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Weakly 2-Absorbing Filters in ADLs

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Abstract

In this paper, the concepts of 2-absorbing filter and weakly 2-absorbing filter in an almost distributive lattice are introduced and obtain certain results of these. Further, the lattice epimorphic images and pre image of weakly (2-absorbing filter) in an ADL is discussed.
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1. INTRODUCTION

Several researchers introduced and analyzed the 2-absorbing and weakly 2-absorbing property in rings (especially in commutative rings), lattices, semi-groups, and modules. Ever since in 2007, Ayami Badawi [3] was introduced the concepts of 2-absorbing ideals on a commutative rings, which is a generalization of prime ideals and some properties of these were studied. Following that, other researchers worked on 2-absorbing ideals in semirings (J.N. Chuadhari [6]), on n-absorbing ideals of commutative rings (D.F. Anderson and A. Badami [1]), on the 2-absorbing ideals (Sh. Payrovi and S. Babali [8]), on 2-absorbing ideals and weakly 2-absorbing ideals of lattices (M.P. Wasadikar and K.T. Gaikevad [10]), on 2-absorbing filter of lattice (S.E. Atani and M.S. Bazari [2]), on weakly 1-absorbing primary ideals of commutative rings (A. Badawi and E. Y. Celikel [5]), on 1-absorbing ideals of commutative rings (A. Yassine, M.J. Nikmehr and R. Nikandish [11]), on weakly 2-absorbing ideals of commutative rings (A. Badawi and A.Y. Darani [4]) and prime and weakly prime ideals in semirings (M.K. Dubey [7]).

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In 1981, Swamy and Rao [9] was introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra and which is an algebra $(A, \wedge, \vee, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of the operations $\wedge$ and $\vee$. It is known that, in any ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. It is well known that, for any lattice $(L, \wedge, \vee)$, interchanging the operations $\wedge$ and $\vee$ again yields a lattice, known as the dual of L. An ideal of the dual $(L, \vee, \wedge)$ is known as a filter of a lattice $(L, \wedge, \vee)$. Unlike the case of a lattice, by interchanging the operations $\wedge$ and $\vee$ in an ADL $(A, \wedge, \vee, 0)$, we do not get an ADL again. In this paper, we introduce and study 2-absorbing filter in an ADL which need not be a prime filter in general. Essentially, it is proved that a proper filter J of an ADL is 2-absorbing filter if and only if $A - J$ is a 2-absorbing ideal of an ADL. Also, it shown that $J \times A$ and $A \times J$ are 2-absorbing filters if J is a 2-absorbing filter in an ADL. Further, we introduce the concept of n-absorbing filters and their properties. On the other hand, we introduce the concept of weakly 2-absorbing filter in an ADL and obtain the inter relationship between this and 2-absorbing filters. It is proved that if $J \times F$ is a weakly 2-absorbing filter, then J and F are 2-absorbing filters and the converse of this is not true. Also, it shown that J is a 2-absorbing filter in an ADL A if and only if $J \times A$ is a weakly(2-absorbing filter) of $A \times A$. Finally, we prove that the image and inverse image of a 2-absorbing filter (resp. weakly 2-absorbing filter) of an ADL is again a 2-absorbing filter (resp. weakly 2-absorbing filter) of an ADL.

Throughout this paper, A stands for an ADL $A = (A, \wedge, \vee, 0)$ with a maximal element.

2. PRELIMINARIES

In this section, we recall definitions and important results from [9].

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

1. $0 \wedge a = 0$
2. $a \vee 0 = a$
3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
5. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

6. $(a \vee b) \wedge b = b$

Each of the axioms (1) through (6) above is independent from the others. The element 0 is called the zero element.

Any bounded below distributive lattice is an ADL.

Example 2.2. Let X be a non-empty set. Fix an arbitrary element $x_0 \in X$. For any $x, y \in X$, define $\wedge$ and $\vee$ on X by,

$$x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases} \quad \text{and} \quad x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element. This ADL is called the **discrete ADL**.

Theorem 2.3. Let $(A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, we have
(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$
(4) $a \vee (b \wedge a) = a$
(5) $a \wedge (a \vee b) = a$
(6) $a \wedge b = a \Leftrightarrow a \vee b = b$
(7) $a \wedge b = b \Leftrightarrow a \vee b = a$
(8) $a \vee (b \vee a) = a \vee b.$

Definition 2.4. Let $(A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define
$a \leq b$ if $a = a \wedge b$ (equivalently $a \vee b = b$).
Then $\leq$ is a **partial order** on A.

Theorem 2.5. The following hold good for any elements a, b, c and d of an ADL $(A, \wedge, \vee, 0)$.
(1) $a \wedge b \leq b \leq b \vee a$
(2) $a \leq b \Rightarrow a \wedge b = a = b \wedge a$ and $a \vee b = b = b \vee a$
(3) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(4) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative on A)
(5) $a \wedge b \wedge c = b \wedge a \wedge c$
(6) The set $\{x \wedge a : x \in A\} = \{y \in A : y \leq a\}$ is a bounded distributive lattice under the induced operations $\wedge$ and $\vee$ with 0 as the smallest element and a as the largest element
(7) $a \vee b = b \vee a$ whenever $a \wedge b = 0$

(8) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(9) $a \leq b \Rightarrow a \wedge c \leq b \wedge c$ and $c \wedge a \leq c \wedge b$
(10) $a \leq b \Rightarrow c \vee a \leq c \vee b$
(11) $\big(a \vee (b \vee c)\big) \wedge d = \big((a \vee b) \vee c\big) \wedge d.$

Theorem 2.6. For any elements a and b of an ADL $(A, \wedge, \vee, 0)$, the following are equivalent to each other.
(1) $(a \wedge b) \vee a = a$
(2) $a \wedge (b \vee a) = a$
(3) $a \wedge b = b \wedge a$
(4) $a \vee b = b \vee a$
(5) $Sup\{a, b\}$ exists in $(A, \leq)$ and is equal to $a \vee b$
(6) There exists $x \in A$ such that $a \leq x$ and $b \leq x$
(7) $inf\{a, b\}$ exists in $(A, \leq)$ is equal to $a \wedge b$.

Theorem 2.7. The following are equivalent to each other for any ADL A.
(1) $a \wedge b = b \wedge a$ for all $a, b \in A$
(2) $a \vee b = b \vee a$ for all $a, b \in A$
(3) $(A, \wedge, \vee)$ is a distributive lattice bounded below
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c \in A$
(5) $b \wedge (a \vee b) = b$ (i.e, $b \leq a \vee b$) for all $a, b \in A$
(6) $(a \wedge b) \vee a = a$ (i.e, $a \wedge b \leq a$) for all $a, b \in A$
(7) For any $a, b, c \in A,\ a \leq b \Rightarrow a \vee c \leq b \vee c$.

An element $m \in A$ is said to be maximal if, for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 2.8. A non-empty subset J of an ADL $A = (A, \wedge, \vee, 0)$ is called a filter of A if for any $a, b \in J$ and $x \in A$, $a \wedge b \in J$ and $x \vee a \in J$.

As a consequence, if J is a filter of A, then $a \vee x \in J$ for any $a \in J$ and $x \in A$.

Theorem 2.9. Let $A = (A, \wedge, \vee, 0)$ be an ADL and $\emptyset \neq X \subseteq A$ and

$$[X\rangle = \big\{a \vee \big(\bigwedge_{i=1}^{n} x_i\big) : n > 0,\ x_i \in X \text{ and } a \in A\big\}.$$

Then $[X\rangle$ is the smallest filter of A containing X and call it the filter generated by X in A.

Theorem 2.10. For any ADL $A = (A, \wedge, \vee, 0)$, $(\mathcal{F}(A), \subseteq)$ is a distributive lattice in which, for any F_1 and $F_2 \in \mathcal{F}(A)$,

$$F_1 \wedge F_2 = F_1 \cap F_2 \text{ and}$$
$$F_1 \vee F_2 = [F_1 \cup F_2\rangle = \{a \wedge b : a \in F_1 \text{ and } b \in F_2\}.$$

Theorem 2.11. Let $A = (A, \wedge, \vee, 0)$ be an ADL and a and $b \in A$. Then the following hold good.

(1) $\langle a] \cap \langle b] = \langle a \wedge b]$

(2) $\langle a] \vee \langle b] = \langle a \vee b]$

(3) $[a\rangle \cap [b\rangle = [a \vee b\rangle$

(4) $[a\rangle \vee [b\rangle = [a \wedge b\rangle$.

Corollary 2.12. *For any a and b in an ADL A,*

(1) $\langle a \wedge b] = \langle b \wedge a]$

(2) $\langle a \vee b] = \langle b \vee a]$

(3) $[a \wedge b\rangle = [b \wedge a\rangle$

(4) $[a \vee b\rangle = [b \vee a\rangle$.

Definition 2.13. Let A_1 and A_2 be ADLs. A mapping $f : A_1 \to A_2$ is called a homomorphism if the following are satisfied, for any $x, y, z \in A_1$.
(1). $f(x \wedge y \wedge z) = f(x) \wedge f(y) \wedge f(z)$
(2). $f(x \vee y \vee z) = f(x) \vee f(y) \vee f(z)$
(3). $f(0) = 0$.

Definition 2.14. Let R be a commutative ring with $1 \neq 0$. A nonzero proper ideal I of R is called a 2-absorbing ideal of R if for any $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

3. 2-ABSORBING FILTER

The concept of 2-absorbing filters is analogous to that of 2-absorbing ideals. In the case of a lattice, we have the duality principle through which we used to prove a result by interchanging the operations $\wedge$ and $\vee$. However, in an ADL we do not have the duality principle. This necessitates a separate study of 2-absorbing filter in an ADL.

Definition 3.1. Let $A = (A, \wedge, \vee, 0)$ be an ADL. A proper filter J of A is said to be a 2-absorbing filter of A if for any $x, y, z \in A$

$$x \vee y \vee z \in J \Rightarrow \text{either } x \vee y \in J \text{ or } y \vee z \in J \text{ or } x \vee z \in J.$$

The next two lemmas are routine verifications.

Lemma 3.2. *Let J be a 2-absorbing filter of A. For all $x, y, z \in A$ whenever $x \vee y \vee z \in J$ implies either $y \vee x \in J$ or $z \vee y \in J$ or $z \vee x \in J$.*

Lemma 3.3. *Let F and G be filters of A and J a 2-absorbing filter of A. Then the following hold, for any $x, y \in A$.*
(1). $[x \vee y\rangle \cap F \subseteq J \Rightarrow [x \vee y\rangle \subseteq J$ *or* $[x\rangle \cap F \subseteq J$ *or* $[y\rangle \cap F \subseteq J$
(2). $[x\rangle \cap (F \cap G) \subseteq J \Rightarrow [x\rangle \cap F \subseteq J$ *or* $[x\rangle \cap G \subseteq J$ *or* $F \cap G \subseteq J$.

Definition 3.4. A proper ideal J of A is said to be a 2-absorbing ideal of an ADL A if for any $x, y, z \in A$, $x \wedge y \wedge z \in J$ implies $x \wedge y \in J$ or $y \wedge z \in J$ or $x \wedge z \in J$.

The following theorem derives necessary and sufficient conditions for 2-absorbing filter of an ADL's to become 2-absorbing ideals.

Theorem 3.5. Let J be a proper filter of A. Then the following are equivalent to each other.
(1). For filters F, G, H of A, $F \cap G \cap H \subseteq J \Rightarrow F \cap G \subseteq J$ or $F \cap H \subseteq J$ or $G \cap H \subseteq J$
(2). For filters F, G, H of A, $J = F \cap G \cap H \Rightarrow J = F \cap G$ or $J = F \cap H$ or $J = G \cap H$
(3). J is a 2-absorbing filter of A
(4). $A - J$ is a 2-absorbing ideal of A.

Proof. $(1) \Rightarrow (2)$: It is clear (refer theorem 2.10).
$(2) \Rightarrow (3)$: Assume the condition (2). We are given that J is a proper filter of A. Now, let x, y and $z \in A$ such that $x \vee y \notin J, y \vee z \notin J$ and $x \vee z \notin J$. Consider the principal filter $[x \vee y\rangle$, $[y \vee z\rangle$ and $[x \vee z\rangle$ generated by $x \vee y$, $y \vee z$ and $x \vee z$ respectively. Then $[x \vee y\rangle \nsubseteq J$, $[y \vee z\rangle \nsubseteq J$ and $[x \vee z\rangle \nsubseteq J$. By (2), we get that $[x \vee y \vee z\rangle = [x\rangle \cap [y\rangle \cap [z\rangle \nsubseteq J$. This implies that $x \vee y \vee z \notin J$. Thus, J is a 2-absorbing filter of A.
$(3) \Rightarrow (4)$: Assume (3). Since J is a proper filter of A, P is a non-empty subset of A and hence $A - J$ is a non-empty proper subset of A. For any x and $y \in A$,

$$\begin{aligned} x, y \in A - J &\Rightarrow x \notin J \text{ and } y \notin J \\ &\Rightarrow x \vee y \notin J \quad \text{(Since } J \text{ is a filter)} \\ &\Rightarrow x \vee y \in A - J \end{aligned}$$

and $x \in A - J$ and $a \in A \Rightarrow x \wedge a \in A - J$ (for, otherwise $x \wedge a \in J$ and $x = x \vee (x \wedge a)$).
Therefore, $A - J$ is a proper ideal of A. Further, for any $x, y, z \in A$,

$x \wedge y \wedge z \in A - J \Rightarrow x \wedge y \wedge z \notin J$

$\Rightarrow x \wedge y \notin J$ or $z \notin J$, or $x \notin J$ or $y \wedge z \notin J$ (since J is a filter)

$\Rightarrow x \wedge y \in A - J$ or $z \in A - J$, or $x \in A - J$ or $y \wedge z \in A - J$.

Thus, $A - J$ is a 2-absorbing ideal of A.

(4) $\Rightarrow$ (1): Assume the condition (4). Let F, G and H be filters of A such that $F \cap G \nsubseteq J$, $G \cap H \nsubseteq J$ and $F \cap H \nsubseteq J$. Now choose $a \in (F \cap G) - J$, $b \in (G \cap H) - J$ and $c \in (F \cap H) - J$. Then $a \in F \cap G$, $b \in G \cap H$, $c \in F \cap H$ and $a, b, c \in A - J$. Since $A - J$ is a 2-absorbing ideal of A, we get that $a \vee b \vee c \in A - J$. Now since F, G and H are filters, $a \in F \cap G$, $b \in G \cap H$ and $c \in F \cap H$, it follows that $a \vee b \vee c \in F \cap G \cap H$. Since $a \vee b \vee c \notin J$, we have $F \cap G \cap H \nsubseteq J$. Hence the result. □

Definition 3.6. Let A and B be ADLs and form the set $A \times B$ by
$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. Define $\wedge$ and $\vee$ in $A \times B$ by, for any $(a, b), (c, d) \in A \times B$,
$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$ and $(a, b) \vee (c, d) = (a \vee c, b \vee d)$. Then $(A \times B, \wedge, \vee, 0)$ is an ADL under the pointwise operations and $0 = (0, 0)$ is the zero element in $A \times B$.

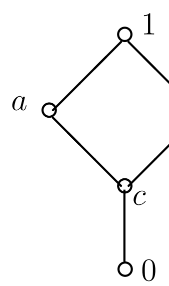
The relationship between the prime filter and the 2-absorbing filter will be discussed in the following theorems. First, let us recall from [9] that a proper filter J of A is said to be a prime filter if, for any x and $y \in A$, $x \vee y \in J \Longrightarrow$ either $x \in J$ or $y \in J$.

Theorem 3.7. Every prime filter of A is a 2-absorbing filter of A.

Proof. Suppose that J is a prime filter of A. Let $x, y, z \in A$ and $x \vee y \vee z \in J$. Since J is prime, either $x \vee y \in J$ or $z \in J$, or $x \in J$ or $y \vee z \in J$. Let $x \vee y \in J$ or $z \in J$. Then clearly $x \vee z \in J$, for any $x \in A$ (since J is a filter). Thus J is a 2-absorbing filter of A. □

2-absorbing filter of an ADL is not a prime filter in general. The following example demonstrates this.

Example 3.8. Let $D = \{0, x, y\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram given below:



Consider $D \times L = \{(t, s) \mid t \in D \text{ and } s \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL (which is not a lattice) under the pointwise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0, 0)$, the zero element in $D \times L$. Let $F = \{(x, 1)\}$. Clearly F is a 2-absorbing filter of $D \times L$ but F is a filter of $D \times L$ which is not prime, since $(x, a) \vee (y, b) = (x, 1)$, for any $(x, a), (y, b) \in D \times L$. From this, we shown that all 2-absorbing filters are not prime filter of $D \times L$.

Theorem 3.9. The intersection of any two prime filters of A is a 2-absorbing filter of A.

Proof. Let F and G be prime filters of A. Clearly $F \cap G$ is a prime filter of A. Let $x, y, z \in A$ and $x \vee y \vee z \in F \cap G$. Since $F \cap G$ is prime, either $x \vee y \in F \cap G$ or $z \in F \cap G$, or $x \in F \cap G$ or $y \vee z \in F \cap G$. Let $z \in F \cap G$. Since $F \cap G$ is a filter, so $x \vee z \in F \cap G$, for any $x \in A$. Therefore, $F \cap G$ is a 2-absorbing filter of A. □

In the following, we prove that the homomorphic image and inverse image of a 2-absorbing filter of an ADL is again a 2-absorbing filter.

Theorem 3.10. Let A and B be ADLs and $h : A \to B$ a lattice homomorphism. Let F and G be 2-absorbing filters of A and B respectively. Then $h(F)$ and $h^{-1}(G)$ are 2-absorbing filters of B and A respectively if h is an epimorphism.

Proof. Let h be an epimorphism and F a 2-absorbing filter of A. Let $x, y, z \in A$ such that $h(x) = a,\ h(y) = b$ and $h(z) = c$, for all $a, b, c \in B$ and suppose $a \vee b \vee c \in h(F)$. Then $a \vee b \vee c = h(x) \vee h(y) \vee h(z) = h(x \vee y \vee z) \in h(F)$ (since h is homomorphism). As $x \vee y \vee z \in F$ and F is a 2-absorbing filter of A, then either $x \vee y \in F$ or $y \vee z \in F$ or $x \vee z \in F$. Which implies that $a \vee b = h(x \vee y) \in h(F)$ or $b \vee c \in (F)$ or $a \vee c \in h(F)$. Thus $h(F)$ is a 2-absorbing filter of B. Let G be a 2-absorbing filter of B and set $h^{-1}(G) = \{a \in A : h(a) \in G \subseteq B\}$. Let $x, y, z \in A$ and $x \vee y \vee z \in h^{-1}(G)$. Since G is a 2-absorbing filter of B, $h(x) \vee h(y) \vee h(z) = h(x \vee y \vee z) \in G$ implies that $h(x \vee y) \in G$ or $h(y \vee z) \in G$ or $h(x \vee z) \in G$. So that, $x \vee y \in h^{-1}(G)$ or $y \vee z \in h^{-1}(G)$ or $x \vee z \in h^{-1}(G)$. Therefore, $h^{-1}(G)$ is a 2-absorbing filter of A. □

Theorem 3.11. Let A and B be ADLs and let J and F be 2-absorbing filters of A and B respectively. Then $A \times F$ and $J \times B$ are 2-absorbing filters of $A \times B$.

Proof. Let F be a 2-absorbing filter of B and $(a, x) \vee (a, y) \vee (a, z) \in A \times F$, for any $(a, x), (a, y), (a, z) \in A \times B$. Then $(a, x) \vee (a, y) \vee (a, z) = (a, x \vee y \vee z) \in A \times F$ implies that $(a, x \vee y) \in A \times F$ or $(a, y \vee z) \in A \times F$ or $(a, x \vee z) \in A \times F$ (since

$x \vee y \vee z \in F$ implies $x \vee y \in F$ or $y \vee z \in F$ or $x \vee z \in F$ and $a \in A$). Hence $A \times F$ is a 2-absorbing filter of $A \times B$. Similarly, $J \times B$ is a 2-absorbing filter of $A \times B$ if J is a 2-absorbing filter of A. □

In the following, we introduce the concept of n-absorbing filter of an ADL A.

Definition 3.12. Let J be a proper filter of A and $n \in Z^+$. Then J is an n-absorbing filter of A if whenever $x_1 \vee x_2 \vee ... \vee x_{n+1} \in J$, for $x_i \in A$, $1 \leq i \leq n+1$, then there are n of the $x_i^{'}s$ whose join is in J.

Lemma 3.13. *Let J be a proper filter of A and $n, m \in Z^+$. Then J is n-absorbing filter if and only if whenever $x_1 \vee x_2 \vee ... \vee x_m \in P$, for $x_1, ..., x_m \in A$ with $m > n$, then there are n of the $x_i^{'}s$ whose join is in P. Also, if J is n-absorbing filter, then J is an m-absorbing filter, for all $m \geq n$.*

Lemma 3.14. *Let $g : A \to B$ be a lattice homomorphism. Let J and F be n-absorbing filters of A and B respectively. Then $g^{-1}(F)$ and $g(J)$ are n-absorbing filters of B and A respectively if g is an isomorphism.*

Theorem 3.15. If $\{J_\alpha\}_{\alpha \in \Delta}$ is a non-empty chain of n-absorbing filter of A, then $\bigvee\limits_{\alpha \in \Delta} J_\alpha$ is an n-absorbing filter of A.

Proof. Let $J = \bigvee\limits_{\alpha \in \Delta} J_\alpha$ and $x_1, x_2, ..., x_{n+1} \in A$ such that $\bigvee\limits_{i=1}^{n+1} x_i \in P$. Let $x_i = \bigvee\limits_{j \neq i} x_j$ and $x_i \notin J$, for all $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, there exist an n-absorbing filter J_{α_i} such that $x_i \notin J_{\alpha_i}$. Assume that $J_{\alpha_1} \subseteq J_{\alpha_2} \subseteq ... \subseteq J_{\alpha_n}$. Let $\beta \in \Delta$. If $J_\beta \subseteq J_{\alpha_1} \subseteq ... \subseteq J_{\alpha_n}$, then $x_i \notin J_\beta$, for each $1 \leq i \leq n$. Since $x_1 \vee x_2 \vee ... \vee x_{n+1} \in P$ and J_β is n-absorbing filter of A, we have $x_{n+1} \in J_\beta$. Again, $x_1 \vee x_2 \vee ... \vee x_{n+1} \in J_{\alpha_1}$ and J_{α_1} is n-absorbing filter of A, then $x_{n+1} \in J_{\alpha_1}$. So, $x_{n+1} \in J_\beta$, for every $\beta \in \Delta$. Thus $x_{n+1} \in J$. Hence the theorem. □

4. WEAKLY 2-ABSORBING FILTER

In this section, we now introduce the concept of weakly 2-absorbing filters of an ADL and obtain the relation between this and 2-absorbing filters. First, we have the following.

Definition 4.1. A proper filter J of A is a weakly prime filter of A if for any $x, y \in A$,

$$1 \neq x \vee y \in J \Rightarrow \text{either } x \in J \text{ or } y \in J.$$

Theorem 4.2. Every prime filter of A is a weakly prime filter of A and the converse of this is not true.

Example 4.3. Let $D \times L = \{(t, s) \mid t \in D \text{ and } s \in L\}$ be an ADL discussed in example 3.8. Let $F = \{(x, 1)\}$. Clearly, F is a weakly prime filter of $D \times L$. On the other hand, F is a filter which is not prime, since $(x, a) \vee (y, b) = (x, 1)$ implies that $(x, a) \notin F$ and $(y, b) \notin F$, for all $(x, a), (y.b) \in D \times L$.

We now introduce the concept of weakly 2-absorbing filter of an ADL in the following.

Definition 4.4. A proper filter J of A is a weakly 2-absorbing filter of A if for any $x, y, z \in A$,

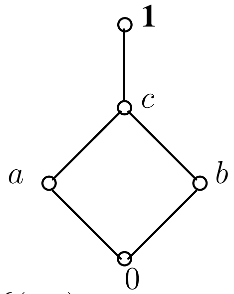
$$1 \neq x \vee y \vee z \in J \Rightarrow \text{either } x \vee y \in J \text{ or } y \vee z \in J \text{ or } x \vee z \in J.$$

In the following discussion, we introduce the sufficient condition for weakly prime filter and 2-absorbing filter of an ADL to become a weakly 2-absorbing filter.

Theorem 4.5. Every weakly prime filter of A is a weakly 2-absorbing filter of A.

The converse of the above corollary is not true; consider the following example.

Example 4.6. Let $D = \{0, x, y\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram given below:



Consider $D \times L = \{(t, s) : t \in D \text{ and } s \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL ($D \times L$ is not a lattice) under the point-wise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0, 0)$, the zero element in $D \times L$. Let $F = \{(x, c), (y, 1), (y, c)\}$. Then $(x, 1) \neq (0, a) \vee (x, b) \vee (y, c) = (x, c) \in F$ implies $(0, a) \vee (x, b) \in F$, $(x, b) \vee (y, c) \in F$ and $(0, a) \vee (y, c) \in F$. Thus F is a weakly 2-absorbing filter of $D \times L$ but F is a filter which is not weakly prime, since $(x, 1) \neq (x, a) \vee (y, b) = (x, c) \in F$ implies $(x, a) \notin F$ and $(y, b) \notin F$. Therefore, every weakly 2-absorbing filter is not a weakly prime filter of $D \times L$.

The following is a consequence of 3.8 and 3.16.

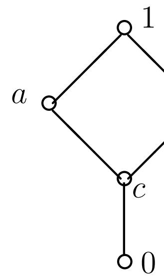
Corollary 4.7. *The intersection of any two weakly prime filters of A is also a weakly 2-absorbing filter of A.*

Theorem 4.8. Every 2-absorbing filter of A is a weakly 2-absorbing filter of A.

Proof. Let J be 2-absorbing filter of A and $1 \neq x \vee y \vee z \in F$, for any $x, y, z \in A$. Then either $x \vee y \in J$ or $y \vee z \in J$ or $x \vee z \in J$. From this, J is a weakly 2-absorbing filter of A. □

In the following example, we show that there are weakly 2-absorbing filters of A which are not 2-absorbing filters of A.

Example 4.9. Let $D = \{0, x, y\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below.



Consider $D \times L = \{(t, s) \mid t \in D \text{ and } s \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL (which is not a lattice) under the pointwise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0, 0)$, the zero element in $D \times L$. Let $J = \{(x, 1)\}$. Let $(0, a), (x, b), (y, c) \in D \times L$. Now, $(0, a) \vee (x, b) \vee (y, c) = (x, 1)$ implies that $(0, a) \vee (x, b) = (x, d) \notin J$, $(x, b) \vee (y, c) = (x, f) \notin J$ and $(0, a) \vee (y, c) = (y, e) \notin J$. Thus, J is a weakly 2-absorbing filter of $D \times L$. But, J is not a 2-absorbing filter of $D \times L$, since $(0, a) \vee (x, b) \vee (y, c) = (x, 1) \in J$ implies that $(0, a) \vee (x, b) \notin J$, $(x, b) \vee (y, c) \notin J$ and $(x, a) \vee (y, c) \notin J$.

Theorem 4.10. Let $J \neq \{1\}$ be a proper filter of A. Then J is a 2- absorbing filter of A if and only if J is a weakly 2-absorbing filter of A.

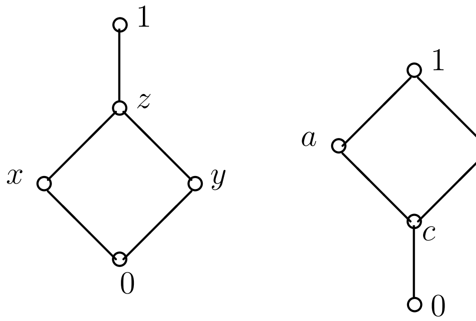
The product of two proper filters is a weakly 2-absorbing filter, but the proper filters themselves may not be weakly 2-absorbing filters, is derived in the following theorem.

Theorem 4.11. Let A and B be ADLs and let J and F be proper filters of A and B respectively. If $J \times F$ is a weakly 2-absorbing filter of $A \times B$, then J and F are weakly 2-absorbing filters of A and B respectively.

Proof. Let $J \times F$ is a weakly 2-absorbing filter of $A \times B$. Let $a, b, c \in A$ and $x, y, z \in B$ such that $1 \neq x \vee y \vee z \in F$. Then $1 \neq (a, x \vee y \vee z) \in J \times F$ implies that either $(a, x \vee y) \in J \times F$ or $(a, x \vee z) \in J \times F$ or $(a, y \vee z) \in J \times F$. From this, either $x \vee y \in F$ or $x \vee z \in F$ or $y \vee z \in F$. Thus F is a weakly 2-absorbing filter of B. Similarly J is a weakly 2-absorbing filter of A. □

The converse of the above theorem is not true; for, consider the following example.

Example 4.12. Let $A = \{0, x, y, z, 1\}$ and $B = \{0, a, b, c, 1\}$ be the lattice represented by the diagram respectively given below:



Let $J = [z\rangle$ and $F = [1\rangle$. Clearly J and F are weakly 2-absorbing filters of A and B respectively. Then $J \times F = [(z, 1)\rangle$. Let $(0, 1), (x, a), (y, a) \in A \times B$. We note that,

$$(0, 1) \vee (x, a) \vee (y, a) = (z, 1) \in J \times F \Rightarrow (0, 1) \vee (x, a) = (x, 1) \notin J \times F,$$

$(x, a) \vee (y, a) = (z, a) \notin J \times F$ and $(0, 1) \vee (y, a) = (y, 1) \notin J \times F$. It follows that, $J \times F$ is not a weakly 2-absorbing filter of $A \times B$.

In the following two theorems, we give another characterization of weakly 2-absorbing filter of an ADL.

Theorem 4.13. Let A and B be ADLs and $J(\neq \{1\})$ be a proper filter of A. Then the following are equivalent to each other.
(1). $J \times B$ is a weakly 2-absorbing filter of $A \times B$
(2). $J \times B$ is a 2-absorbing filter of $A \times B$
(3). J is a 2-absorbing filter of A.

Proof. $(1) \Rightarrow (2)$ is clear by Theorem 4.10.
$(2) \Rightarrow (3)$: Assume (2). Let $x, y, z \in A$ such that $x \vee y \vee z \in J$. Since $J \times B$ is a 2-absorbing filter of $A \times B$, $(x \vee y \vee z, t) \in J \times B$, for every $t \in B$, which implies that either $(x \vee y, t) \in J \times B$ or $(x \vee z, t) \in J \times B$ or $(y \vee z, t) \in J \times B$. It follows that, $x \vee y \in J$ or $x \vee z \in J$ or $y \vee z \in J$. Therefore, J is a 2-absorbing filter of A.
$(3) \Rightarrow (1)$. Suppose J is a 2-absorbing filter and $1 \neq (x, b) \vee (y, b) \vee (z, b) = (x \vee y \vee z, b) \in J \times B$, for $x, y, z \in A$ and $b \in B$. By (3), we have $x \vee y \in J$

or $x \vee z \in J$ or $y \vee z \in J$. Which implies that $(x \vee y, b) \in J \times B$ or $(x \vee z, b) \in J \times B$ or $(y \vee z, b) \in J \times B$, for every $b \in B$. Thus $J \times B$ is a weakly 2-absorbing filter of $A \times B$. □

Theorem 4.14. Let A and B be ADLs and let $J(\neq \{1\})$ and $F(\neq \{1\})$ be proper filters of A and B respectively. Then the following are equivalent to each other.
(1). $J \times F$ is a weakly 2-absorbing filter of $A \times B$
(2). $F = B$ and J is a 2-absorbing filter of A or F is a prime filter of B and J is a prime filter of A
(3). $J \times F$ is a 2-absorbing filter of $A \times B$.

Proof. $(1) \Rightarrow (2)$: Suppose $J \times F$ is a weakly 2-absorbing filter of $A \times B$. Then by Theorem 3.21, J and F are weakly 2-absorbing filters of A and B respectively and, $J \neq \{1\}$ and $F \neq \{1\}$, so by Theorem 4.10, J and F are 2-absorbing filters of A and B respectively. If $F = B$, then by Theorem 3.23 J is a 2-absorbing filter of A. Suppose $F \neq B$. Let $x, y \in B$ such that $x \vee y \in F$ and let $0 \neq t \in J$. Then $(t, 0) \vee (0, x) \vee (0, y) = (t, x \vee y) \in J \times F$. Since $(0, x) \vee (0, y) = (0, x \vee y) \notin J \times F$, we conclude that either $(t, 0) \vee (0, x) = (t, x) \in J \times F$ or $(t, 0) \vee (0, y) = (t, y) \in J \times F$ and hence either $x \in F$ or $y \in F$. Thus F is a prime filter of B. Similarly, J is a prime filter of A.
$(2) \Rightarrow (3)$: Suppose $F = B$ and J is a 2-absorbing filter of A. Then by the above theorem, $J \times F$ is a 2-absorbing filter of A. Suppose also that J and F are prime filers of A and B respectively. Then clearly $J \times F$ is a prime filter of $A \times B$. Let $(x, y), (z, t), (a, b) \in A \times B$ such that $(x, y) \vee (z, t) \vee (a, b) \in J \times F$. Then either $(x, y) \vee (z, t) \in J \times F$ or $(a, b) \in J \times F$, or $(x, y) \vee (a, b) \in J \times F$ or $(z, t) \in J \times F$, or $(x, y) \in J \times F$ or $(z, t) \vee (a, b) \in J \times F$. Thus $J \times F$ is a 2-absorbing filter of $A \times B$.
$(3) \Rightarrow (1)$: Suppose $J \times F$ is a 2-absorbing filter of $A \times B$. Let $J(\neq \{1\})$ and $F(\neq \{1\})$ be proper filters of A and B respectively. Then by Theorem 4.10, $J \times F$ is a weakly 2-absorbing filter of $A \times B$. □

In Theorem 3.10, we prove that the image and inverse image of a 2-absorbing filter of an ADL is again a 2-absorbing filter. In the case of a weakly 2-absorbing ADL filter, we have the following.

Theorem 4.15. Let A and B be ADLs, $h : A \to B$ be a lattice homomorphism and, let F and G be weakly 2-absorbing filters of A and B respectively. Then $h(F)$ and $h^{-1}(G)$ are weakly 2-absorbing filters of B and A respectively if h is an epimorphism.

5. CONCLUSION

In this study, we introduce the concept of 2-absorbing filters in an almost distributive lattice(ADL). The characterization of weakly 2-absorbing filters in an ADL is obtained. The Hull kernel topology of the foregoing notions will be the focus of our future research.

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