Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 17, Number 2, (2022) pp. 467-481

http://www.ripublication.com/adsa.htm

Delay Logistic Model with Harvesting on Weighted Average Population

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Abstract

The study examines the dynamics of a delay-logistic model incorporated with proportionate harvesting on weighted average population function. The stability of our model is compared with the existing model. The existing model is found to be highly unstable and shows no signs of convergence. However, when the harvested part is replaced with a weighted average population function with harvest, the oscillations are reduced drastically. In the neighborhood of the maximum harvest rate set for the systems; as the time-space is extended, while our model converges to the actual population value, the existing model on the other hand, exhibits its worst chaotic state and could not predict the expected fish population size. In conclusion, the new model estimates the expected growth of the fish population in a short period with precision while the existing model has to be adjusted to be able to provide estimates near the ones projected from the new model. It is recommended that physical systems with a long inherent gestation period should be modeled by our model rather than the existing model as it fails to estimate the expected population size precisely.

Keywords: delay differential equation, logistic model, stability analysis, positive solution.

1. INTRODUCTION

Understanding the dynamics of mathematical formulations on population has been a major concern in areas such as cell biology, genetics and demography. Most of the mathematical models on growth and development of population are highly simplified in nature. Analytic solutions to most of these models have been established, yet the solutions to these models differ from the real situations described by the models [1]. This is serving as a motivation for this study to revisit some specific population models that have been considered in many researches and yet researchers have failed to identify specific deficiency associated with them. Thus, the purpose of the study is to point out the best method of estimating expected growth in population model with little error.

Verhulst in [2] proposed a simpler logistic model for population growth given by

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{c}\right),\ for\ x\left(t_0\right) = x_0, \tag{1}$$

where c is a carrying capacity which is the stable population level: if the population x(t) (at time t) goes beyond c, then it will decrease, but if it stays below it the population increases. The r is a constant which defines the growth rate.

Another interesting model is where the carrying capacity is considered as a function of the population in time past which captures the delay the population adjust to its environment. This paved way for the well-known Hutchinson equation for population growth expressed by

$$x'(t) = rx(t)\left(1 - \frac{x(t-\tau)}{c}\right),\ \text{for } t > 0 \tag{2}$$

$$x(t) = \varphi(t), \quad \text{for } t \in [-\tau, 0],$$

where r, τ, c are positive numbers, with $x(t) = \varphi(t)$, where $\varphi(t) \in ([-\tau, 0], \mathbb{R})$ as the initial history function [3].

Piotrowska and Bodnar in [4] and Bodnar in [5] introduced another time delay parameter on the growth rate to transform equation (2) to the following form

$$x'(t) = rx(t-\tau)\left(1 - \frac{x(t-\tau)}{c}\right), \tag{3}$$

where r the growth rate, c the population carrying capacity, and τ time delay are all positive constants.

The assumptions under which logistic equation is operating can also affect its dynamics especially when we incorporate in the logistic models outside interfering factors including harvesting, drugs for treatment, poaching just to mention a few [6]. Usually, logistic growth equations are modeled with constant-rate harvest and density- dependent harvest or proportional harvesting. Assuming population modeled by equation (1) is affected by proportional harvesting then the model introduces the term $hx(t)$, where $h \geq 0$ is the harvest proportionality constant, yielding the system of the form [6-7],

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{c}\right) - hx(t). \tag{4}$$

Holling's studies in [8] identified three basic types of outside influencing factors on population in the form:
$TypeI(linear) : E(N,t) = \alpha N + \beta,$
$TypeII(cyrtoid) : E(N,t) = \alpha N/(N + \beta),$
$TypeIII(sigmoid) : E(N,t) = \alpha N^2/(\gamma^2 + \beta^2)$, ($\alpha$, β, and γ are positive functions of t),
that stand out as the most comprehensive population outside affected factors and have been used in different forms in several researches including Berezansky et al. in [9] who incorporated $h(t)$ in equation (2) and obtained the following equation

$$x'(t) = rx(t)\left(1 - \frac{x(t-\tau)}{c}\right) - h(t)x(t-\tau), t \in [0,\tau]. \tag{5}$$

In this model, $h(t)$ is the rate at which individual in the population at time $t - \tau$ is harvested. Thus the model supports the arguments that any interacting species in the system are dependent on amassed resources and hunting or harvesting effects in the past.
Similarly, Piotrowska and Bodnar in [4] and Cooke et al. in [10] used the model below by introducing time delay on the growth rate $rx(t)$ to postulate that the intrinsic growth rate depends on past time $(t - \tau)$, where τ, the developmental time of the population is in the system with model given by

$$x'(t) = rx(t-\tau)\left(1 - \frac{x(t-\tau)}{c}\right) - h(t)x(t), t \in [0,\tau], \tag{6}$$

where h is constant function at a step time t.
Doust and Saraj [11] on the other hand used the following logistic model

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{c}\right) - f(x), \tag{7}$$

where $f(x) = h(\frac{x}{1+x})$ is the rate of outside influence which is inversely related to the density of the population at time t.
From the literature reviewed so far one can observe that no author has yet considered the

rate of outside interference on the population in the form of weighted average of current and previous population as $g(x) = h(\gamma x(t) + (1-\gamma)x(t-\tau))$, where $0 < \gamma < 1$ in any given logistic model as given by

$$x'(t) = rx(t-\tau)\left(1 - \frac{x(t-\tau)}{c}\right) - g(x), \quad \mathrm{t} \in [0,\tau] \tag{8}$$
$$x(t) = \varphi(t), \quad \text{for } t \in [-\tau, 0],$$

where a, τ are positive numbers, with $x(t) = \varphi(t)$, as the initial history function.

In this study, we will also explore the possible effects of changing the model parameters as means to compare the stability of equation (8) to the existing equation (5). We will also fix $g(x)$ in place of the harvesting part in the equation (5) and compare their estimated population sizes with our model. We will discuss the stability of these models through numerical simulations using different values of the outside interference $h \geq 0$ which will be noted as the harvest rate. Thus we will use the harvest rate as bifurcation parameter so that by its variation we will be able to determine the state of the models and estimate the expected population size right.

The paper is structured as follows: Section 2 discusses the existence, stability, and bifurcation of equation (8). The section will also consider conditions for the positivity of the solution in equation (8). In Section 3, detailed numerical solutions and stability analyses will be done comparing model (8) to model (5) and its transformation when the function $g(x)$ is incorporated into it using the same parameter values. Section 4 outlines the findings and conclusions derived from the analyses of the models.

2. MATERIALS AND METHODS

2.1 Stability Analysis of Equation (8)

As we normalize the system, then we can observe that if $h < r$, the two possible equilibrium of equation (8) will be $x_e = 0$ and $x_e = c(r-h)/r$. The second part $x_e = c(r-h)/r$, is stable for say $c = 1$ and so for any initial condition $x_0 > 0$, all solutions of equation (8) will converge towards it as $t \to \infty$. We let c_h denote the positive equilibrium of the equation (8) to emphasize the fact that h is the bifurcation parameter of the system as stated in the objective of this study. On the other hand if $h \geq r$, then all solutions will be attracted to the former equilibrium point $x_e = 0$ as the only solution of equation (8). This means the population should be driven towards extinction.

In order to understand the overview of stability given the equilibrium points above, we linearize the equation (8) in the neighbourhood of the equilibrium point $x_e = x(t) =$

$x(t-\tau)$ and obtain the following expression

$$\mathrm{x}'(t) + \alpha \mathrm{x}(t) + \beta \mathrm{x}(t-\tau) = 0, \tag{9}$$

where the constants $\alpha = h\gamma$, $\beta = r - 2rx_e - h(1-\gamma)$, with $\tau > 0$. The actual values of β will be determined by either $x_e = 0$ or $x_e = c(r-h)/r$ as shown in the expression for β depending on the value of the carrying capacity. Now, we seek for nontrivial solution of equation (9) in the form of $x(t) = \kappa e^{\lambda t}$, $\kappa \neq 0$ where λ is complex and κ a constant. In substituting $x(t) = \kappa e^{\lambda t}$ into equation (9) we obtain the corresponding characteristic equation (10) from which the stability of equation (9) is determined through the locations of its eigenvalues λ. The characteristic equation is derived as follows

$$\lambda + \alpha + \beta e^{-\lambda\tau} = 0. \tag{10}$$

If we let $\tau = 0$, then for $\lambda = -(\alpha+\beta) < 0$ the steady state x_e is asymptotically stable. However, when we assume that $\tau > 0$ then there exists $\lambda = i\omega$ with $\omega > 0$ such that the characteristic equation (10) is broken down into real and imaginary parts as given by

$$\begin{cases} \alpha + \beta\cos\omega\tau = 0, \\ \omega - \beta\sin\omega\tau = 0. \end{cases} \tag{11}$$

Separating the constants and the circular functions to either side of the equations, then by adding the squares of resulting equations provides the following expression

$$\omega^2 = (\beta+\alpha)(\beta-\alpha). \tag{12}$$

The necessary condition for stability change should be $|\alpha| \leq |\beta|$. From equation (12), if $\beta \leq \alpha$, then it contradicts the fact that $\omega > 0$, which implies that the delay parameter is harmless.

Theorem 2.1: *If $\beta \leq \alpha$, then steady state of equation (8) is asymptotically stable for any positive value of τ.*

On the contrary if $\beta > \alpha$, then we can define $\omega > 0$ from equation (12) as follows

$$\omega = \sqrt{(\beta+\alpha)(\beta-\alpha)}. \tag{13}$$

When equation (13) is substituted into the first of the pairwise equation (11) we obtain the threshold value of the delay parameter $\hat{\tau}$:

$$\hat{\tau} = \frac{[cos^{-1}(-\frac{\alpha}{\beta})]}{\sqrt{(\beta^2-\alpha^2)}}. \tag{14}$$

2.1.1 Stability Switches and Hopf Bifurcation of (8)

We now determine the direction of the stability switch. Assuming the roots of equation (10) are continuous function of the time delay parameter, then by taking derivative of the equation (10) with respect to the time delay τ and solving for $\frac{d\lambda}{d\tau}$ we arrive at the following results

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1-\tau\beta e^{-\lambda\tau}}{\lambda\beta e^{-\lambda\tau}}. \tag{15}$$

Then

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega} = Re\left(\frac{1-\tau\beta e^{-\lambda\tau}}{\lambda\beta e^{-\lambda\tau}}\right)_{\lambda=i\omega}$$
$$\left[\frac{d(Re\lambda)}{d\tau}\right]^{-1}_{\lambda=i\omega} = Re\left(\frac{-1}{\lambda(\lambda+\alpha}\right)_{\lambda=i\omega} = Re\left(\frac{\omega(\omega+i\alpha)}{\omega^2(\omega^2+\alpha^2)}\right) = \frac{1}{\omega^2+\alpha^2}.$$

Therefore

$$\left[\frac{d(Re\lambda)}{d\tau}\right]^{-1}_{\lambda=i\omega} > 0. \tag{16}$$

The results in equation (16) indicate that at $i\omega$ on the imaginary axis, all roots of the characteristic equation (10) near the critical value $\hat{\tau}$, will cross from left to right as the delay parameter τ varies continuously from a number less than $\hat{\tau}$ to that greater than $\hat{\tau}$. Furthermore, we partially differentiate equation (14) with respect to the values α and β to determine the critical value of the delay as follows

$$\frac{\delta\hat{\tau}}{\delta\alpha} = \frac{[(\beta^2-\alpha^2)^{1/2}+\alpha cos^{-1}(-\frac{\alpha}{\beta})]}{(\beta^2-\alpha^2)^{3/2}} > 0. \tag{17}$$

with

$$\frac{\delta\hat{\tau}}{\delta\beta} = -\frac{[\alpha(\beta^2-\alpha^2)^{1/2}+\beta^2 cos^{-1}(-\frac{\alpha}{\beta})]}{\beta(\beta^2-\alpha^2)^{3/2}} < 0. \tag{18}$$

Therefore, as $\frac{\delta\hat{\tau}}{\delta\alpha} > 0$ and $\frac{\delta\hat{\tau}}{\delta\alpha} < 0$ then it implies that by increasing the value of α and decreasing the value of β, the stability switching curves is shifted upwards. Thus the variation of the parameters in the directions indicated have significant effects on stability. Theorem 2.1.1. summaries this assertion as follows:

Theorem 2.1.1: *If $\alpha+\beta < 0$ and $\alpha < \beta$, then there exists $\hat{\tau} > 0$ such that the steady state x_e of equation (8) is asymptotically stable for $0 < \tau < \hat{\tau}$, loses stability at*

$\tau = \hat{\tau}$ and becomes unstable or bifurcates to a limit cycle if $\tau > \hat{\tau}$.
For proof see [12,13].

2.1.2 Existence and Uniqueness Solution of (8)

From equation (8), if we let $\bar{x}(t) = \frac{\gamma r}{r-h} x(t)$, then we transform it into the following initial value problem after dropping the bars on the new variable $\bar{x}$

$$x'(t) = ax(t-\tau)[k - \rho x(t-\tau)] - \mu x(t), \text{ for } t > 0 \tag{19}$$
$$x(t) = \varphi(t), \quad \text{for } t \in [-\tau, 0],$$

where $a = (r-h), \tau, \mu = h\gamma$, $k = \frac{a+\mu}{a}$ and $\rho = \frac{1}{\gamma c}$ are positive numbers, with $x(t) = \varphi(t)$, as the initial history function. From the interval $[0, \tau]$ we generate a non-negative solution to equation (19) from an equivalent expression given by

$$x(t) = \varphi(0) + \int_0^t [ax(s-\tau)(k - \rho x(s-\tau)) - \mu x(s)]ds.$$

Since $\varphi(0) \geq 0$ and $a > 0$, the solution exists and it is unique and non-negative in the neighbourhood considered. Again in the interval $t \in [(n-1)\tau, n\tau]$, if we let $x_n : [(n-1)\tau, n\tau] \to \mathbb{R}^+$ be the solution of equation (19) then for $t \in [n\tau, (n+1)\tau]$, it implies that

$$x(t) = x_n(n\tau) + \int_{n\tau}^t [ax_n(s-\tau)(k - \rho x(s-\tau)) - \mu x(s)]ds.$$

Thus, it is observed that for every non-negative initial function $\varphi(t)$, the solution of equation (19) is defined for $t \geq 0$ in $t \in [n\tau, (n+1)\tau]$.

2.1.3 Existence and Positivity Solution of (8)

We now study the conditions which will guarantee non-negative solutions of equation (19) for every positive initial function $\varphi(t)$, since as a consequence of the Theorem 1.2 in [5], there is a possibility of solution of (19) having negative values for a positive initial condition. This study is very important due to biological reasons. If we had let every variable for substitution in equation (19) to have the delay parameter τ as a result of letting $t = \bar{t}\tau$, then by dropping the bar, equation (19) would have been in the following form

$$x'(t) = a\tau x(t-1)[k - \rho x(t-1)] - \mu x(t), \text{ for } t > 0, \tag{20}$$
$$x(t) = \varphi(t), \quad \text{for } t \in [-1, 0].$$

We adopt the condition below following the proof of Theorem 1.2 in [5]:

Let

$$0 \leq \varphi(t) \leq 1, \quad \text{for } t \in [-1,0] \tag{21}$$

and

$$\varphi(t) = \begin{cases} \frac{1}{2}, \text{ for } t \in [-1,0] \\ 1, \text{ for } t = 0 \end{cases} \tag{22}$$

then on the interval $[n-1,n]$, we let x_n be the solution of equation (20). We can now observe that

$$x_1(t) = \varphi(0) + \int_0^t [a\varphi(s)[k - \varphi(s)] - \mu x(s)]ds.$$

Hence,

$$x_1(t) = 1 + \int_0^t \left[a(\frac{1}{2})(k - \rho(\frac{1}{2})) - \mu(\frac{1}{2}) \right] ds = 1 + \frac{akt}{2} - \frac{a\rho t}{2} - \frac{\mu t}{2},$$

and in the next interval [1,2],

$$x_2(t) = x_1(1) + \int_0^{t-1} [ax_1(s)(k - x_1(s)) - \mu x_1(s)]\, ds,$$

and then

$$x_2(t) = 1 + \frac{ak}{2} - \frac{a\rho}{2} - \frac{\mu}{2} + \int_0^{t-1} [ax_1(s)(k - x_1(s)) - \mu x_1(s)]\, ds.$$

In summary, we group the results of the expression under the integral according to the degree of a as shown beneath:

$$a^3 : \frac{a^3\rho^2 k}{6} + \frac{a^3\rho}{8} - \frac{a^3\rho^3}{48},$$

$$a^2 : -\frac{5a^2\rho k}{8} + \frac{a^2k^2}{6} + \frac{a^2\rho^2}{8} - \frac{a^2k^3}{12} + \frac{a^2\rho\mu k}{12} - \frac{a^2\rho^2\mu}{12} + \frac{a^2\rho\mu^2}{12},$$

$$a : \frac{5ak}{4} - \frac{9a\rho}{8} + \frac{a\mu\rho}{2} - \frac{ak\mu}{4} - \frac{a\mu^2\rho}{12},$$

$$constant : -\frac{\mu}{4}.$$

We now drop all the variables aside $a = \tau(r-h)$ as the dynamics of the equation (19) and for that matter equation (8) revolves around it. This gives us the following parameters which will form the polynomial $P_1(a)$:

$$a^3 : \frac{a^3}{24} + \frac{a^3}{4} - \frac{a^3}{48} = \frac{13a^3}{48},$$

$$a^2 : -\frac{5a^2}{8} + \frac{a^2}{6} - \frac{a^2}{8} = \frac{-a^2}{3},$$

$$a : \frac{5a}{4} - \frac{9a}{8} + \frac{a}{2} - \frac{a}{4} - \frac{a}{12} = \frac{7a}{24},$$

$$constant : -\frac{1}{4}.$$

Therefore, we obtain

$$X(2) = X_2(2) = -\frac{1}{4} + \frac{7a}{24} - \frac{a^2}{3} + \frac{13a^3}{48} = P_1(a). \tag{23}$$

If the condition (21) is satisfied, then the following inequality also holds:

$$\forall t > 0, \quad x(t) \leq 1 + \frac{ak}{2} - \frac{a\rho}{2} - \frac{\mu}{2}$$

If $x(t) > 1$, then there exists $t_0 < t$ such that $x(t_0) = 1$, and

$$x(t) = x(t_0) + \int_{t_0-1}^{t-1} [ax(s)(k - x(s)) - \mu x(s)]\, ds,$$

so that we can have

$$x(t) \leq 1 + \int_{t_0-1}^{t_0} [\frac{ak}{2} - \frac{a\rho}{2} - \frac{\mu}{2}]ds = 1 + \frac{ak}{2} - \frac{a\rho}{2} - \frac{\mu}{2},$$

or else we have the following after dropping all the variables with the exception of a:

$$x(t) \geq 1 + \int_{t_0-1}^{t_0} [a(\frac{1}{2} + \frac{a}{4})(1 + -\frac{1}{2} - \frac{a}{4}) - (\frac{1}{2} + \frac{a}{4})]ds = P_2(s)ds = \frac{1}{2} - \frac{a^3}{16} = P_2(a). \tag{24}$$

Then as a consequence of Theorem 2.1 in [9], if $a < r_2$ it implies the polynomial $P_2(a) > 0$ and $P_1(a) < 0$ for $a > r_1$, where r_1 and r_2 are the greatest roots of P_1 and P_2 respectively. That is $r_1 = 1.0453$ and $r_2 = 2$.

3.0 RESULTS AND DISCUSSION

3.1 Numerical Solution: Stability Analysis of Model (8) and (25)

In this section, we will examine the stability analysis of equation (8) through numerical simulations and then compare it with that of equation (5). We will also replace the harvesting part of the equation (5) with the harvesting function of the weighted average population from the current and previous population as presented below and then compare it with our model (8). The aim is to help researchers make the appropriate choice when modeling certain physical systems. The comparison will be done by applying the same parameter values for the two models. It is believed that by so doing, we will also be able to estimate the expected fish population value right. All these numerical simulations are done using MatLab Software.

$$x'(t) = rx(t)\left(1 - \frac{x(t-\tau)}{c}\right) - h(\gamma x(t) + (1-\gamma)x(t-\tau)), t \in [0, \tau]. \quad (25)$$

where $h > 0$ is the harvest proportionality constant and $0 < \gamma < 1$.
Now setting the common parameter values; the growth rate $r = 0.51$, harvesting rate $h = 0.05$, time-delay $\tau = 3$, the weighting coefficient $\gamma = 0.30$ and the carrying capacity $c = 1$, we simulate the models as shown below.

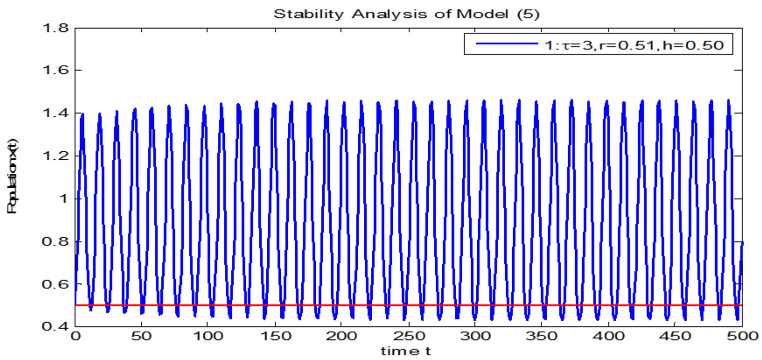

Figure 1: Stability Analysis of the model (5)

In figure 1, it shows that for $\tau r = 3 * 0.51 > 0$ and with the other parameter values given above such that $\beta = -0.44 \leq \alpha = 0.015$, the model (5) is highly unstable around the equilibrium population of $x_e = 0.902$ and displays no signs of convergence. The system with time seems to oscillate more from the equilibrium point . This behaviour contradicts the assertion in the Theorem 2.1.

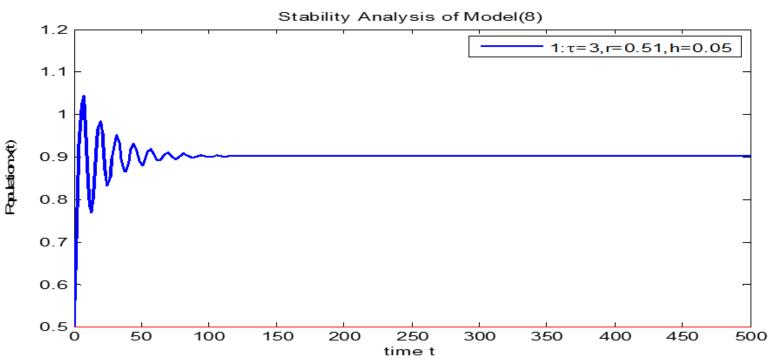

Figure 2: Stability Analysis of the model (8)

From figure 2, it complements that for $\tau r = 3 * 0.51 > 0$, and with the parameters given from which $\beta = -0.44 \leq \alpha = 0.015$, as stated in the Theorem 2.1, the system should be asymptotically stable about $x_e = 0.902$, after few oscillations.

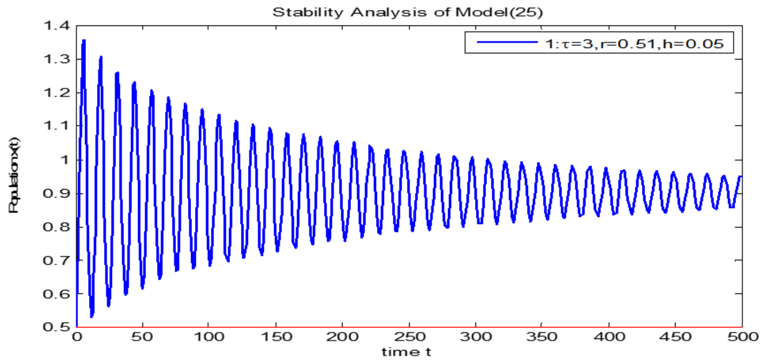

Figure 3: Stability Analysis of the model (25)

For the same parameter values as used for model (8), it was expected that in figure 3, the system will converge to equilibrium point $x_e = 0.902$. However, there are a lot of oscillations that make it difficult to converge to equilibrium with the time limit set for the system. In comparison with figure 1, there seems to be a drastic reduction in oscillations in the same time-space. The reduction is caused by the replacement of the second part of the model (5) with the weighted average population function with a harvest.

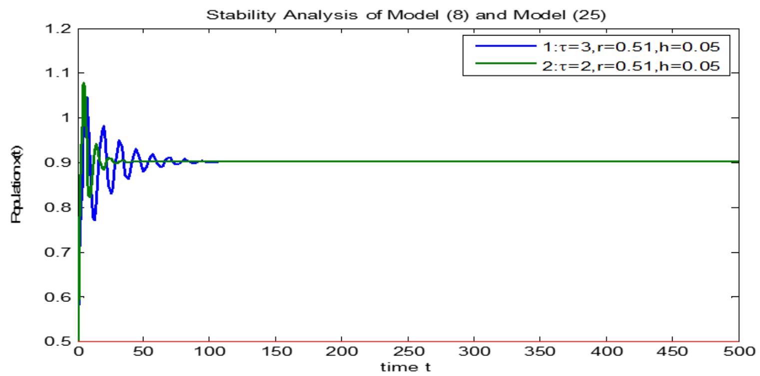

Figure 4: Stability Analysis of the model (25)

In figure 4, stability has been restored for both model (8) and model (25) toward the equilibrium point $x_e = 0.902$ after reducing the time delay τ_2 associated with the model (25) to $\tau_2 = 2$. This means the model (25) works with only physical systems with a short inherent gestation period.

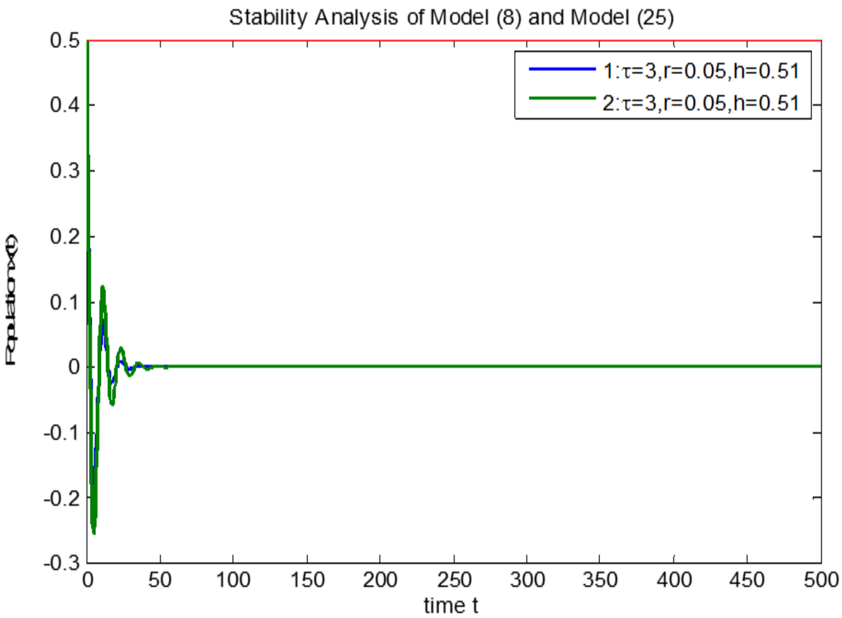

Figure 5: Stability Analysis of both model (8) and model (25)

In figure 5, stability has been achieved for both models at $x_e = 0$, a condition that satisfies Theorem 2.1 for $\beta = -0.307 \leq \alpha = 0.153$. However, this means that the fish population size will be driven to extinction with time because the harvesting rate is greater than the growing population rate. In this situation, the model (25) did not need adjustment to be able to estimate population size precisely.

3.2. Maximum harvesting rate and expected population growth

Using the positive equilibrium fish population size $c_h = x_e = c(r-h)/r$, then for $c = 1$ and $r = 0.51$ as applied in previous simulations, we can determine maximum safe harvesting value and maximum population estimate through the following equation

$$hc_h = h(1 - 1.961h). \tag{26}$$

Differentiating equation (26) with respect to h results in

$$(hc_h)' = 1 - 3.922h.$$

Therefore, at the critical point the optimal harvesting rate $h = 0.255$. By substituting $h = 0.255$ in into the positive equilibrium equation, we have

$$c_h = x_e = [1 - 1.96(0.255)] = 0.5002,$$

which offers the expected fish population value almost half of the maximum carrying $c = 1$ of the system. We now study the dynamics of the model (25) and model (8) in the neighbourhood of the maximum harvesting rate $h = 0.255$.

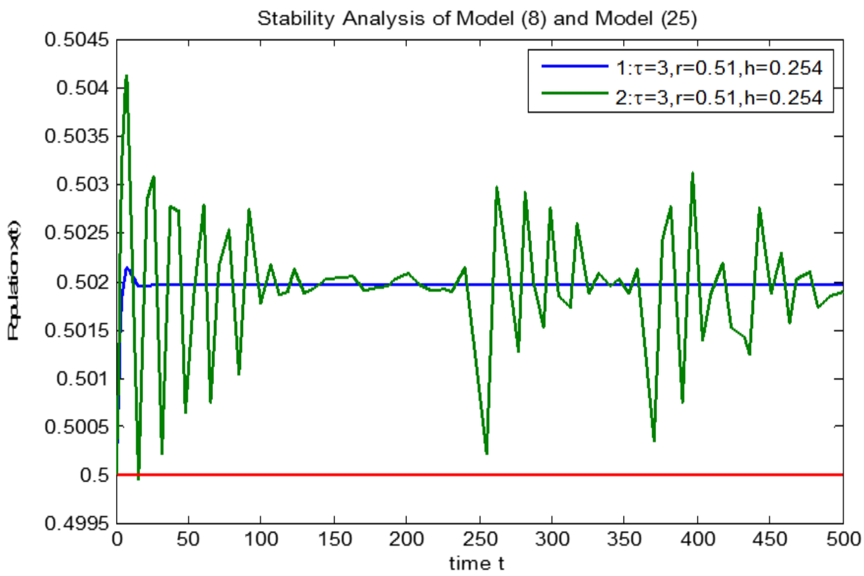

Figure 6: Stability Analysis of both model (8) and model (25)

For $h = 0.254$, 0.001 less than the maximum harvest rate of $h = 0.255$, in figure 6 there is chaotic behaviour emanating from the model (25) around the new expected fish size shown by the model (8) at 0.502 which is 0.002 more than the estimated population at 0.5002 (i.e. expected fish population value for harvest rate at h=0.255).

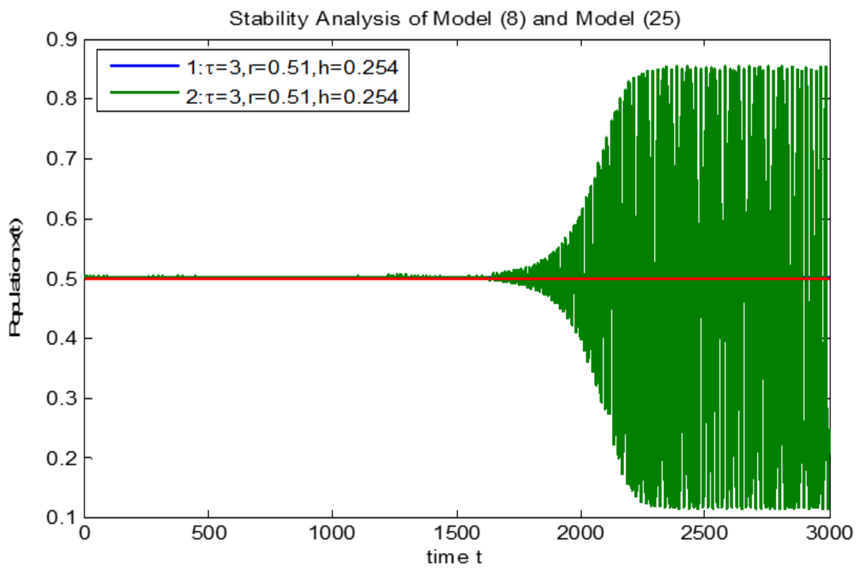

Figure 7: Stability Analysis of both model (8) and model (25)

In figure 7, it shows that while the model (8) converges to the maximum population value of 0.5002 (which is almost half of the carrying capacity) as the time-space is extended to 3000 (compared to figure 6), the model (25) on the other hand displays the worst chaotic state and that we cannot predict the expected fish population size.

4. CONCLUSION

The paper studied the dynamics of a delay-logistic population model with a harvesting rate proportional to the weighted average fish population. In this paper, we used the harvest rate as a bifurcation parameter to estimate the expected growth of the fish population. The model (8) used in this study was found to be less oscillatory compared to the existing model given by (5). When the harvesting part in model (5), was replaced by the function $g(x)$ as provided by the model (25), the two models converged to the fish population equilibrium set for the system by reducing the time-delay parameter associated with the model (25). This denotes that physical systems with a long gestation period modeled by (25) will fail to estimate the expected population size correctly. However, as the fish population is driven to extinction at zero (0), both the model (8) and (25), moved towards the fish population equilibrium of 0 without any parameter adjustment. In the neighbourhood of the maximum harvest rate set for the systems; as the time-space is extended, while our model converged to the actual population value, the existing model on the other hand, reached the worst chaotic state and we could not use it to predict the expected fish population size. Since our logistic growth model is less oscillatory, it is believed that the expected fish growth estimated in the study reflects the true fish size (population).

STATEMENT AND DECLARATIONS:

The research did not receive any specific funding but was performed as part of the employment requirements by the employer, University of Cape Coast, Ghana and Kumasi Technical University, Ghana for the promotion of the authors. Therefore, the authors have no conflict of interest with any organization.

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