \frac{1}{2}m_1\right)$$

$$m_2 = hR\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_1, y_i + \frac{1}{2}l_1, z_i + \frac{1}{2}m_1\right), \quad k_3 = hP\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_2, y_i + \frac{1}{2}l_2, z_i + \frac{1}{2}m_2\right)$$

$$l_3 = hQ\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_2, y_i + \frac{1}{2}l_2, z_i + \frac{1}{2}m_2\right), \quad m_3 = hR\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_2, y_i + \frac{1}{2}l_2, z_i + \frac{1}{2}m_2\right)$$

$$k_4 = hP\left(t_i + h, x_i + k_3, y_i + l_3, z_i + m_3\right), \qquad l_4 = hQ\left(t_i + h, x_i + k_3, y_i + l_3, z_i + m_3\right)$$

$$m_4 = hR\left(t_i + h, x_i + k_3, y_i + l_3, z_i + m_3\right)\ ,\ i = 0,1,2,\cdots$$

2.2 Runge-Kutta of Fifth Order Method

In [3], the general formula for RK5 approximation of IVP (1) can be written as

$$y_{n+1} = y_n + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$$

where

$$x_{n+1} = x_n + h, \quad k_1 = hf\left(x_n, y_n\right), \quad k_2 = hf\left(x_n + \frac{h}{4}, y_n + \frac{1}{4}k_1\right),$$

$$k_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{1}{8}k_1 + \frac{1}{8}k_2\right), \quad k_4 = hf\left(x_n + \frac{h}{2}, y_n - \frac{1}{2}k_2 + k_3\right),$$

$$k_5 = hf\left(x_n + \frac{3}{4}h, y_n + \frac{3}{16}k_1 + \frac{9}{16}k_4\right)$$

$$k_6 = hf\left(x_n + h, y_n - \frac{3}{7}k_1 + \frac{2}{7}k_2 + \frac{12}{7}k_3 - \frac{12}{7}k_4 + \frac{8}{7}k_5\right), \; n = 0,1,2,\cdots$$

The fifth order RK formulation of the system of equations (2) will be given as:

$$x_{i+1} = x_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$$

$$y_{i+1} = y_i + \frac{1}{90}(7l_1 + 32l_3 + 12l_4 + 32l_5 + 7l_6) \tag{4}$$

$$z_{i+1} = z_i + \frac{1}{90}(7m_1 + 32m_3 + 12m_4 + 32m_5 + 7m_6)$$

where

$$t_{i+1} = t_i + h, \quad k_1 = hP\left(t_i, x_i, y_i, z_i\right), \quad l_1 = hQ\left(t_i, x_i, y_i, z_i\right), \quad m_1 = hR\left(t_i, x_i, y_i, z_i\right)$$

$$k_2 = hP\left(t_i + \frac{h}{4}, x_i + \frac{1}{4}k_1, y_i + \frac{1}{4}l_1, z_i + \frac{1}{4}m_1\right), \; l_2 = hQ\left(t_i + \frac{h}{4}, x_i + \frac{1}{4}k_1, y_i + \frac{1}{4}l_1, z_i + \frac{1}{4}m_1\right)$$

$$m_2 = hR\left(t_i + \frac{1}{4}h, x_i + \frac{1}{4}k_1, y_i + \frac{1}{4}l_1, z_i + \frac{1}{4}m_1\right)$$

$$k_3 = hP\left(t_i + \frac{1}{4}h, x_i + \frac{1}{8}k_1 + \frac{1}{8}k_2, y_i + \frac{1}{8}l_1 + \frac{1}{8}l_2, z_i + \frac{1}{8}m_1 + \frac{1}{8}m_2\right)$$

$$l_3 = hQ\left(t_i + \frac{1}{4}h, x_i + \frac{1}{8}k_1 + \frac{1}{8}k_2, y_i + \frac{1}{8}l_1 + \frac{1}{8}l_2, z_i + \frac{1}{8}m_1 + \frac{1}{8}m_2\right)$$

$$m_3 = hR\left(t_i + \frac{1}{4}h, x_i + \frac{1}{8}k_1 + \frac{1}{8}k_2, y_i + \frac{1}{8}l_1 + \frac{1}{8}l_2, z_i + \frac{1}{8}m_1 + \frac{1}{8}m_2\right)$$

$$k_4 = hP\left(t_i + \frac{1}{2}h, x_i - \frac{1}{2}k_2 + k_3, y_i - \frac{1}{2}l_2 + l_3, z_i - \frac{1}{2}m_2 + m_3\right)$$

$$l_4 = hQ\left(t_i + \frac{1}{2}h, x_i - \frac{1}{2}k_2 + k_3, y_i - \frac{1}{2}l_2 + l_3, z_i - \frac{1}{2}m_2 + m_3\right)$$

$$m_4 = hR\left(t_i + \frac{1}{2}h, x_i - \frac{1}{2}k_2 + k_3, y_i - \frac{1}{2}l_2 + l_3, z_i - \frac{1}{2}m_2 + m_3\right)$$

$$k_5 = hP\left(t_i + \frac{3}{4}h, x_i + \frac{3}{16}k_1 + \frac{9}{16}k_4, y_i + \frac{3}{16}l_1 + \frac{9}{16}l_4, z_i + \frac{3}{16}m_1 + \frac{9}{16}m_4\right)$$

$$l_5 = hQ\left(t_i + \frac{3}{4}h, x_i + \frac{3}{16}k_1 + \frac{9}{16}k_4, y_i + \frac{3}{16}l_1 + \frac{9}{16}l_4, z_i + \frac{3}{16}m_1 + \frac{9}{16}m_4\right)$$

$$m_5 = hR\left(t_i + \frac{3}{4}h, x_i + \frac{3}{16}k_1 + \frac{9}{16}k_4, y_i + \frac{3}{16}l_1 + \frac{9}{16}l_4, z_i + \frac{3}{16}m_1 + \frac{9}{16}m_4\right)$$

$$k_6 = hP\left(\begin{array}{l} t_i + h, x_i - \frac{3}{7}k_1 + \frac{2}{7}k_2 + \frac{12}{7}k_3 - \frac{12}{7}k_4 + \frac{8}{7}k_5, y_i - \frac{3}{7}l_1 + \frac{2}{7}l_2 \\ + \frac{12}{7}l_3 - \frac{12}{7}l_4 + \frac{8}{7}l_5, z_i - \frac{3}{7}m_1 + \frac{2}{7}m_2 + \frac{12}{7}m_3 - \frac{12}{7}m_4 + \frac{8}{7}m_5 \end{array}\right)$$

$$l_6 = hQ\left(\begin{array}{l} t_i + h, x_i - \frac{3}{7}k_1 + \frac{2}{7}k_2 + \frac{12}{7}k_3 - \frac{12}{7}k_4 + \frac{8}{7}k_5, y_i - \frac{3}{7}l_1 + \frac{2}{7}l_2 \\ + \frac{12}{7}l_3 - \frac{12}{7}l_4 + \frac{8}{7}l_5, z_i - \frac{3}{7}m_1 + \frac{2}{7}m_2 + \frac{12}{7}m_3 - \frac{12}{7}m_4 + \frac{8}{7}m_5 \end{array}\right)$$

$$m_6 = hR\left(\begin{array}{l} t_i + h, x_i - \frac{3}{7}k_1 + \frac{2}{7}k_2 + \frac{12}{7}k_3 - \frac{12}{7}k_4 + \frac{8}{7}k_5, y_i - \frac{3}{7}l_1 + \frac{2}{7}l_2 \\ + \frac{12}{7}l_3 - \frac{12}{7}l_4 + \frac{8}{7}l_5, z_i - \frac{3}{7}m_1 + \frac{2}{7}m_2 + \frac{12}{7}m_3 - \frac{12}{7}m_4 + \frac{8}{7}m_5 \end{array}\right)$$

$i = 0,1,2,\cdots$

2.3 Runge-Kutta of Sixth Order Method

The general formula for the RK6 approximation for IVP (1) [4] is

$$y_{n+1} = y_n + \frac{1}{120}(11k_1 + 81k_3 + 81k_4 - 32k_5 - 32k_6 + 11k_7)$$

where

$$x_{n+1} = x_n + h, \quad k_1 = hf\left(x_n, y_n\right), \quad k_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{1}{3}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}k_2\right), \qquad k_4 = hf\left(x_n + \frac{h}{3}, y_n + \frac{1}{12}k_1 + \frac{1}{3}k_2 - \frac{1}{12}k_3\right)$$

$$k_5 = hf\left(x_n + \frac{h}{2}, y_n - \frac{1}{16}k_1 + \frac{9}{8}k_2 - \frac{3}{16}k_3 - \frac{3}{8}k_4\right)$$

$$k_6 = hf\left(x_n + \frac{h}{2}, y_n + \frac{9}{8}k_2 - \frac{3}{16}k_3 - \frac{3}{4}k_4 + \frac{1}{2}k_5\right)$$

$$k_7 = hf\left(x_n + h, y_n + \frac{9}{44}k_1 - \frac{9}{11}k_2 + \frac{63}{44}k_3 + \frac{18}{11}k_4 - \frac{16}{11}k_6\right), n = 0,1,2,\cdots$$

The sixth order RK formulation of the system of equations (2) can be written as:

$$\begin{aligned} x_{i+1} &= x_i + \frac{1}{120}(11k_1 + 81k_3 + 81k_4 - 32k_5 - 32k_6 + 11k_7) \\ y_{i+1} &= y_i + \frac{1}{120}(11l_1 + 81l_3 + 81l_4 - 32l_5 - 32l_6 + 11l_7) \qquad (5) \\ z_{i+1} &= z_i + \frac{1}{120}(11m_1 + 81m_3 + 81m_4 - 32m_5 - 32m_6 + 11m_7), \end{aligned}$$

where

$$t_{i+1}=t_i+h,\quad k_1=hP\left(t_i,x_i,y_i,z_i\right),\quad l_1=hQ\left(t_i,x_i,y_i,z_i\right),\quad m_1=hR\left(t_i,x_i,y_i,z_i\right)$$

$$k_2=hP\left(t_i+\frac{1}{3}h,x_i+\frac{1}{3}k_1,y_i+\frac{1}{3}l_1,z_i+\frac{1}{3}m_1\right),\ l_2=hQ\left(t_i+\frac{1}{3}h,x_i+\frac{1}{3}k_1,y_i+\frac{1}{3}l_1,z_i+\frac{1}{3}m_1\right)$$

$$m_2=hR\left(t_i+\frac{1}{3}h,x_i+\frac{1}{3}k_1,y_i+\frac{1}{3}l_1,z_i+\frac{1}{3}m_1\right),\ k_3=hP\left(t_i+\frac{2}{3}h,x_i+\frac{2}{3}k_2,y_i+\frac{2}{3}l_2,z_i+\frac{2}{3}m_2\right)$$

$$l_3=hQ\left(t_i+\frac{2}{3}h,x_i+\frac{2}{3}k_2,y_i+\frac{2}{3}l_2,z_i+\frac{2}{3}m_2\right),\ m_3=hR\left(t_i+\frac{2}{3}h,x_i+\frac{2}{3}k_2,y_i+\frac{2}{3}l_2,z_i+\frac{2}{3}m_2\right)$$

$$k_4=hP\left(t_i+\frac{1}{3}h,x_i+\frac{1}{12}k_1+\frac{1}{3}k_2-\frac{1}{12}k_3,y_i+\frac{1}{12}l_1+\frac{1}{3}l_2-\frac{1}{12}l_3,z_i+\frac{1}{12}m_1+\frac{1}{3}m_2-\frac{1}{12}m_3\right)$$

$$l_4=hQ\left(t_i+\frac{1}{3}h,x_i+\frac{1}{12}k_1+\frac{1}{3}k_2-\frac{1}{12}k_3,y_i+\frac{1}{12}l_1+\frac{1}{3}l_2-\frac{1}{12}l_3,z_i+\frac{1}{12}m_1+\frac{1}{3}m_2-\frac{1}{12}m_3\right)$$

$$m_4=hR\left(t_i+\frac{1}{3}h,x_i+\frac{1}{12}k_1+\frac{1}{3}k_2-\frac{1}{12}k_3,y_i+\frac{1}{12}l_1+\frac{1}{3}l_2-\frac{1}{12}l_3,z_i+\frac{1}{12}m_1+\frac{1}{3}m_2-\frac{1}{12}m_3\right)$$

$$k_5=hP\left(t_i+\frac{1}{2}h,x_i-\frac{1}{16}k_1+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{8}k_4,y_i-\frac{1}{16}l_1+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{8}l_4,z_i-\frac{1}{16}m_1+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{8}m_4\right)$$

$$l_5=hQ\left(t_i+\frac{1}{2}h,x_i-\frac{1}{16}k_1+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{8}k_4,y_i-\frac{1}{16}l_1+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{8}l_4,z_i-\frac{1}{16}m_1+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{8}m_4\right)$$

$$m_5=hR\left(t_i+\frac{1}{2}h,x_i-\frac{1}{16}k_1+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{8}k_4,y_i-\frac{1}{16}l_1+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{8}l_4,z_i-\frac{1}{16}m_1+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{8}m_4\right)$$

$$k_6=hP\left(t_i+\frac{1}{2}h,x_i+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{4}k_4+\frac{1}{2}k_5,y_i+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{4}l_4+\frac{1}{2}l_5,z_i+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{4}m_4+\frac{1}{2}m_5\right)$$

$$l_6=hQ\left(t_i+\frac{1}{2}h,x_i+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{4}k_4+\frac{1}{2}k_5,y_i+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{4}l_4+\frac{1}{2}l_5,z_i+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{4}m_4+\frac{1}{2}m_5\right)$$

$$m_6=hR\left(t_i+\frac{1}{2}h,x_i+\frac{9}{8}k_2-\frac{3}{16}k_3-\frac{3}{4}k_4+\frac{1}{2}k_5,y_i+\frac{9}{8}l_2-\frac{3}{16}l_3-\frac{3}{4}l_4+\frac{1}{2}l_5,z_i+\frac{9}{8}m_2-\frac{3}{16}m_3-\frac{3}{4}m_4+\frac{1}{2}m_5\right)$$

$$k_7=hP\begin{pmatrix}t_i+h,x_i+\frac{9}{44}k_1-\frac{9}{11}k_2+\frac{63}{44}k_3+\frac{18}{11}k_4-\frac{16}{11}k_6,y_i+\frac{9}{44}l_1-\frac{9}{11}l_2\\+\frac{63}{44}l_3+\frac{18}{11}l_4-\frac{16}{11}l_6,z_i+\frac{9}{44}m_1-\frac{9}{11}m_2+\frac{63}{44}m_3+\frac{18}{11}m_4-\frac{16}{11}m_6\end{pmatrix}$$

$$l_7=hQ\begin{pmatrix}t_i+h,x_i+\frac{9}{44}k_1-\frac{9}{11}k_2+\frac{63}{44}k_3+\frac{18}{11}k_4-\frac{16}{11}k_6,y_i+\frac{9}{44}l_1-\frac{9}{11}l_2\\+\frac{63}{44}l_3+\frac{18}{11}l_4-\frac{16}{11}l_6,z_i+\frac{9}{44}m_1-\frac{9}{11}m_2+\frac{63}{44}m_3+\frac{18}{11}m_4-\frac{16}{11}m_6\end{pmatrix}$$

$$m_7 = hR\begin{pmatrix} t_i + h, x_i + \frac{9}{44}k_1 - \frac{9}{11}k_2 + \frac{63}{44}k_3 + \frac{18}{11}k_4 - \frac{16}{11}k_6, y_i + \frac{9}{44}l_1 - \frac{9}{11}l_2 \\ + \frac{63}{44}l_3 + \frac{18}{11}l_4 - \frac{16}{11}l_6, z_i + \frac{9}{44}m_1 - \frac{9}{11}m_2 + \frac{63}{44}m_3 + \frac{18}{11}m_4 - \frac{16}{11}m_6 \end{pmatrix}$$

$i = 0,1,2,\cdots$

2.4 Runge-Kutta of Seventh Order Method

The general formula for RK7 approximation for IVP (1) [20] is

$$y_{n+1} = y_n + \frac{1}{840}(41k_1 + 216k_4 + 27k_5 + 272k_6 + 27k_7 + 216k_8 + 41k_9)$$

where

$$x_{n+1} = x_n + h, \qquad k_1 = hf\left(x_n, y_n\right), \qquad k_2 = hf\left(x_n + \frac{1}{12}h, y_n + \frac{1}{12}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{12}h, y_n - \frac{10}{12}k_1 + \frac{11}{12}k_2\right), \quad k_4 = hf\left(x_n + \frac{2}{12}h, y_n + \frac{2}{12}k_3\right)$$

$$k_5 = hf\left(x_n + \frac{4}{12}h, y_n + \frac{1}{9}\left(157k_1 - 318k_2 + 4k_3 + 160k_4\right)\right)$$

$$k_6 = hf\left(x_n + \frac{6}{12}h, y_n + \frac{1}{30}\left(-161k_1 + 199k_2 + 108k_3 - 131k_4\right)\right)$$

$$k_7 = hf\left(x_n + \frac{8}{12}h, y_n + \frac{3158}{45}k_1 - \frac{683}{6}k_2 - \frac{69}{6}k_3 + \frac{314}{6}k_4 + \frac{157}{45}k_6\right)$$

$$k_8 = hf\left(x_n + \frac{10}{12}h, y_n - \frac{265}{70}k_1 + \frac{379}{70}k_2 - \frac{15}{70}k_3 - \frac{65}{72}k_5 + \frac{29}{90}k_7\right)$$

$$k_9 = hf\left(x_n + h, y_n + \frac{56}{25}k_1 + \frac{849}{42}k_2 - \frac{833}{42}k_3 - \frac{156}{42}k_4 - \frac{39}{45}k_5 + \frac{149}{32}k_6 - \frac{125}{45}k_7 + \frac{27}{25}k_8\right)$$

$n = 0,1,2,\cdots$

The seventh order RK d formulation of the system of equations (2) can be written as:

$$\begin{aligned} x_{i+1} &= x_i + \frac{1}{840}(41k_1 + 216k_4 + 27k_5 + 272k_6 + 27k_7 + 216k_8 + 41k_9) \\ y_{i+1} &= y_i + \frac{1}{840}(41l_1 + 216l_4 + 27l_5 + 272l_6 + 27l_7 + 216l_8 + 41l_9) \qquad (6) \\ z_{i+1} &= z_i + \frac{1}{840}(41m_1 + 216m_4 + 27m_5 + 272m_6 + 27m_7 + 216m_8 + 41m_9) \end{aligned}$$

where

$$t_{i+1} = t_i + h, \quad k_1 = hP\left(t_i, x_i, y_i, z_i\right), \quad l_1 = hQ\left(t_i, x_i, y_i, z_i\right), \quad m_1 = hR\left(t_i, x_i, y_i, z_i\right)$$

$$k_2 = hP\left(t_i + \frac{1}{12}h, x_i + \frac{1}{12}k_1, y_i + \frac{1}{12}l_1, z_i + \frac{1}{12}m_1\right),$$

$$l_2 = hQ\left(t_i + \frac{1}{12}h, x_i + \frac{1}{12}k_1, y_i + \frac{1}{12}l_1, z_i + \frac{1}{12}m_1\right)$$

$$m_2 = hR\left(t_i + \frac{1}{12}h, x_i + \frac{1}{12}k_1, y_i + \frac{1}{12}l_1, z_i + \frac{1}{12}m_1\right)$$

$$k_3 = hP\left(t_i + \frac{1}{12}h, x_i - \frac{10}{12}k_1 + \frac{11}{12}k_2, y_i - \frac{10}{12}l_1 + \frac{11}{12}l_2, z_i - \frac{10}{12}m_1 + \frac{11}{12}m_2\right)$$

$$l_3 = hQ\left(t_i + \frac{1}{12}h, x_i - \frac{10}{12}k_1 + \frac{11}{12}k_2, y_i - \frac{10}{12}l_1 + \frac{11}{12}l_2, z_i - \frac{10}{12}m_1 + \frac{11}{12}m_2\right)$$

$$m_3 = hR\left(t_i + \frac{1}{12}h, x_i - \frac{10}{12}k_1 + \frac{11}{12}k_2, y_i - \frac{10}{12}l_1 + \frac{11}{12}l_2, z_i - \frac{10}{12}m_1 + \frac{11}{12}m_2\right)$$

$$k_4 = hP\left(t_i + \frac{2}{12}h, x_i + \frac{2}{12}k_3, y_i + \frac{2}{12}l_3, z_i + \frac{2}{12}m_3\right)$$

$$l_4 = hQ\left(t_i + \frac{2}{12}h, x_i + \frac{2}{12}k_3, y_i + \frac{2}{12}l_3, z_i + \frac{2}{12}m_3\right)$$

$$m_4 = hR\left(t_i + \frac{2}{12}h, x_i + \frac{2}{12}k_3, y_i + \frac{2}{12}l_3, z_i + \frac{2}{12}m_3\right)$$

$$k_5 = hP\left(\begin{array}{l} t_i + \frac{4}{12}h, x_i + \frac{1}{9}\left(157k_1 - 318k_2 + 4k_3 + 160k_4\right), \\ y_i + \frac{1}{9}\left(157l_1 - 318l_2 + 4l_3 + 160l_4\right), z_i + \frac{1}{9}\left(157m_1 - 318m_2 + 4m_3 + 160m_4\right) \end{array}\right)$$

$$l_5 = hQ\left(\begin{array}{l} t_i + \frac{4}{12}h, x_i + \frac{1}{9}\left(157k_1 - 318k_2 + 4k_3 + 160k_4\right), \\ y_i + \frac{1}{9}\left(157l_1 - 318l_2 + 4l_3 + 160l_4\right), z_i + \frac{1}{9}\left(157m_1 - 318m_2 + 4m_3 + 160m_4\right) \end{array}\right)$$

$$m_5 = hR\left(\begin{array}{l} t_i + \frac{4}{12}h, x_i + \frac{1}{9}\left(157k_1 - 318k_2 + 4k_3 + 160k_4\right), \\ y_i + \frac{1}{9}\left(157l_1 - 318l_2 + 4l_3 + 160l_4\right), z_i + \frac{1}{9}\left(157m_1 - 318m_2 + 4m_3 + 160m_4\right) \end{array}\right)$$

$$k_6 = hP\left(\begin{array}{l} t_i + \frac{6}{12}h, x_i + \frac{1}{30}\left(-161k_1 + 199k_2 + 108k_3 - 131k_4\right), \\ y_i + \frac{1}{30}\left(-161l_1 + 199l_2 + 108l_3 - 131l_4\right), z_i + \frac{1}{30}\left(-161m_1 + 199m_2 + 108m_3 - 131m_4\right) \end{array}\right)$$

$$l_6 = hQ\left(\begin{array}{l} t_i + \frac{6}{12}h, x_i + \frac{1}{30}\left(-161k_1 + 199k_2 + 108k_3 - 131k_4\right), \\ y_i + \frac{1}{30}\left(-161l_1 + 199l_2 + 108l_3 - 131l_4\right), z_i + \frac{1}{30}\left(-161m_1 + 199m_2 + 108m_3 - 131m_4\right) \end{array}\right)$$

$$m_6 = hR\left(\begin{array}{l} t_i + \frac{6}{12}h, x_i + \frac{1}{30}\left(-161k_1 + 199k_2 + 108k_3 - 131k_4\right), \\ y_i + \frac{1}{30}\left(-161l_1 + 199l_2 + 108l_3 - 131l_4\right), z_i + \frac{1}{30}\left(-161m_1 + 199m_2 + 108m_3 - 131m_4\right) \end{array}\right)$$

$$k_7 = hP\begin{pmatrix} t_i + \frac{8}{12}h, x_i + \frac{3158}{45}k_1 - \frac{683}{6}k_2 - \frac{69}{6}k_3 + \frac{314}{6}k_4 + \frac{157}{45}k_6, y_i + \frac{3158}{45}l_1 - \frac{683}{6}l_2 \\ -\frac{69}{6}l_3 + \frac{314}{6}l_4 + \frac{157}{45}l_6, z_i + \frac{3158}{45}m_1 - \frac{683}{6}m_2 - \frac{69}{6}m_3 + \frac{314}{6}m_4 + \frac{157}{45}m_6 \end{pmatrix}$$

$$l_7 = hQ\begin{pmatrix} t_i + \frac{8}{12}h, x_i + \frac{3158}{45}k_1 - \frac{683}{6}k_2 - \frac{69}{6}k_3 + \frac{314}{6}k_4 + \frac{157}{45}k_6, y_i + \frac{3158}{45}l_1 - \frac{683}{6}l_2 \\ -\frac{69}{6}l_3 + \frac{314}{6}l_4 + \frac{157}{45}l_6, z_i + \frac{3158}{45}m_1 - \frac{683}{6}m_2 - \frac{69}{6}m_3 + \frac{314}{6}m_4 + \frac{157}{45}m_6 \end{pmatrix}$$

$$m_7 = hR\begin{pmatrix} t_i + \frac{8}{12}h, x_i + \frac{3158}{45}k_1 - \frac{683}{6}k_2 - \frac{69}{6}k_3 + \frac{314}{6}k_4 + \frac{157}{45}k_6, y_i + \frac{3158}{45}l_1 - \frac{683}{6}l_2 \\ -\frac{69}{6}l_3 + \frac{314}{6}l_4 + \frac{157}{45}l_6, z_i + \frac{3158}{45}m_1 - \frac{683}{6}m_2 - \frac{69}{6}m_3 + \frac{314}{6}m_4 + \frac{157}{45}m_6 \end{pmatrix}$$

$$k_8 = hP\begin{pmatrix} t_i + \frac{10}{12}h, x_i - \frac{265}{70}k_1 + \frac{379}{70}k_2 - \frac{15}{70}k_3 - \frac{65}{72}k_5 + \frac{29}{90}k_7, y_i - \frac{265}{70}l_1 + \frac{379}{70}l_2 \\ -\frac{15}{70}l_3 - \frac{65}{72}l_5 + \frac{29}{90}l_7, z_i - \frac{265}{70}m_1 + \frac{379}{70}m_2 - \frac{15}{70}m_3 - \frac{65}{72}m_5 + \frac{29}{90}m_7 \end{pmatrix}$$

$$l_8 = hQ\begin{pmatrix} t_i + \frac{10}{12}h, x_i - \frac{265}{70}k_1 + \frac{379}{70}k_2 - \frac{15}{70}k_3 - \frac{65}{72}k_5 + \frac{29}{90}k_7, y_i - \frac{265}{70}l_1 + \frac{379}{70}l_2 \\ -\frac{15}{70}l_3 - \frac{65}{72}l_5 + \frac{29}{90}l_7, z_i - \frac{265}{70}m_1 + \frac{379}{70}m_2 - \frac{15}{70}m_3 - \frac{65}{72}m_5 + \frac{29}{90}m_7 \end{pmatrix}$$

$$m_8 = hR\begin{pmatrix} t_i + \frac{10}{12}h, x_i - \frac{265}{70}k_1 + \frac{379}{70}k_2 - \frac{15}{70}k_3 - \frac{65}{72}k_5 + \frac{29}{90}k_7, y_i - \frac{265}{70}l_1 + \frac{379}{70}l_2 \\ -\frac{15}{70}l_3 - \frac{65}{72}l_5 + \frac{29}{90}l_7, z_i - \frac{265}{70}m_1 + \frac{379}{70}m_2 - \frac{15}{70}m_3 - \frac{65}{72}m_5 + \frac{29}{90}m_7 \end{pmatrix}$$

$$k_9 = hP\begin{pmatrix} t_i + h, x_i + \frac{56}{25}k_1 + \frac{849}{42}k_2 - \frac{833}{42}k_3 - \frac{156}{42}k_4 - \frac{39}{45}k_5 + \frac{149}{32}k_6 - \frac{125}{45}k_7 + \frac{27}{25}k_8, \\ y_i + \frac{56}{25}l_1 + \frac{849}{42}l_2 - \frac{833}{42}l_3 - \frac{156}{42}l_4 - \frac{39}{45}l_5 + \frac{149}{32}l_6 - \frac{125}{45}l_7 + \frac{27}{25}l_8, \\ z_i + \frac{56}{25}m_1 + \frac{849}{42}m_2 - \frac{833}{42}m_3 - \frac{156}{42}m_4 - \frac{39}{45}m_5 + \frac{149}{32}m_6 - \frac{125}{45}m_7 + \frac{27}{25}m_8 \end{pmatrix}$$

$$l_9 = hQ\begin{pmatrix} t_i + h, x_i + \frac{56}{25}k_1 + \frac{849}{42}k_2 - \frac{833}{42}k_3 - \frac{156}{42}k_4 - \frac{39}{45}k_5 + \frac{149}{32}k_6 - \frac{125}{45}k_7 + \frac{27}{25}k_8, \\ y_i + \frac{56}{25}l_1 + \frac{849}{42}l_2 - \frac{833}{42}l_3 - \frac{156}{42}l_4 - \frac{39}{45}l_5 + \frac{149}{32}l_6 - \frac{125}{45}l_7 + \frac{27}{25}l_8, \\ z_i + \frac{56}{25}m_1 + \frac{849}{42}m_2 - \frac{833}{42}m_3 - \frac{156}{42}m_4 - \frac{39}{45}m_5 + \frac{149}{32}m_6 - \frac{125}{45}m_7 + \frac{27}{25}m_8 \end{pmatrix}$$

$$m_9 = hR\begin{pmatrix} t_i + h, x_i + \frac{56}{25}k_1 + \frac{849}{42}k_2 - \frac{833}{42}k_3 - \frac{156}{42}k_4 - \frac{39}{45}k_5 + \frac{149}{32}k_6 - \frac{125}{45}k_7 + \frac{27}{25}k_8, \\ y_i + \frac{56}{25}l_1 + \frac{849}{42}l_2 - \frac{833}{42}l_3 - \frac{156}{42}l_4 - \frac{39}{45}l_5 + \frac{149}{32}l_6 - \frac{125}{45}l_7 + \frac{27}{25}l_8, \\ z_i + \frac{56}{25}m_1 + \frac{849}{42}m_2 - \frac{833}{42}m_3 - \frac{156}{42}m_4 - \frac{39}{45}m_5 + \frac{149}{32}m_6 - \frac{125}{45}m_7 + \frac{27}{25}m_8 \end{pmatrix}$$

3. Error Analysis

In this section, to finding the numerical solution of ordinary differential equations, two types of errors occur such as round-off errors and truncation errors. This type of errors occur when ODEs are solved numerically. Rounding errors originate from the fact that computer can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is called round-off error. Truncation error in numerical analysis arises when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we take the step size h.

In this paper, we consider a system of first order ODEs for initial value problem (IVP) to verify accuracy of the proposed method. Then using this method, we find numerical approximations for desired IVP. A numerical method is said to be convergent if $\lim_{h\to 0}\max_{1\le n\le N}|y(x_n)-y_n|=0$ where $y(x_n)$ denotes the approximate solution and y_n denotes the exact solution. To show the accuracy of the present method, the maximum error is defined by $e_r = \max_{1\le n\le steps}\left(|y(x_n)-y_n|\right)$ ([14],[15]) .

4. Applications and Comparisons:

In this section, we consider the following example of IVP to test the efficiency and convergence of numerical solution along with actual solution for home heating system where Runge-Kutta fourth, fifth, sixth and seventh order methods were applied separately. The tables are containing the approximated result, and computed absolute error. The outcomes are displayed in the graph also. For comparison purpose, one step size is used $h = 0.05$.

Illustrative Example

The problem of home heating is chosen to numerically validate the comparison of the methods to solve systems of ODEs. For this purpose, we used a system of three first order ordinary differential equation of the form ([10]):

$$x' = \frac{1}{2}(45 - x) + \frac{1}{2}(y - x)$$
$$y' = \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20$$
$$z' = \frac{1}{4}(y - z) + \frac{1}{2}(35 - z)$$

with initial conditions $x(0) = 45, y(0) = 35, z(0) = 35.$ The exact solution of the given this systems is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -\frac{85}{24} e^{-t/2} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \left(-\frac{25}{24} - \frac{115\sqrt{21}}{168}\right) e^{-\left((7-\sqrt{21})/8\right)t} \begin{pmatrix} 4 \\ -1+\sqrt{21} \\ 2 \end{pmatrix} + \left(-\frac{25}{24} + \frac{115\sqrt{21}}{168}\right) e^{-\left((7+\sqrt{21})/8\right)t} \begin{pmatrix} 4 \\ -1-\sqrt{21} \\ 2 \end{pmatrix} + \begin{pmatrix} 455/8 \\ 275/4 \\ 185/4 \end{pmatrix}$$

The approximate results, exact results and maximum errors for step size 0.05 are shown in Tables 1-6 and the graphs 1-3 of the numerical solutions.

Table 1. Numerical test results for RK4 and RK5 the described techniques to evaluate X.

t	Approximation of RK4 Method h=0.05		Approximation of RK5 Method h=0.05		X Exact
	x by RK4	Error x RK4	x by RK5	Error x RK5	
0	45	0	45	0	45
0.05	44.7712370198568	0.0184184158035663	44.7712368173782	0.0184182133249990	44.7528186040532
0.10	44.5824946618616	0.0419043420768404	44.5824942870815	0.0419039672967188	44.5405903197848
0.15	44.4303076708338	0.0697369657721154	44.4303071505654	0.0697364455036720	44.3605707050617
0.20	44.3114791908865	0.101266723701364	44.3114785489121	0.101266081726962	44.2102124671851
0.25	44.2230605738176	0.135908984213877	44.2230598311908	0.135908241587060	44.0871515896037
0.30	44.1623326929887	0.173138263386420	44.1623318683082	0.173137438705858	43.9891944296023
0.35	44.1267886507896	0.212482931677755	44.1267877604474	0.212482041335527	43.9143057191118
0.40	44.1141177760958	0.253520370567443	44.1141168345024	0.253519428974016	43.8605974055284
0.45	44.1221908158165	0.295872541981005	44.1221898356045	0.295871561769012	43.8263182738355
0.50	44.1490462317500	0.339201936323718	44.1490452239586	0.339200928532357	43.8098442954263
0.55	44.1928775205571	0.383207867722398	44.1928764947989	0.383206841964295	43.8096696528347
0.60	44.2520214807635	0.427623087630010	44.2520204453756	0.427622052242114	43.8243983931335
0.65	44.3249473563549	0.472210690297672	44.3249463185348	0.472209652477567	43.8527366660573
0.70	44.4102467917564	0.516761285779623	44.4102457576854	0.516760251708682	43.8934855059768
0.75	44.5066245378313	0.561090418124167	44.5066235127859	0.561089393078689	43.9455341197072
0.80	44.6128898530201	0.605036208230956	44.6128888414715	0.605035196682415	44.0078536447891
0.85	44.7279485478847	0.648457202535198	44.7279475535901	0.648456208240603	44.0794913453495
0.90	44.8507956251732	0.691230410223767	44.8507946512564	0.691229436307047	44.1595652149494
0.95	44.9805084710707	0.733249513108575	44.9805075200959	0.733248562133809	44.2472589579621
1	45.1162405565993	0.774423233587299	45.1162396306366	0.774422307624597	44.3418173230120

Table 2. Numerical test results for RK6 and RK7 the described techniques to evaluate X.

t	Approximation of RK6 Method h=0.05		Approximation of RK7 Method h=0.05		X Exact
	x by RK6	Error x RK6	x by RK7	Error x RK7	
0	45	0	45	0	45
0.05	44.7712368123881	0.0184182083349000	44.7712628917056	0.0184442876523576	44.7528186040532
0.10	44.5824942778429	0.0419039580581355	44.5825423890275	0.0419520692427113	44.5405903197848
0.15	44.4303071377374	0.0697364326756471	44.4303736902732	0.0698029852114530	44.3605707050617
0.20	44.3114785330793	0.101266065894130	44.3115603479553	0.101347880770135	44.2102124671851
0.25	44.2230598128708	0.135908223267066	44.2231540813722	0.136002491768437	44.0871515896037
0.30	44.1623318479584	0.173137418356063	44.1624360943027	0.173241664700392	43.9891944296023
0.35	44.1267877384710	0.212482019359122	44.1268997859459	0.212594066834086	43.9143057191118
0.40	44.1141168112539	0.253519405725541	44.1142347515427	0.253637346014330	43.8605974055284
0.45	44.1221898113948	0.295871537559329	44.1223119768029	0.295993702967401	43.8263182738355
0.50	44.1490451990595	0.339200903633198	44.1491701373812	0.339325841954917	43.8098442954263
0.55	44.1928764694470	0.383206816612365	44.1930029212335	0.383333268398829	43.8096696528347
0.60	44.2520204197762	0.427622026642766	44.2521472977865	0.427748904653072	43.8243983931335
0.65	44.3249462928654	0.472209626808173	44.3250726635027	0.472335997445427	43.8527366660573
0.70	44.4102457320984	0.516760226121647	44.4103707986492	0.516885292672455	43.8934855059768
0.75	44.5066234874113	0.561089367704170	44.5067465749236	0.561212455216371	43.9455341197072
0.80	44.6128888164199	0.605035171630803	44.6130093580678	0.605155713278698	44.0078536447891
0.85	44.7279475289542	0.648456183604722	44.7280650537540	0.648573708404534	44.0794913453495
0.90	44.8507946271135	0.691229412164148	44.8509087488639	0.691343533914463	44.1595652149494
0.95	44.9805074965095	0.733248538547393	44.9806179038427	0.733358945880596	44.2472589579621
1	45.1162396076580	0.774422284646008	45.1163460550989	0.774528732086857	44.3418173230120

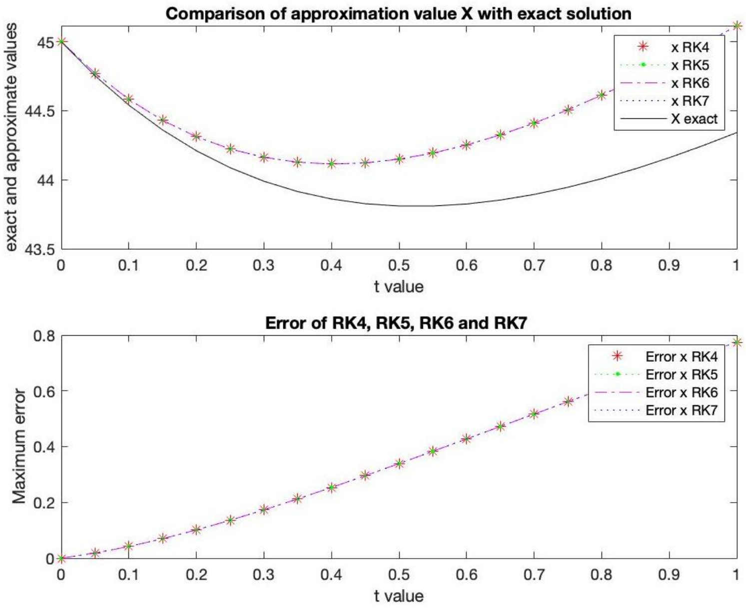

Figure1: Graphs of the numerical and exact solution of X and their absolute error expiration.

Table 3. Numerical test results for RK4 and RK5 the described techniques to evaluate Y.

t	Approximation of RK4Method h=0.05		Approximation of RK5 Method h=0.05		Y Exact
	y by RK4	Error y RK4	y by RK5	Error y RK5	
0	35	0	35	0	35
0.05	36.2163980916341	0.000215142436843507	36.2163983138182	0.000215364620885339	36.2161829491973
0.10	37.3685739556164	0.000932829386186995	37.3685743671336	0.000933240903442822	37.3676411262302
0.15	38.4607581520908	0.00224321708468977	38.4607587237407	0.00224378873455322	38.4585149350062
0.20	39.4968786851030	0.00421546155973829	39.4968793909743	0.00421616743103925	39.4926632235433
0.25	40.4805831942136	0.00690040869638864	40.4805840113573	0.00690122584006758	40.4736827855172
0.30	41.4152595062063	0.0103330026930735	41.4152604143385	0.0103339108253024	41.4049265035132
0.35	42.3040546683527	0.0145344395964102	42.3040556495858	0.0145354208294819	42.2895202287563
0.40	43.1498925756925	0.0195140902186850	43.1498936142909	0.0195151288171402	43.1303784854738
0.45	43.9554902964532	0.0252712145603837	43.9554913786130	0.0252722967201251	43.9302190818929
0.50	44.7233731920163	0.0317964878669912	44.7233743056647	0.0317976015153505	44.6915767041493
0.55	45.4558889206886	0.0390733566288191	45.4558900553034	0.0390744912436389	45.4168155640598
0.60	46.1552204079261	0.0470792411700316	46.1552215543721	0.0470803876160701	46.1081411667561
0.65	46.8233978595287	0.0557865999551979	46.8233990099098	0.0557877503362576	46.7676112595735
0.70	47.4623098886559	0.0651638693559491	47.4623110361813	0.0651650168813305	47.3971460193000
0.75	48.0737138222610	0.0751762913564562	48.0737149611249	0.0751774302203998	47.9985375309045
0.80	48.6592452476814	0.0857866405224570	48.6592463729543	0.0857877657953807	48.5734586071589
0.85	49.2204268556217	0.0969558605073289	49.2204279631522	0.0969569680377873	49.1234709951144
0.90	49.7586766315973	0.108643619409264	49.7586777179237	0.108644705735635	49.6500330121880
0.95	50.2753154440501	0.120808792419766	50.2753165063211	0.120809854690734	50.1545066516303
1	50.7715740737751	0.133409879407004	50.7715751096782	0.133410915310094	50.6381641943681

Table 4. Numerical test results for RK6 and RK7 the described techniques to evaluate Y

t	Approximation of RK6 Method h=0.05		Approximation of RK7 Method h=0.05		Y Exact
	y by RK6	Error y RK6	y by RK7	Error y RK7	
0	35	0	35	0	35
0.05	36.2163983192496	0.000215370052345065	36.2163663203892	0.000183371191937454	36.2161829491973
0.10	37.3685743771918	0.000933250961651311	37.3685149479042	0.000873821674019837	37.3676411262302
0.15	38.4607587377104	0.00224380270427105	38.4606759440733	0.00216100906719419	38.4585149350062
0.20	39.4968794082209	0.00421618467765939	39.4967768634462	0.00411363990288294	39.4926632235433
0.25	40.4805840313188	0.00690124580159335	40.4804649406598	0.00678215514257374	40.4736827855172
0.30	41.4152604365183	0.0103339330050645	41.4151276380038	0.0102011344905861	41.4049265035132
0.35	42.3040556735459	0.0145354447895372	42.3039116749118	0.0143914461554573	42.2895202287563
0.40	43.1498936396461	0.0195151541722822	43.1497406518052	0.0193621663314261	43.1303784854738
0.45	43.9554914050254	0.0252723231324978	43.9553313723860	0.0251122904931407	43.9302190818929
0.50	44.7233743328388	0.0317976286895174	44.7232089607553	0.0316322566059668	44.6915767041493
0.55	45.4558900829820	0.0390745189221633	45.4557208625960	0.0389052985361573	45.4168155640598
0.60	46.1552215823316	0.0470804155755005	46.1550498130406	0.0469086462845567	46.1081411667561
0.65	46.8233990379570	0.0557877783835039	46.8232258477252	0.0556145881516912	46.7676112595735
0.70	47.4623110641504	0.0651650448503958	47.4621374278580	0.0649914085580789	47.3971460193000
0.75	48.0737149888739	0.0751774579694100	48.0735417448857	0.0750042139812095	47.9985375309045
0.80	48.6592464003629	0.0857877932039273	48.6590742654773	0.0856156583183889	48.5734586071589
0.85	49.2204279901189	0.0969569950045184	49.2202575730489	0.0967865779344592	49.1234709951144
0.90	49.7586777443641	0.108644732176096	49.7585095578835	0.108476545695432	49.6500330121880
0.95	50.2753165321658	0.120809880535425	50.2751510040466	0.120644352416249	50.1545066516303
1	50.7715751348708	0.133410940502706	50.7714126177234	0.133248423355305	50.6381641943681

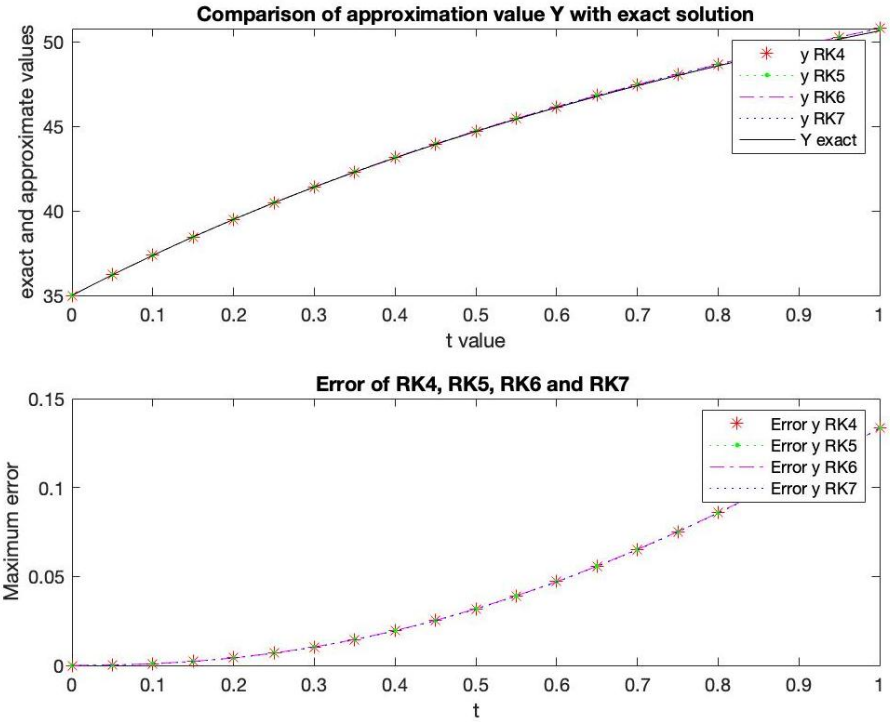

Figure 2: Graphs of the numerical and exact solution OF Y and their absolute error expiration.

Table 5. Numerical test results for RK4 and RK5 the described techniques to evaluate Z.

t	Approximation of RK4 Method h=0.05		Approximation of RK5 Method h=0.05		Z Exact
	z by RK4	Error z RK4	z by RK5	Error z RK5	
0	35	0	35	0	35
0.05	35.0075763346354	9.40259326398518e-07	35.0075762630787	8.68702620948625e-07	35.0075753943761
0.10	35.0293963204691	7.47823676050530e-06	35.0293961878694	7.34563710125258e-06	35.0293888422323
0.15	35.0641742346286	2.61532182150859e-05	35.0641740503376	2.59689271828734e-05	35.0641480814104
0.20	35.1107216876100	6.42373335040247e-05	35.1107214599315	6.40096549844316e-05	35.1106574502765
0.25	35.1679404676724	0.000129467851166964	35.1679402039665	0.000129204145260076	35.1678109998212
0.30	35.2348159106055	0.000229820562452687	35.2348156173828	0.000229527339719482	35.2345860900430
0.35	35.3104107561741	0.000373319454631371	35.3104104391811	0.000373002461543592	35.3100374367195
0.40	35.3938594554001	0.000567878362680574	35.3938591196965	0.000567542659076992	35.3932915770374
0.45	35.4843628954904	0.000821170731796883	35.4843625455196	0.000820820761056496	35.4835417247586
0.50	35.5811835116707	0.00114052402442155	35.5811831513233	0.00114016367698611	35.5800429876463
0.55	35.6836407574532	0.00153283566847762	35.6836403901241	0.00153246833937715	35.6821079217847
0.60	35.7911069069671	0.00200450777200700	35.7911065356080	0.00200413641285735	35.7891023991951
0.65	35.9030031649315	0.00256139812636746	35.9030027920975	0.00256102529239399	35.9004417668051
0.70	36.0187960616457	0.00320878528796698	36.0187956895382	0.00320841318049503	36.0155872763577
0.75	36.1379941120474	0.00395134577085088	36.1379937425522	0.00395097627568219	36.1340427662766
0.80	36.2601447194308	0.00479314160057243	36.2601443541528	0.00479277632258857	36.2553515778302
0.85	36.3848313058489	0.00573761667644135	36.3848309461433	0.00573725697083916	36.3790936891724
0.90	36.5116706525508	0.00678760056678129	36.5116702995512	0.00678724756721749	36.5048830519840
0.95	36.6403104350313	0.00794531852069724	36.6403100896752	0.00794497316460507	36.6323651165106
1	36.7704269384064	0.00921240662374601	36.7704266014578	0.00921206967514365	36.7612145317826

Table 6. Numerical test results for RK6 and RK7 the described techniques to evaluate Z.

t	Approximation of RK6 Method h=0.05		Approximation of RK7 Method h=0.05		Z Exact
	z by RK6	Error z RK6	z by RK7	Error z RK7	
0	35	0	35	0	35
0.05	35.0075762613408	8.66964725787511e-07	35.0075858961090	1.05017329303791e-05	35.0075753943761
0.10	35.0293961846498	7.34241747579745e-06	35.0294139883872	2.51461548259613e-05	35.0293888422323
0.15	35.0641740458640	2.59644536058090e-05	35.0641987146241	5.06332137177878e-05	35.0641480814104
0.20	35.1107214544062	6.40041296477989e-05	35.1107518302447	9.43799681891733e-05	35.1106574502765
0.25	35.1679401975685	0.000129197747305909	35.1679752541521	0.000164254330897506	35.1678109998212
0.30	35.2348156102706	0.000229520227534863	35.2348544398158	0.000268349772802878	35.2345860900430
0.35	35.3104104314944	0.000372994774878066	35.3104522329190	0.000414796199471823	35.3100374367195
0.40	35.3938591115583	0.000567534520911295	35.3939031797359	0.000611602698498359	35.3932915770374
0.45	35.4843625370379	0.000820812279336280	35.4844082530585	0.000866528299951597	35.4835417247586
0.50	35.5811831425925	0.00114015494617803	35.5812299649382	0.00118697729190131	35.5800429876463
0.55	35.6836403812265	0.00153245944185443	35.6836878377802	0.00157991599556340	35.6821079217847
0.60	35.7911065266153	0.00200412742019296	35.7911542074274	0.00205180823230933	35.7891023991951
0.65	35.9030027830716	0.00256101626649041	35.9030503338164	0.00260856701130763	35.9004417668051
0.70	36.0187956805324	0.00320840417464297	36.0188427965913	0.00325552023359421	36.0155872763577
0.75	36.1379937336120	0.00395096733548428	36.1380401547270	0.00399738845045050	36.1340427662766
0.80	36.2601443453170	0.00479276748680491	36.2601898507616	0.00483827293138717	36.2553515778302
0.85	36.3848309374446	0.00573724827216182	36.3848753416659	0.00578165249348928	36.3790936891724
0.90	36.5116702910170	0.00678723903293843	36.5117134397048	0.00683038772080380	36.5048830519840
0.95	36.6403100813279	0.00794496481725560	36.6403518478718	0.00798673136112882	36.6323651165106
1	36.7704265933157	0.00921206153304155	36.7704668756137	0.00925234383109341	36.7612145317826

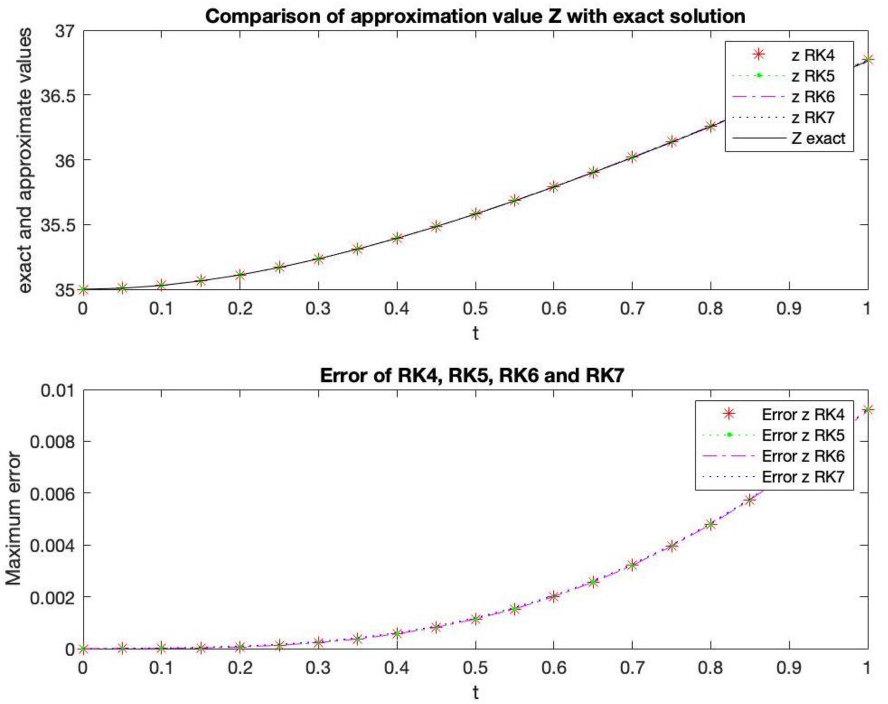

Figure 3: Graphs of the numerical and exact solution of Z and their absolute error expiration.

5. Discussion of Results:

In this paper, The acquired results are displayed in Tables 1-6 and graphically presented in Figures (1-3). The approximate solutions and maximum errors are calculated by using Matlab programming with the step size 0.05 and also compared to the exact solutions. The numerical experiments reveal that the numerical solutions converge to the exact solution and the absolute errors between the RK5 and RK6 very converge, we note that from the tables. Furthermore, we observed from Tables 2 and 6 that in solving the system home heating, the method RK6 gives a more accurate result than the method RK7 and it has been clarified with the adjacent graphs Figures 1 and 3. In addition, we observed from Tables 1 and 5 that the method RK5 gives a more accurate result than the method RK4 in solving such system.

6. Conclusion

In this paper, we have applied Runge-Kutta of fourth, fifth, sixth and seventh order methods to solve system home heating. The convergence rate of RK5, RK6 and RK7 is superior to RK4 in comparison to the exact solutions. Hence from this study it is clear that to find more accurate results higher order methods are more appropriate than lower order methods

Acknowledgements

Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project. The authors are also grateful to the anonymous referees for suggestions that have improved manuscript.

References

[1] Aliyi Koroche, K. (2021). Numerical Solution of First Order Ordinary Differential Equation by Using Runge-Kutta Method. *International Journal of Systems Science and Applied Mathematics*, *6*(1), 1-8.

[2] Anidu, A. O., Arekete, S. A., Adedayo, A. O., & Adekoya, A. O. (2015). Dynamic Computation of Runge Kutta Fourth Order Algorithm for First and Second Order Ordinary Differential Equation Using Java. *IJCSI International Journal*, *12*(3), 211–218.

[3] Babul Hossain, M. (2017). A Comparative Study on Fourth Order and Butcher's Fifth Order Runge-Kutta Methods with Third Order Initial Value Problem (IVP). *Applied and Computational Mathematics*, *6*(6), 243–253.

[4] Banu, M. S., Raju, I., & Mondal, S. (2021). A Comparative Study on Classical Fourth Order and Butcher Sixth Order Runge-Kutta Methods with Initial and Boundary Value Problems. *International Journal of Material and Mathematical Sciences*, *3*(1), 8–21.

[5] Benjamin, O. C., & Okisamen, A. B. (2020). On The Derivation and Implementation of a Diagonally Implicit Runge Kutta Method for Solving

Initial Value Problems in Ordinary Differential Equation. *IOSR Journal of Mathematics (IOSR-JM*, *16*(4), 29–39.

[6] Butcher, J. C. (2016). Numerical Methods for Ordinary Differential Equations. In *John Wiley & Sons Ltd*.

[7] Demsie Abraha, J. (2020). Comparison of Numerical Methods for System of First Order Ordinary Differential Equations. *Pure and Applied Mathematics Journal*, *9*(2), 32–36.

[8] Dhokrat, V. T. (2021). Comparative Analysis of Different Numerical Methods for the Solution of Initial Value Problems in First Order Ordinary Differential Equations. *International Journal of Trend in Scientific Research and Development (IJTSRD)*, *5*(5), 2456–6470.

[9] Elamin, M. A., & Altoum, S. H. (2022). A Comparison of Different Numerical Schemes to Solve Nonlinear First Order ODE. *Journal of Applied Mathematics and Physics*, *10*(03), 865–876.

[10] Gustafson, G. B. (2022). Differential Equations and Linear Algebra. *The American Mathematical Monthly*, *II : Chapt*(2) .

[11] Habtamu, garoma D., & Kabeto, M. J. (2017). NUMERICAL SOLUTION OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS USING FIFTH ORDER RUNGE – KUTTA METHOD * HabtamuGaromaDebela and Masho Jima Kabeto. *Asian Journal of Science and Technology*, *8*(2), 4332–4339.

[12] Hetmaniok, E., & Pleszczyński, M. (2022). Comparison of the Selected Methods Used for Solving the Ordinary Differential Equations and Their Systems. *Mathematics*, *10*(3), 1–15.

[13] Hossain, M. J., Alam, M. S., & Hossain, M. B. (2017). A study on numerical solutions of second order initial value problems (IVP) for ordinary differential equations with fourth order and Butcher?s fifth order Runge--Kutta methods. *American Journal of Computational and Applied Mathematics*, *7*(5), 129–137.

[14] Islam, M. A. (2015a). A Comparative Study on Numerical Solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE) with Euler and Runge Kutta Methods. *American Journal of Computational Mathematics*, *05*(03), 393–404.

[15] Islam, M. A. (2015b). Accurate Solutions of Initial Value Problems for Ordinary Differential Equations with the Fourth Order Runge Kutta Method. *Journal of Mathematics Research*, *7*(3), 41–45.

[16] L.Ross, S. (2021). *Introduction to ordinary differential equations*.

[17] Okeke, A. A., Tumba, P., Anorue, O. F., & Dauda, A. (2019). Analysis and Comparative Study of Numerical Solutions of Initial Value Problems (IVP) in Ordinary Differential Equations (ODE) With Euler and Runge Kutta Methods. *American Journal of Engineering Research (AJER)*, *8*(8), 6–15.

[18] Senthilnathan, C. (2018). A Numerical Solutions Of Initial Value Problems (Ivp) For Ordinary Differential Equations (Ode) With Euler And Higher Order Of Runge Kutta Methods Using Matlab. *International Journal of Engineering Science Invention (IJESI)*, *7*(4), 25–31.

[19] Suwan, I. (2014). Analytical solution to an attic, basement, and insulated main floor home heating systems. *Advanced Studies in Theoretical Physics*, *8*(10), 463–469.

[20] Trikkaliotis, G. D., & Gousidou-Koutita, M. C. (2022). Production of the Reduction Formula of Seventh Order Runge-Kutta Method with Step Size Control of an Ordinary Differential Equation. *Applied Mathematics*, *13*(04), 325–337.