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Backward doubly stochastic differential equations driven by fractional Brownian motion with integral-Lipschitz coefficients

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Abstract

This paper deals with a class of backward doubly stochastic differential equation driven by fractional Brownian motion with Hurst parameter H greater than $1/2$. We essentially establish existence and uniqueness of a solution in the case of Lipschitz coefficients and integral-Lipschitz coefficients. The stochastic integral used throughout the paper is the divergence type integral.

Keywords: backward stochastic differential equation, Lipschitz coefficients, integral-Lipschitz coefficients, Malliavin derivative and fractional Itô's formula.

AMS Subject Classification: 60H05, 60H07, 60G22.

1. INTODUCTION

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [8]. They proved the celebrated existence and uniqueness result under Lipschitz assumption. This pioneer work was extensively used in many fields like stochastic interpretation of solutions of PDEs and financial mathematics.

Few years later, several authors investigated BSDEs with respect to fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter H. Since B^H is not a semimartingale when $H \neq \frac{1}{2}$, we cannot use the beautiful classical theory of stochastic calculus to define the fractional stochastic integral. It is a significant and challenging problem to extend the results in the classical stochastic calculus to this fractional Brownian motion.

Essentially, two different types of integrals with respect to a fractional Brownian motion have been defined and studied. The first one is the pathwise Riemann-Stieltjes integral (see Young [11]). This integral has a proprieties of Stratonovich integral, which leads to difficulties in applications. The second one, introduced in Decreusefond [4] is the divergence operator (or Skorohod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. Since this stochastic integral satisfies the zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products, it was later developed by many authors. Recently, new classes of BSDEs driven by both standard and fractional Brownian motions were introduced by Fei et al [5]. They established the existence and uniqueness of solutions.

In this paper, our aim is to generalize the result established in [5] to the following equation called Backward doubly stochastic differential equations driven by fractional Brownian motion:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)ds + \int_t^T g(s, \eta_s, Y_s, Z_s)dW_t - \int_t^T Z_s dB_t^H, \quad t \in [0, T], \tag{1.1}$$

where the integral with respect to $\left(B_t^H\right)_{t\geq 0}$ is a divergence-type integral and the integral with respect to $(W_t)_{t\geq 0}$ is a standard Itô integral. Inspired by the works of Aidara and Sow [1], we prove that under integral-Lipschitz coefficients, the solution of the above Backward doubly stochastic differential equations exists uniquely.

The paper is organized as follows: In Section 2, we introduce some preliminaries, before studying the solvability of our equation under Lipschitz conditions on the generator in Section 3. Using this result, we prove existence and uniqueness of the solution with a coefficient satisfying integral-Lipschitz conditions in Section 4.

2. PRELIMINARIES

2.1. Fractional Stochastic calculus

Let Ω be a non-empty set, $\mathcal{F}$ a $\sigma-$algebra of sets Ω, $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space. Suppose that the process $\left(B_t^H\right)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be two mutually independent processes, where $\left(B_t^H\right)_{t\geq 0}$ is a one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and $(W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion.

We consider the family $(\mathscr{F}_t)_{0\leq t\leq T}$ given by

$$\mathscr{F}_t = \mathscr{F}_t^{B^H} \vee \mathscr{F}_{t,T}^{W}, \qquad 0 \leq t \leq T,$$

where for any process $\{\phi_t\}_{t\geq 0}$, $\mathscr{F}_{s,t}^{\phi} = \sigma\{\phi_r - \phi_s,\ s \leq r \leq t\} \vee \mathscr{N}$, $\mathscr{F}_t^{\phi} = \mathscr{F}_{0,t}^{\phi}$.

$\mathscr{N}$ denotes the class of $\mathbf{P}-$null sets of $\mathscr{F}$. Note that $(\mathscr{F}_t)_{0\leq t\leq T}$ does not constitute a classical filtration.

Denote $\phi(t,s) = H(2H-1)|t-s|^{2H-2},\ (t,s)\in\mathbf{R}^2$. Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle\xi,\eta\rangle_t = \int_0^t\int_0^t \phi(u,v)\xi(u)\eta(v)dudv \ \text{ and } \ \|\xi\|_t^2 = \langle\xi,\xi\rangle_t.$$

Note that, for any $t\in[0,T]$, $\langle\xi,\eta\rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle\xi_i,\xi_j\rangle_T = \delta_{ij}$. Let $\mathscr{P}_T^H$ be the set of all polynomials of fractional Brownian motion. Namely, $\mathscr{P}_T^H$ contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right)$$

where f is a polynomial function of n variables. The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right)\xi_i(s) \quad 0\leq s\leq T.$$

Similarly, we can define the Malliavin derivative D_tG of the Brownian functional

$$G(\omega) = f\left(\int_0^T \xi_1(t)dW_t, \int_0^T \xi_2(t)dW_t, \ldots, \int_0^T \xi_n(t)dW_t\right).$$

The divergence operator D^H is closable from $L^2(\Omega,\mathcal{F},\mathbf{P})$ to $L^2(\Omega,\mathcal{F},\mathbf{P},\mathcal{H})$. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of $\mathscr{P}_T^H$ with the norm

$$||F||_{1,2}^2 = \mathbf{E}|F|^2 + \mathbf{E}||D_s^H F||_T^2.$$

Now we introduce the Malliavin ϕ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t,s)D_s^H F ds.$$

Let us recall the following result which is a useful tool in our work (see [[6] , Proposition 6.25])

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathbf{P}) \longrightarrow \mathcal{H}$ be a stochastic processes such that*

$$\mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt \right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ and satisfies

$$\mathbf{E}\left(\int_0^T F_s dB_s^H \right) = 0 \quad \textit{and} \quad \mathbf{E}\left(\int_0^T F_s dB_s^H \right)^2 = \mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s ds dt \right).$$

We now give the fractional Itô formula of an Itô type process involving the stochastic integral with respect to both standard and fractional Brownian motions.

Theorem 2.2. *Let $\sigma_1, \sigma_2 \in \mathcal{H}$ be deterministic continuous functions. Denote*

$$X_t = X_0 + \int_0^t \alpha(s) ds + \int_0^t \sigma_1(s) dW_s + \int_0^t \sigma_2(s) dB_s^H,$$

where X_0 is a constant, $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)| ds < +\infty$.
Let $F(t, x)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) dX_s \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \left[\sigma_1^2(s) + \frac{d}{ds} \|\sigma_2\|_s^2 \right] ds, \quad 0 \le t \le T. \end{aligned}$$

Let us finish this section by giving a fractional Itô chain rule (see [[5], Theorem 3.2]).

Theorem 2.3. *Assume that the processes μ, α_1, α_2, μ', α_1' and α_2', satisfy*

$$\mathbf{E}\left[\int_0^T \mu(s) ds + \int_0^T \alpha_1(s) ds + \int_0^T \alpha_2(s) ds + \int_0^T \mu'(s) ds + \int_0^T \alpha_1'(s) ds + \int_0^T \alpha_2'(s) ds \right] < +\infty.$$

Suppose that $D_t \alpha_1(s)$, $D_t \alpha_1'(s)$, $\mathbb{D}_t^H \alpha_2(s)$ and $\mathbb{D}_t^H \alpha_2'(s)$ are continuously differentiable with respect to $(s, t) \in [0, T]^2$ for almost all $\omega \in \Omega$. Let X_t and Y_t be two processes satisfying

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu(s) ds + \int_0^t \alpha_1(s) dW_s + \int_0^t \alpha_2(s) dB_s^H, \qquad 0 \le t \le T, \\ Y_t &= Y_0 + \int_0^t \mu'(s) ds + \int_0^t \alpha_1'(s) dW_s + \int_0^t \alpha_2'(s) dB_s^H, \qquad 0 \le t \le T. \end{aligned}$$

If the following conditions hold:

$$\mathbf{E}\left[\int_0^T\int_0^T |D_t\alpha_1(s)|^2 dsdt\right] < +\infty, \qquad \mathbf{E}\left[\int_0^T\int_0^T |\mathbb{D}_t^H\alpha_2(s)|^2 dsdt\right] < +\infty,$$

$$\mathbf{E}\left[\int_0^T\int_0^T |D_t\alpha_1'(s)|^2 dsdt\right] < +\infty, \qquad \mathbf{E}\left[\int_0^T\int_0^T |\mathbb{D}_t^H\alpha_2'(s)|^2 dsdt\right] < +\infty,$$

then

$$\begin{aligned} X_tY_t &= X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s \\ &\quad + \int_0^t \left[\alpha_1(s)D_sY_s + \alpha_1'(s)D_sX_s + \alpha_2(s)\mathbb{D}_s^H Y_s + \alpha_2'(s)\mathbb{D}_s^H X_s\right] ds, \end{aligned}$$

which may be written formally as

$$d\left(X_tY_t\right) = X_t dY_t + Y_t dX_t + \left[\alpha_1(t)D_tY_t + \alpha_1'(t)D_tX_t + \alpha_2(t)\mathbb{D}_t^H Y_t + \alpha_2'(t)\mathbb{D}_t^H X_t\right] dt.$$

2.2. Definitions and notations

Let us consider

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma_1(s)dW_s + \int_0^t \sigma_2(s)dB_s^H, \quad 0 \le t \le T$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant,
- $b, \sigma_1, \sigma_2 \ : \ [0,T] \ \to \ \mathbb{R}$ are deterministic continuous functions, σ_1 and σ_2 are differentiable and $\sigma_1(t) \neq 0, \ \sigma_2(t) \neq 0$ such that $\hat{\sigma}_2(t) = \int_0^t \phi(t,v)\sigma_2(v)dv$ and
$$\|\sigma_2\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma_2(u)\sigma_2(v)dudv.$$

The next Remark will be useful in the sequel

Remark 2.4. *There exists a constant* C_0 *such that* $\inf_{0\le t\le T}\frac{\hat{\sigma}_2(t)}{\sigma_2(t)} \ge C_0$.

We introduce the following sets (where $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $L^2\left(\mathcal{F}_T, \mathbf{R}\right) = \left\{\xi : \Omega \to \mathbf{R} \middle| \xi \text{ is } \mathcal{F}_T - \text{measurable}, \mathbf{E}\left[|\xi|^2\right] < +\infty\right\}$;
- $\mathcal{V}_{[0,T]} = \left\{Y = \psi(\cdot,\eta) : \ \psi \in \mathscr{C}_{pol}^{1,2}([0,T]\times\mathbf{R}), \frac{\partial\psi}{\partial t} \text{ is bounded}, \ t \in [0,T]\right\},$

- $\widetilde{\mathcal{V}}^{H}_{[0,T]}$ the completion of $\mathcal{V}_{[0,T]}$ under the following norm ($\beta > 0$)

$$\|Y\|_\beta = \left(\int_0^T e^{\beta t}\mathbf{E}|Y_t|^2 dt\right)^{1/2} = \left(\int_0^T e^{\beta t}\mathbf{E}|\psi(t,\eta_t)|^2 dt\right)^{1/2}.$$

- $\mathscr{B}^2_H(0,T) = \widetilde{\mathcal{V}}^{\frac{1}{2}}_{[0,T]} \times \widetilde{\mathcal{V}}^{\frac{1}{2}}_{[0,T]}$ is a Banach space with the norm

$$\|(Y,Z)\|_{\mathscr{B}^2} = \left(\mathbf{E}\int_0^T e^{\beta t}(|Y_t|^2 + |Z_t|^2)dt\right)^{1/2}.$$

Let $f : \Omega \times [0,T] \times \mathbf{R}^3 \to \mathbf{R}$ *and* $g : \Omega \times [0,T] \times \mathbf{R}^3 \to \mathbf{R}$. We are interested in the following BDSDEs driven by fractional Brownian motion (BDSDEF):

$$Y_t = \xi + \int_t^T f(s,\eta_s,Y_s,Z_s)ds + \int_t^T g(s,\eta_s,Y_s,Z_s)dW_s - \int_t^T Z_s dB^H_s, \quad 0 \le t \le T. \tag{2.1}$$

Definition 2.5. *A triplet of processes $(Y_t,Z_t)_{0\le t\le T}$ is called a solution to BDSDEF* (2.1) *if $(Y,Z) \in \mathscr{B}^2_H(0,T)$ and satisfies* (2.1).

The next proposition will be useful in the sequel.

Proposition 2.6. *Let $(Y_t,Z_t)_{0\le t\le T}$ be a solution of the BDSDEF* (2.1). *Then, we have the stochastic representation*

$$\mathbb{D}^H_t Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} Z_t, \quad 0 \le t \le T,$$

Proof. Since (Y_t,Z_t) satisfies the BDSDEF (2.1) then we have $Y \in \widetilde{\mathcal{V}}^{1/2}_{[0,T]}$. This implies $Y = \psi(\cdot,\eta)$ where $\psi \in \mathscr{C}^{1,2}_{\text{pol}}([0,T]\times\mathbf{R})$. Note that $Y_T = \xi = \psi(T,\eta_T)$. Applying Itô's formula, we obtain

$$\begin{aligned}
\psi(t,\eta_t) - \xi = &-\int_t^T \psi'_s(s,\eta_s)ds - \int_t^T \psi'_x(s,\eta_s)b(s)ds - \int_t^T \psi'_x(s,\eta_s)\sigma_1(s)dW_s \\
&- \int_t^T \psi'_x(s,\eta_s)\sigma_2(s)dB^H_s - \frac{1}{2}\int_t^T \psi''_{xx}(s,\eta_s)\sigma_1^2(s)ds - \frac{1}{2}\int_t^T \psi''_{xx}(s,\eta_s)\frac{d}{ds}\|\sigma_2\|^2_s ds \\
= &-\int_t^T \left(\psi'_s(s,\eta_s) + \psi'_x(s,\eta_s)b(s) + \frac{1}{2}\psi''_{xx}(s,\eta_s)\sigma_1(s) + \frac{1}{2}\psi''_{xx}(s,\eta_s)\frac{d}{ds}\|\sigma_2\|^2_s\right)ds \\
&- \int_t^T \psi'_x(s,\eta_s)\sigma_1(s)dW_s - \int_t^T \psi'_x(s,\eta_s)\sigma_2(s)dB^H_s.
\end{aligned}$$

Using eq. (2.1), we derive that the hand right side is equal to

$$\int_t^T f(s,\eta_s,Y_s,Z_s)ds + \int_t^T g(s,\eta_s,Y_s,Z_s)dW_s - \int_t^T Z_s dB^H_s, \quad 0 \le t \le T.$$

We deduce that $g\left(s, \eta_s, Y_s, Z_s\right)=\psi_x^{\prime}(t, \eta_t) \sigma_1(t)$ and $Z_t=\psi_x^{\prime}(t, \eta_t) \sigma_2(t)$. Then we can write

$$
\begin{aligned}
\mathbb{D}_t^H Y_t & =\int_0^T \phi(t-r) D_r^H \psi(t, \eta_t) d r \\
& =\psi_x^{\prime}(t, \eta_t) \int_0^T \phi(t-r) \sigma_2(r) d r \\
& =\widehat{\sigma}_2(t) \psi_x^{\prime}(t, \eta_t) \\
& =\frac{\widehat{\sigma}_2(t)}{\sigma_2(t)} Z_t .
\end{aligned}
$$

□

We are now in position to move on to study our main subject. First we investigate the case of Lipschitz coefficients.

3. THE CASE OF LIPSCHITZ COEFFICIENTS

We assume that the coefficients f and g of the BDSDEF are continuous functions and satisfy the following assumption **(H1)**:

(H1.1): There exists a constant $L>0$ such that for any $(\omega, t) \in \Omega \times[0, T]$, $x \in \mathbf{R}$, $(y, y^{\prime}) \in \mathbf{R}^2$ and $(z, z^{\prime}) \in \mathbf{R}^2$; we have

$$
\begin{aligned}
\Big|f\left(t, x, y, z\right)-f\left(t, x, y^{\prime}, z^{\prime}\right)\Big|^2 \vee \Big|g\left(t, x, y, z\right)-g\left(t, x, y^{\prime}, z^{\prime}\right)\Big|^2 \\
\leq L\Big(|y-y^{\prime}|^2+|z-z^{\prime}|^2\Big)
\end{aligned}
$$

(H1.2): There exists $\beta>0$ and a function h with bounded derivative $\xi=h(\eta_T)$, $\mathbf{E}\left(e^{\beta T}|\xi|^2\right)<+\infty$

The main result of this section is the following theorem:

Theorem 3.1. *Let the assumptions* **(H1)** *be satisfied. Then the BDSDEF* (2.1) *has a unique solution* (Y, Z) *in the space* $\mathscr{B}^2([0, T], \mathbf{R})$.

Proof. Let us consider the mapping $\Psi: \mathscr{B}^2([0, T], \mathbf{R}) \rightarrow \mathscr{B}^2([0, T], \mathbf{R})$ driven by $(U, V) \longmapsto \Psi(U, V)=(Y, Z)$.

We will show that the mapping Ψ is a contraction, where (Y, Z) is a solution of the following BDSDEF:

$$Y_t = \int_t^T f(s, \eta_s, U_s, V_s)ds - \int_t^T g(s, \eta_s, U_s, V_s)dW_s - \int_t^T Z_s dB_s^H, \quad t \in [0, T]. \tag{3.1}$$

Let us define for a process $\delta \in \{Y, Z, U, V\}$, $\overline{\delta} = \delta - \delta'$ and the functions

$$\Delta f(t) = f(t,\eta_t, U_t, V_t) - f(t,\eta_t, U_t', V_t').$$

$$\Delta g(t) = g(t,\eta_t, U_t, V_t) - g(t,\eta_t, U_t', V_t').$$

Then, the couple $(\overline{Y}, \overline{Z})$ solves the BDSDEF

$$\overline{Y}_t = \int_t^T \Delta f(s)ds - \int_t^T \Delta g(s)dW_s - \int_t^T \overline{Z}_s dB_s^H, \quad t \in [0, T]. \tag{3.2}$$

By the fractional Itô chain rule, we have

$$\begin{aligned}|\overline{Y}_t|^2 &= 2\int_t^T \overline{Y}_s \Delta f(s)ds + 2\int_t^T \Delta g(s) D_s \overline{Y}_s ds - 2\int_t^T \overline{Z}_s \mathbb{D}_s^H \overline{Y}_s ds \\ &\quad + 2\int_t^T \overline{Y}_s \Delta g(s)dW_s - 2\int_t^T \overline{Y}_s \overline{Z}_s dB_s^H.\end{aligned}$$

Applying Itô formula to $e^{\beta t}|\overline{Y}_t|^2$, we obtain that

$$\begin{aligned}e^{\beta t}|\overline{Y}_t|^2 &= 2\int_t^T e^{\beta s}\overline{Y}_s \Delta f(s)ds + 2\int_t^T e^{\beta s}\Delta g(s) D_s \overline{Y}_s ds - 2\int_t^T e^{\beta s}\overline{Z}_s \mathbb{D}_s^H \overline{Y}_s ds \\ &\quad + 2\int_t^T e^{\beta s}\overline{Y}_s \Delta g(s)dW_s - 2\int_t^T e^{\beta s}\overline{Y}_s \overline{Z}_s dB_s^H - \beta\int_t^T e^{\beta s}|\overline{Y}_s|^2 ds.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|\overline{Y}_t|^2\right] &+ \beta\mathbf{E}\left[\int_t^T e^{\beta s}|\overline{Y}_s|^2 ds\right] + 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Z}_s \mathbb{D}_s^H \overline{Y}_s ds\right] \\ &= 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Y}_s \Delta f(s)ds\right] + 2\mathbf{E}\left[\int_t^T e^{\beta s}\Delta g(s) D_s \overline{Y}_s ds\right]\end{aligned}$$

It is know that, by Proposition 2.6, $\mathbb{D}_s^H \overline{Y}_s = \frac{\hat{\sigma}_2(s)}{\sigma_2(s)}\overline{Z}_s$. Then, we have

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|\overline{Y}_t|^2\right] &+ \beta\mathbf{E}\left[\int_t^T e^{\beta s}|\overline{Y}_s|^2 ds\right] + 2\mathbf{E}\left[\int_t^T e^{\beta s}\frac{\hat{\sigma}_2(s)}{\sigma_2(s)}|\overline{Z}_s|^2 ds\right] \\ &= 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Y}_s \Delta f(s)ds\right] + 2\mathbf{E}\left[\int_t^T e^{\beta s}|\Delta g(s)|^2 ds\right]\end{aligned} \tag{3.3}$$

Using standard estimates $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$ (where $a, b, \epsilon > 0$) and asumption **(H1.1)**, we obtain that

$$\mathbf{E}\left[e^{\beta t}|\overline{Y}_t|^2\right] + \beta \mathbf{E}\left[\int_t^T e^{\beta s}|\overline{Y}_s|^2 ds\right] + 2\mathbf{E}\left[\int_t^T e^{\beta s}\frac{\hat{\sigma}_2(s)}{\sigma_2(s)}|\overline{Z}_s|^2 ds\right]$$
$$\leq \frac{2L}{\epsilon}\mathbf{E}\int_t^T e^{\beta s}|\overline{Y}_s|^2 ds + \frac{(4L+\epsilon)}{2}\mathbf{E}\left[\int_t^T e^{\beta s}(|\overline{U}|^2 + |\overline{V}|^2)ds\right]$$

Taking β such that $\beta \geq 2C_0 + \frac{2L}{\epsilon}$, we get

$$\mathbf{E}\int_0^T e^{\beta s}\left(|\overline{Y}_s|^2 + |\overline{Z}_s|^2\right) ds \leq \left(\frac{4L+\epsilon}{4C_0}\right)\mathbf{E}\int_0^T e^{\beta s}\left|\overline{U}_s|^2 + |\overline{V}_s|^2\right) ds. \tag{3.4}$$

Hence, if we choose $\epsilon = \epsilon_0$ satisfying $C_0 = \frac{\epsilon_0 + 4L}{3}$, we have

$$\mathbf{E}\int_0^T e^{\beta s}\left(|\overline{Y}_s|^2 + |\overline{Z}_s|^2\right) ds \leq \frac{3}{4}\mathbf{E}\int_0^T e^{\beta s}\left|\overline{U}_s|^2 + |\overline{V}_s|^2\right) ds. \tag{3.5}$$

Thus, the mapping $(U, V) \longmapsto \Psi(U, V) = (Y, Z)$ determined by the fractional BDSDEs (2.1) is a strict contraction on $\mathscr{B}^2([0, T], \mathbf{R})$. Using the fixed point principle, we deduce the solution to the fractional BDSDEs (2.1) that exists uniquely. This completes the proof. $\square$

4. THE CASE OF INTEGRAL-LIPSCHITZ COEFFICIENTS

We assume that the coefficients f and g of the BDSDEFs are continuous functions and satisfy the following assumption $(\mathbf{H2})$:

(H2.1) : There exists $K > 0$ s.t. for $0 \leq t \leq T$, $(y, y') \in \mathbf{R}^2$, $(z, z') \in \mathbf{R}^2$, $x \in \mathbf{R}$,

$$|f(t, x, y, z) - f(t, x, y', z')|^2 \vee |g(t, x, y, z) - g(t, x, y', z')|^2 \leq \rho(t, |y - y'|^2) + K|z - z'|^2$$

where $\rho(t, v) : [0, T] \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies

- For fixed $t \in [0, T]$, $\rho(t, \cdot)$ is a continuous, concave and nondecreasing s.t.

$$\rho(t, 0) = 0, \quad \text{and} \quad \forall \alpha > 0, \;\; \alpha\rho(t, v) = \rho(t, \alpha v).$$

- The ordinary differential equation

$$v'(t) = -\rho(t, v(t)), \quad v(T) = 0, \tag{4.1}$$

has a unique solution $v(t) = 0, \quad 0 \leq t \leq T.$

- There exists two continuous and non-negative functions a and b such that

$$\rho(t,v) \leq a(t) + b(t)v \quad \text{and} \quad \int_0^T [a(t)+b(t)]dt < +\infty.$$

(H2.2) : the integrability condition holds

$$\mathbf{E}\bigg(|\xi|^2 + \int_0^T |f(s,\eta_s,0,0)|^2 ds\bigg) < +\infty.$$

Our strategy in the proof of existence of solutions of the fractional BDSDEs (2.1) is to use the Picard approximate sequence. To this end, we consider now the sequence $(Y_t^n, Z_t^n)_{n\geq 0}$ given by

$$\begin{cases} Y_t^0 = 0; \\ Y_t^n = \xi + \displaystyle\int_t^T f(s,\eta_s,Y_s^{n-1},Z_s^n)ds + \int_t^T g(s,\eta_s,Y_s^{n-1},Z_s^n)dW_s - \int_t^T Z_s^n dB_s^H, \quad n\geq 1. \end{cases} \tag{4.2}$$

In what follows, we establish two results which will be useful in the sequel.

Lemma 4.1. *Let the assumption* **(H2)** *be satisfied. Then, for* $n,\ m\geq 1$ *we have*

$$\mathbf{E}\left(|Y_t^{n+m} - Y_t^n|^2\right) \leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0-2K-1}} \int_t^T \rho\left(s, \mathbf{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2]\right) ds, \quad 0\leq t\leq T$$

Proof. Let us define for a process $\delta\in\{Y,Z\},\ n,m\geq 1,\ \overline{\delta}^{n,m} = \delta^{n+m} - \delta^n$ and the functions

$$\begin{aligned} \Delta f^{(n,m)}(s) &= f(s,\eta_s,Y_s^{n+m-1},Z_s^{n+m}) - f(s,\eta_s,Y_s^{n-1},Z_s^n), \\ \Delta g^{(n,m)}(s) &= g(s,\eta_s,Y_s^{n+m-1},Z_s^{n+m}) - g(s,\eta_s,Y_s^{n-1},Z_s^n). \end{aligned}$$

It is easily seen that the pair of processes $(\overline{Y}_t^{n,m}, \overline{Z}_t^{n,m})_{0\leq t\leq T}$ solves the BDSDEF

$$\overline{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s)ds + \int_t^T \Delta g^{(n,m)}(s)dW_s - \int_t^T \overline{Z}_s^{n,m} dB_s^H, \quad 0\leq t\leq T.$$

Applying again Itô's formula to $|\overline{Y}_t^{n,m}|^2$, we obtain for $n,m\geq 1$ and $0\leq t\leq T$,

$$\begin{aligned} |\overline{Y}_t^{n,m}|^2 &= 2\int_t^T \overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds + \int_t^T \overline{Y}_s^{n,m}\Delta g^{(n,m)}(s)dW_s + 2\int_t^T |\Delta g^{(n,m)}(s)|^2 ds \\ &\quad - 2\int_t^T \overline{Y}_s^{n,m}\overline{Z}_s^{n,m} dB_s^H - 2\int_t^T \mathbb{D}_s^H \overline{Y}_s^{n,m}\overline{Z}_s^{n,m} ds. \end{aligned} \tag{4.3}$$

Hence we deduce from (4.3)

$$\mathbf{E}\bigg(|\overline{Y}_t^{n,m}|^2 + 2\int_t^T \mathbb{D}_s^H \overline{Y}_s^{n,m} \overline{Z}_s^{n,m} ds\bigg)$$
$$= 2\mathbf{E}\int_t^T \overline{Y}_s^{n,m} \Delta f^{(n,m)}(s) ds + 2\int_t^T |\Delta g^{(n,m)}(s)|^2 ds \qquad (4.4)$$

Using standard estimates and assumption (H2.1), we have

$$2\overline{Y}_s^{n,m} \Delta f^{(n,m)}(s) \leq \epsilon \left|\overline{Y}_s^{n,m}\right|^2 + \frac{1}{\epsilon}\rho\left(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\right) + \frac{K}{\epsilon}|\overline{Z}_s^{n,m}|^2,$$

and

$$2\left|\Delta g^{(n,m)}(s)\right|^2 \leq 2\rho\left(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\right) + 2K|\overline{Z}_s^{n,m}|^2,$$

Putting pieces together, we deduce from eq. (4.4),

$$\mathbf{E}\bigg(|\overline{Y}_t^{n,m}|^2 + 2C_0\int_t^T |\overline{Z}_s^{n,m}|^2 ds\bigg) \qquad (4.5)$$
$$\leq \int_t^T \mathbf{E}\left(\epsilon\left|\overline{Y}_s^{n,m}\right|^2 + \frac{(1+2\epsilon)}{\epsilon}\rho\left(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\right) + \frac{K(1+2\epsilon)}{\epsilon}|\overline{Z}_s^{n,m}|^2\right) ds$$

which implies

$$\mathbf{E}\bigg(|\overline{Y}_t^{n,m}|^2 + \frac{2C_0\epsilon - K(1+2\epsilon)}{\epsilon}\int_t^T |\overline{Z}_s^{n,m}|^2 ds + \bigg) \qquad (4.6)$$
$$\leq \epsilon\mathbf{E}\int_t^T |\overline{Y}_s^{n,m}|^2 ds + \frac{(1+2\epsilon)}{\epsilon}\int_t^T \mathbf{E}[\rho\left(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\right)] ds.$$

Choosing $\epsilon = \frac{K}{2C_0 - 2K - 1}$, we obtain

$$\mathbf{E}\bigg(|\overline{Y}_t^{n,m}|^2 + \int_t^T |\overline{Z}_s^{n,m}|^2 ds + \bigg) \qquad (4.7)$$
$$\leq \frac{K}{2C_0 - 2K - 1}\mathbf{E}\int_t^T |\overline{Y}_s^{n,m}|^2 ds + \frac{(2C_0 - 1)}{K}\int_t^T \mathbf{E}[\rho\left(s, |Y_s^{n+m-1} - Y_s^{n-1}|^2\right)] ds.$$

Applying Gronwall's lemma and Jensen inequality (since $\rho(t,\cdot)$ is concave), we obtain

$$\mathbf{E}\left(|\overline{Y}_t^{n,m}|^2\right) \leq \frac{(2C_0 - 1)}{K} e^{\frac{K(T-t)}{2C_0 - K - 1}} \int_t^T \rho\left(s, \mathbf{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2]\right) ds$$
$$\leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0 - 2K - 1}} \int_t^T \rho\left(s, \mathbf{E}[|Y_s^{n+m-1} - Y_s^{n-1}|^2]\right) ds.$$

$\square$

Lemma 4.2. *Let the assumption* **(H2)** *be satisfied. Then there exist* $\Gamma > 0$ *and* $0 \le T_1 < T$ *not depending on* ξ *and such that*

$$\forall n \geq 1, \quad \mathbf{E}\left(|Y_t^n|^2\right) \leq \Gamma, \qquad T_1 \leq t \leq T.$$

Proof. Using the same method as in the the proof of Lemma 4.1, we have

$$\begin{aligned} |Y_t^n|^2 = |\xi|^2 &+ 2\int_t^T Y_s^n f(s,\eta_s,Y_s^{n-1},Z_s^n)ds + 2\int_t^T Y_s^n g(s,\eta_s,Y_s^n)dW_s \\ &- 2\int_t^T Y_s^n Z_s^n dB_s^H - 2\int_t^T \mathbb{D}_s^H Y_s^n Z_s^n ds + 2\int_t^T |g(s,\eta_s,Y_s^{n-1},Z_s^n)|^2 ds. \end{aligned} \tag{4.8}$$

Assumption (H2.1) and the same computations as before imply

$$\mathbf{E}(|Y_t^n|^2) \leq \frac{2C_0}{K}\left[\mu_t + \mathbf{E}\int_t^T \rho(s,|Y_s^{n-1}|^2)ds\right] + \frac{2K}{2C_0-2K-1}\mathbf{E}\int_t^T |Y_s^n|^2 ds \tag{4.9}$$

where

$$\mu_t = \left[\frac{K}{2C_0}\mathbf{E}[|\xi|^2] + \mathbf{E}\int_t^T |f(s,\eta_s,0,0)|^2 ds\right].$$

Applying once again Gronwall's lemma, we deduce that

$$\mathbf{E}(|Y_t^n|^2) \leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0-2K-1}}\left[\mu_t + \mathbf{E}\int_t^T \rho(s,|Y_s^{n-1}|^2)ds\right] \tag{4.10}$$

Let $\overline{T}_1 = \max\left\{T - \dfrac{(2C_0-2K-1)\ln\left(\frac{K}{2C_0}\right)}{2K}, 0\right\}.$

Then we have

$$\mathbf{E}\left(|Y_t^n|^2\right) \leq \mu_t + \int_t^T \rho\left(s, \mathbf{E}[|Y_s^{n-1}|^2]\right)ds, \quad \overline{T}_1 \leq t \leq T. \tag{4.11}$$

Define

$$\Gamma = \mu_0 + \int_0^T a(s)\,ds \geq 0. \tag{4.12}$$

Arguing as in [10, Lemma 2], we choose $\hat{T}_1$ such that

$$\mu_0 + \int_t^T \rho(s,\Gamma)\,ds \leq \Gamma, \quad t \in [\hat{T}_1, T]. \tag{4.13}$$

Define $T_1 = \max\left\{\overline{T}_1, \hat{T}_1\right\}$. By inequalities (4.11) and (4.13), we have for $T_1 \leq t \leq T$,

$$\begin{aligned}
\mathbf{E}\left[\left|Y_t^1\right|^2\right] &\leq \mu_t + \int_t^T \rho\,(s,0)\,ds \leq \mu_0 \leq \Gamma,\\
\mathbf{E}\left[\left|Y_t^2\right|^2\right] &\leq \mu_t + \int_t^T \rho\left(s, \mathbf{E}[\left|Y_s^1\right|^2]\right) ds \leq \mu_0 + \int_t^T \rho\,(s,\Gamma)\,ds \leq \Gamma,\\
\mathbf{E}\left[\left|Y_t^3\right|^2\right] &\leq \mu_t + \int_t^T \rho\left(s, \mathbf{E}[\left|Y_s^2\right|^2]\right) ds \leq \mu_0 + \int_t^T \rho\,(s,\Gamma)\,ds \leq \Gamma.
\end{aligned}$$

Hence by induction, one can prove that for all $n \geq 1, t \in [T_1, T]$

$$\mathbf{E}\left[|Y_t^n|^2\right] \leq \Gamma. \tag{4.14}$$

□

We are now in position to prove our main result

Theorem 4.3. *Let the assumption* **(H2)** *be satisfied. Then, the BDSDEF* (2.1) *has a unique solution* (Y, Z) *in the space* $\mathscr{B}^2([0,T], \mathbf{R})$.

Proof. **Existence**. Let us consider the sequence $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_0(t) = \int_t^T \rho\,(s,\Gamma)\,ds, \qquad \varphi_{n+1}(t) = \int_t^T \rho\,(s, \varphi_n(s))\,ds.$$

Then for all $t \in [T_1, T]$, from the proof of Lemma 4.2 we have the following inequalities

$$\begin{aligned}
\varphi_0(t) &= \int_t^T \rho\,(s,\Gamma)\,ds \leq \Gamma,\\
\varphi_1(t) &= \int_t^T \rho\,(s,\varphi_0(s))\,ds \leq \int_t^T \rho\,(s,\Gamma)\,ds = \varphi_0(t) \leq \Gamma,\\
\varphi_2(t) &= \int_t^T \rho\,(s,\varphi_1(s))\,ds \leq \int_t^T \rho\,(s,\varphi_0(s))\,ds = \varphi_1(t) \leq \Gamma.
\end{aligned}$$

By induction, one can prove that for all $n \geq 1$, $\varphi_n(t)$ satisfies

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq ... \leq \varphi_1(t) \leq \varphi_0(t) \leq \Gamma.$$

Then $\{\varphi_n(t), t \in [T_1, T]\}_{n\geq 1}$ is uniformly bounded. For all $n \geq 1$ and $t_1, t_2 \in [T_1, T]$, we obtain

$$|\varphi_n(t_1) - \varphi_n(t_2)| = \left|\int_{t_1}^{t_2} \rho\,(s, \varphi_{n-1}(s))\,ds\right| \leq \left|\int_{t_1}^{t_2} \rho\,(s,\Gamma)\,ds\right|.$$

Since, for fixed v, $\int_0^T \rho(s,v)\,ds < +\infty$. Therefore

$$\sup_n |\varphi_n(t_1) - \varphi_n(t_2)| \to 0 \quad as \quad |t_1 - t_2| \to 0,$$

which means that $\{\varphi_n(t), t \in [T_1, T]\}_{n\geq 1}$ is an equicontinuous family of functions. Therefore, by the Ascoli-Arzela theorem, we can define $\varphi(t)$ the limit function of $(\varphi_n(t))_{n\geq 1}$. By (4.1), we have $\varphi(t) = 0, t \in [T_1, T]$.
Now for all $t \in [T_1, T]$, $n, m \geq 1$ in view of Lemma 4.1 and Lemma 4.2, we have the inequalities

$$\begin{aligned}
\mathbf{E}\left[\left|Y_t^{1+m} - Y_t^1\right|^2\right] &\leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0-2K-1}} \int_t^T \rho\left(s, \mathbf{E}[|Y_s^m|^2]\right) ds \leq \int_t^T \rho(s,\Gamma)\,ds = \varphi_0(t) \leq \Gamma,\\
\mathbf{E}\left[\left|Y_t^{2+m} - Y_t^2\right|^2\right] &\leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0-2K-1}} \int_t^T \rho\left(s, \mathbf{E}[\left|Y_s^{1+m} - Y_s^1\right|^2]\right) ds \leq \varphi_1(t) \leq \Gamma,\\
\mathbf{E}\left[\left|Y_t^{3+m} - Y_t^3\right|^2\right] &\leq \frac{2C_0}{K} e^{\frac{2K(T-t)}{2C_0-2K-1}} \int_t^T \rho\left(s, \mathbf{E}[\left|Y_s^{2+m} - Y_s^2\right|^2]\right) ds \leq \varphi_2(t) \leq \Gamma.
\end{aligned}$$

By induction, we can derive that

$$\forall n \geq 1, \quad \mathbf{E}\left[\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \varphi_{n-1}(t), \qquad T_1 \leq t \leq T$$

which implies in particular

$$\sup_{T_1\leq t\leq T} \mathbf{E}\left[\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \sup_{T_1\leq t\leq T} \varphi_{n-1}(t) = \varphi_{n-1}(T_1) \to 0, \text{ as } \quad n \to \infty.$$

We see immediately that (Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathscr{B}^2([T_1, T], \mathbf{R})$.

Then there exists a pair of processes $(Y, Z) \in \mathscr{B}^2([T_1, T], \mathbf{R})$ being a limit of $(Y^n, Z^n)_{n\geq 1}$ i.e.

$$\lim_{n\to+\infty} \mathbf{E} \int_0^T |Y_s^n - Y_s|^2 ds = 0 \quad \text{and} \quad \lim_{n\to+\infty} \mathbf{E} \int_0^T |Z_s^n - Z_s|^2 ds = 0.$$

It remains to show that the pair of processes (Y, Z) satisfies eq.(2.1) on the interval $[T_1, T]$. We have for any $t \in [T_1, T]$,

$$\begin{aligned}
&\lim_{n\to+\infty} \bigg(-Y_t^n + \xi + \int_t^T f(s, \eta_s, Y_s^{n-1}, Z_s^n) ds + \int_t^T g(s, \eta_s, Y_s^{n-1}, Z_s^n) dW_s \bigg)\\
&\qquad = -Y_t + \xi + \int_t^T f(s, \eta_s, Y_s, Z_s) ds + \int_t^T g(s, \eta_s, Y_s, Z_s) dW_s := \Phi(t) \text{ in } \ L^2(\Omega, \mathcal{F}, \mathbf{P}).
\end{aligned}$$

We show that $(Y, Z)_{0\leq t\leq T}$ satisfies eq.(2.1) on $[T_1, T]$. Note that from Lemma 4.2, T_1 does not depend on the final value ξ. Hence one we can deduce by iteration the existence on $[T-\tau(T-T_1), T]$, for each τ, and therefore the existence on the whole $[0,T]$.

Uniqueness. Let (Y,Z) and $(\widetilde{Y},\widetilde{Z})$, be two solutions of eq.(2.1). Define $\overline{\delta}=\delta-\widetilde{\delta}$ for $\delta\in\{Y,Z\}$. By the Itô formula, we have

$$\mathbf{E}\left(|\overline{Y_t}|^2+2C_0\int_t^T|\overline{Z_s}|^2ds\right)\leq\int_t^T\mathbf{E}\left(\epsilon|\overline{Y_s}|^2+\frac{(1+\epsilon)}{\epsilon}\rho\left(s,|\overline{Y_s}|^2\right)+\frac{K(1+\epsilon)}{\epsilon}|\overline{Z_s}|^2\right)ds. \tag{4.15}$$

Using the same computations as in Lemma 4.1, we deduce that

$$\mathbf{E}\left[|\overline{Y_t}|^2\right]\leq\frac{2C_0}{K}e^{\frac{2K(T-t)}{2C_0-K-1}}\int_t^T\rho\left(s,\mathbf{E}[|\overline{Y_s}|^2]\right)ds,\quad 0\leq t\leq T.$$

Define $\delta=\dfrac{(2C_0-K-1)\ln\left(K/2C_0\right)}{2K}$ and $N=[T/\delta]+1$. If $(t_j)_{0\leq j\leq N}$ denotes the uniform subdivision of $[0,T]$ given by $T_0=0,\quad T_j=T-(N-j)\delta,\ j\ \geq 1$, we have

$$\mathbf{E}\left[|\overline{Y_t}|^2\right]\leq\int_t^T\rho\left(s,\mathbf{E}[|\overline{Y_s}|^2]\right)ds,\quad T_{N-1}\leq t\leq T.$$

From the comparison theorem of ODEs, we deduce that $\mathbf{E}\left(|\overline{Y_t}|^2\right)\leq r(t)$, where $r(t)$ is the maximum of solution of equation (4.1) on $[T_{N-1},T]$. As a consequence, we have $Y_t=\widetilde{Y}_t$ for $t\in[T_{N-1},T]$. From (4.15), we deduce $Z_t=\widetilde{Z}_t$ for $t\in[T_{N-1},T]$. Then we can use the same argument to prove that uniqueness of the solution also holds on $[T_j,T_{j+1}],\ \ j=0,\ldots,N-2$. This completes the proof. □

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