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Best Approximation and Interpolation of Entire Functions of Slow Growth in Several Complex Variables

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Abstract

The problem of generalized type of entire functions of slow growth in terms of best approximation and interpolation errors in several complex variables have been investigated.

Keywords: Entire functions, best approximation and interpolation errors, generalized type, Siciak extremal function and slow growth.

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1. INTRODUCTION

The growth of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ can be measured in terms of the order ρ defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

where $M(r,f) := \sup_{|z|\leq r} |f(z)|$. If the order is a positive real number the type T of the function is defined by

$$T = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}}.$$

A function has slow growth if the order is equal to 0. For entire functions there is a large literature concerning the growth of this topic (see e.g. [1-3], [7-9], [11-12], [15-16], [18-20]) and others.

The concept of order and type was generalized in the literature (see e.g. Seremeta [13], Shah [14]), in particular to discuss subclasses of entire function of order $\rho = 0$ (functions of slow growth). Here one replaces the log function in the above formulae by more general functions α, β defined on an interval (r_0, ∞), which are assumed to be positive, strictly increasing and tending to infinity as $r \to \infty$, and satisfying properties of class L^0 and Λ defined below:

Let a function $h(x)$ is defined on $[a, \infty)$ for some $a \geq 0$ and it is strongly monotonically increasing and tends to ∞ as $x \to \infty$. According to Seremeta [13], this function belongs to the class L^0 if, for any real function ϕ such that $\phi(x) \to \infty$ as $x \to \infty$, the equality

(i) $\lim_{x\to\infty} \frac{h[(1+\frac{1}{\phi(x)})x]}{h(x)} = 1.$

It belongs to the class Λ if, for all $c, 0 < c < \infty$,

(ii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1.$

By using functions α and β from the classes L^0 and Λ, by analogy with [13], the generalized order of an entire function $f(z)$ is given by the formula:

$$\rho(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r, f)]}{\beta[\log r]} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(-\frac{1}{n}\log|a_n|)}.$$

Further, for $\alpha(x), \beta^{-1}(x)$ and $\gamma(x) \in L^0$, the generalized type of an entire transcendental function $f(z)$ is given by

$$T(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r, f)]}{\beta[\{\gamma(r)\}^\rho]} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta[\{\gamma(e^{\frac{1}{\rho}}|a_n|^{-\frac{1}{n}})\}^\rho]},$$

where $0 < \rho < \infty$.

The historical background of my work is an old result of Bernstein characterization when a continuous real-valued function f on the interval [-1,1] can be extended to an entire (holomorphic) function in terms of the best approximation by polynomials of degree $\leq n$. In 1968, Varga [20] has obtained results giving the order and type of the entire extension.

Let $g : \mathbb{C}^N, N \geq 1$, be an entire transcendental function. For $z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, we put $S(r, g) = \sup\{|g(z)| : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = r^2\}, r > 0$. Then we define the generalized order and generalized type of $g(z)$ as:

$$\rho(\alpha, \beta, g) = \limsup_{r\to\infty} \frac{\alpha[\log S(r, g)]}{\beta[\log r]}$$

and

$$T(\alpha, \beta, g) = \limsup_{r\to\infty} \frac{\alpha[\log S(r, g)]}{\beta[\{\gamma(r)\}^{\rho}]}.$$

Let K be a compact set in $\mathbb{C}^N$ and let $||.||_K$ denote the supnorm on K. The function $\Phi_K(z) = \sup[|p(z)|^{\frac{1}{n}} : p - polynomial, deg.p \leq n, ||p||_K \leq 1, n = 1, 2, \ldots \ and \ z \in \mathbb{C}^N]$, is called Siciak extremal function of the compact set K (see [5] and [6]). Given a function f defined and bounded on K, we put for $n = 1, 2, \ldots$

$$E_n^1(f, K) = ||f - t_n||_K;$$
$$E_n^2(f, K) = ||f - l_n||_K;$$
$$E_{n+1}^3(f, K) = ||l_{n+1} - l_n||_K$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of K.

Janik ([4],[6]), investigated the order and generalized order of entire functions in terms of the approximation errors defined above. Srivastava and Kumar [17] obtained the characterizations of generalized type. For $N = 1$ these results were obtained by Shah [14]. All these results mentioned above were obtained under certain conditions which fail to hold if $\alpha = \beta = \gamma$. To overcome this difficulty, we will use here the concept of slow growth introduced by Kapoor and Nautiyal [10] and investigate the generalized type in terms of approximation and interpolation errors defined above in several complex variables.

Kapoor and Nautiyal [10] defined generalized order $\rho(\alpha, f)$ of slow growth with the help of general functions as:

Let Ω be the class of functions $h(x)$ defined above and

(iii) there exists a $\delta(x) \in \Omega$ and x_0, C_1 and C_2 such that

$$0 < C_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq C_2 < \infty$$

for all $x > x_0, \delta(x) \in \Omega$.

Let $\overline{\Omega}$ be the class of functions $h(x)$ and

$$\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = C, 0 < C < \infty.$$

It has been shown [10] that classes Ω and $\overline{\Omega}$ are contained in Λ. Further $\Omega \cap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha, f)$ of entire function $f(z)$ of slow growth as

$$\rho(\alpha, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r, f)]}{\alpha[\log r]},$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$.

We define the generalized type $T(\alpha, g)$ of an entire function $g(z)$ having finite generalized order $\rho(\alpha, g)$ as

$$T(\alpha, g) = \limsup_{r\to\infty} \frac{\alpha[\log M(r, g)]}{[\alpha[\log r]]^{\rho}}$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$.

To the best of our knowledge, coefficient characterization for generalized type in terms of approximation and interpolation errors defined above for slow growth have not been obtained so far. In this paper, we have tried to fill this gap.

Notations:
1. $F[x; \overline{T}, \rho] = \alpha^{-1}\{\overline{T}[\alpha(x)]^{\frac{1}{\rho}}\}$ where ρ is fixed number, $0 < \rho < \infty$ and $\overline{T} = T + \varepsilon$.
2. $E[F[x; \overline{T}, \rho]]$ is an integer part of the function F.

2. AUXILLIARY RESULTS

Lemma 2.1. Let K be a compact set in $\mathbb{C}^N$ such that Φ_K is locally bounded in $\mathbb{C}^N$. For all $T, 0 < T < \infty$,
(i) If $\alpha(x) \in \overline{\Omega}$, then

$$\frac{dF[x; T, \rho]}{d\log x} = O(1) \;\; as \;\; x \to \infty.$$

Let $(p_n)_{n\in N}$ be a sequence of polynomials in $\mathbb{C}^N$ such that
(ii) $deg.p_n \leq n, n \in N$.
(iii) there exists $n_0 \in N$ such that

$$||p_n|| \leq [\exp\{(\frac{\rho-1}{\rho})nF[\frac{n}{\rho}; \frac{1}{\overline{T}}, \rho-1]\}]^{-1}$$

where $\overline{T} = T + \varepsilon$ for small $\varepsilon > 0$.
Then $\sum_{n=0}^{\infty} p_n$ is an entire function and the generalized type $\sigma(\alpha, \sum_{n=0}^{\infty} p_n)$ of this entire function satisfies

$$\sigma(\alpha, \sum_{n=0}^{\infty} p_n) \leq T$$

provided $\sum_{n=0}^{\infty} p_n$ is not a polynomial.
Proof. By assumption, we have

$$||p_n||r^n \leq r^n[\exp\{(\frac{\rho-1}{\rho})nF[\frac{n}{\rho};\frac{1}{\overline{\overline{T}}},\rho-1]\}]^{-1}$$

The inequality

$$(||p_n||r^n)^{\frac{1}{n}} \leq r[\exp\{-(\frac{\rho-1}{\rho})F[\frac{n}{\rho};\frac{1}{\overline{\overline{T}}},\rho-1]\}] \leq \frac{1}{2} \tag{2.1}$$

is fulfilled beginning with some $n = n(r)$. Then

$$\sum_{n=n(r)+1}^{\infty} ||p_n||_K r^n \leq \sum_{n=n(r)+1}^{\infty} \frac{1}{2^n} \leq 1. \tag{2.2}$$

Now from inequality (2.1) we have

$$2r \leq \exp\{(\frac{\rho-1}{\rho})nF[\frac{n}{\rho};\frac{1}{\overline{\overline{T}}},\rho-1]\},$$

we can take $n(r) = E[\rho\alpha^{-1}\{\overline{T}(\alpha(\log r + \log 2))^{\rho-1}\}]$.
Let us consider the function

$$\varphi(x) = r^x[\exp\{-(\frac{\rho-1}{\rho})xF[\frac{x}{\rho};\frac{1}{\overline{\overline{T}}},\rho-1]\}].$$

The maximum of $\varphi(x)$ is attained for a value of x by $x^*(r)$ where

$$\varphi(x^*(r)) = \max_{n_0 \leq x \leq n_1(r)} \varphi(x),$$
$$x^*(r) = \rho\alpha^{-1}\{\overline{T}(\alpha(\log r - a(r)))^{\rho-1}\},$$

where $A > 0$ and

$$-A < a(r) = \frac{dF[\frac{x}{\rho};\frac{1}{\overline{\overline{T}}},\rho-1]}{d\log x}|_{x=x^*(r)} < A.$$

Further

$$\begin{aligned} ||p_n||_K r^n &= \max_{n_0 < x < n_1(r)} \varphi(x) = \frac{r^{\rho\alpha^{-1}\{\overline{T}(\alpha(\log r - a(r)))^{\rho-1}\}}}{e^{\rho\alpha^{-1}\{\overline{T}(\alpha(\log r - a(r)))^{\rho-1}\}(\log r - a(r))}} \\ &= \exp\{a(r)\rho\alpha^{-1}\{\overline{T}(\alpha(\log r - a(r)))^{\rho-1}\}\} \\ &\leq \exp\{A\rho\alpha^{-1}\{\overline{T}(\alpha(\log r + A))^{\rho-1}\}\}, n > n_0, r > 0. \end{aligned} \tag{2.3}$$

Put $K_r = \{z \in \mathbb{C}^N : \Phi_K(z) < r, r > 1\}$, then for every polynomial p of degree$\leq n$, we have (see [6], pp. 323)

$$|p_n(z)| \leq ||p_n||_K \Phi_K^n(z), z \in \mathbb{C}^N. \tag{2.4}$$

The series $\sum_{n=0}^{\infty} p_n$ is convergent in every $K_r, r > 1$, so $\sum_{n=0}^{\infty} p_n$ is an entire function. Put

$$\tilde{M}(r) = \sup\{||p_n||_K r^n : n \in N, r > 0\}.$$

In view of inequality (2.3), for every $r > 0$, there exists a positive integer $\nu(r)$ such that

$$\tilde{M}(r) = ||p_{\nu(r)}||_K r^{\nu(r)}.$$

and

$$\tilde{M}(r) > ||p_n||_K r^n, n > \nu(r).$$

It is clear that $\nu(r)$ increases with r. Suppose that $\nu(r) \to \infty$ as $r \to \infty$. Then putting $n = \nu(r)$ in (2.3) we get for sufficiently large r

$$\tilde{M}(r) \leq \exp\{A\rho\alpha^{-1}\{\overline{T}(\alpha(\log r + A))^{\rho-1}\}\}. \tag{2.5}$$

Let us put

$$\gamma_r = \{z \in \mathbb{C}^N : \Phi_K(z) = r, r > 1\}$$

and

$$M(r) = \sup\{|\sum_{n=0}^{\infty} p_n(z)| : z \in \gamma_r\}, r > 1.$$

Now taking into account the facts of Janik (see [6], pp. 323), we have for some positive constant k,

$$S(r, \sum_{n=0}^{\infty} p_n) \leq M(kr) \leq 2\tilde{M}(2kr). \tag{2.6}$$

Considering (2.5) and (2.6), we obtain

$$S(r, \sum_{n=0}^{\infty} p_n) \leq 2\exp\{A\rho\alpha^{-1}\{\overline{T}(\alpha(\log r + A))^{\rho-1}\}\}.$$

or

$$\begin{aligned}\alpha[\frac{1}{A\rho}\log\{\frac{1}{2}S(r, \sum_{n=0}^{\infty} p_n)\}] &\leq \overline{T}(\alpha(\log r + A))^{\rho-1} \\ &\leq \overline{T}(\alpha(\log r + A))^{\rho}.\end{aligned}$$

Thus, we have

$$\frac{\alpha[\frac{1}{A\rho}\log\{\frac{1}{2}S(r, \sum_{n=0}^{\infty} p_n)\}]}{(\alpha(\log r + A))^{\rho}} \leq \overline{T} = T + \varepsilon.$$

Since $\alpha(x) \in \overline{\Omega} \subseteq \Lambda$, proceeding to limits we get

$$\limsup_{r\to\infty} \frac{\alpha[\frac{1}{A\rho}\log\{\frac{1}{2}S(r,\sum_{n=0}^{\infty}p_n)\}]}{(\alpha(\log r + A))^{\rho}} \leq T.$$

or

$$\sigma(\alpha, \sum_{n=0}^{\infty} p_n) \leq T. \tag{2.7}$$

In case $\nu(r)$ is bounded then $\tilde{M}(r)$ is also bounded, whence $\sum_{n=0}^{\infty} p_n$ reduces to a polynomial. Hence the proof of lemma is completed.

3. MAIN RESULTS

Theorem 3.1. Let K be a compact set in $\mathbb{C}^N$ such that Φ_K is locally bounded in $\mathbb{C}^N$ and $\alpha(x) \in \overline{\Omega}$. Then the function f, defined and bounded on K, is the restriction of an entire function g of generalized order $\rho; 1 < \rho < \infty$, is of the generalized type $\sigma(\alpha, g)$ if and only if

$$\sigma(\alpha, g) = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^s(f,K)]^{-\frac{1}{n}}]\}^{\rho-1}}; s = 1, 2, 3,$$

provided $\frac{dF[x;T,\rho]}{d\log x} = O(1) \ \ as \ \ x \to \infty$, for all $T, 0 < T < \infty$.

Proof. Let f has an entire function extension g of generalized type $\sigma = \sigma(\alpha, g)$. Put

$$\delta_s = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^s(f,K)]^{-\frac{1}{n}}]\}^{\rho-1}}; s = 1, 2, 3.$$

Here $E_n^s \equiv E_n^s(g|_K, K)$. we have to show that $\sigma = \delta_s, s = 1, 2, 3$. From [21] it is known that

$$E_n^1 \leq E_n^2 = (n_* + 2)E_n^1, n \geq 0, \tag{3.1}$$

$$E_n^3 \leq 2(n_* + 2)E_{n-1}^1, n \geq 1, \tag{3.2}$$

where $n_* = (n + N)_{C_n}$. Using Stirling formula we have

$$n! \approx e^{-n}n^{n+\frac{1}{2}}\sqrt{2\pi},$$

and

$$n_* \approx \frac{n^N}{N!} \ \ for \ \ all \ \ large \ \ values \ \ of \ \ n.$$

Hence, we have $E_n^1 \leq E_n^2 \leq \frac{n^N}{N!}[1 + o(1)]E_n^1$ and $E_n^3 \leq 2\frac{n^N}{N!}[1 + o(1)]E_n^1$.
Thus, $\delta_3 \leq \delta_2 \leq \delta_1$ and it suffices to prove that $\delta_1 \leq \sigma \leq \delta_3$.

First we prove that $\delta_1 \leq \sigma$. Using the definition of generalized type for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we get

$$S(r,g) \leq \exp\{\alpha^{-1}\{\overline{\sigma}(\alpha(\log r))^{\rho}\}\},$$

where $\overline{\sigma} = \sigma + \varepsilon$ provided r is sufficiently large. Without loss of generality we may suppose that

$$K \subset B = \{z \in \mathbb{C}^N : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 \leq 1\}.$$

Then $E_n^1 \leq E_n^1(g,B)$. Now following Janik (see [6], pp.324) we get

$$E_n^1(g,B) \leq r^{-n} S(r,g), r \geq 2, n \geq 0,$$

or

$$E_n^1 \leq r^{-n} \exp\{\alpha^{-1}\{\overline{\sigma}(\alpha(\log r))^{\rho}\}\}.$$

Putting $r = r(n) = \exp[\alpha^{-1}[(\frac{1}{\overline{\sigma}}\alpha(\frac{n}{\rho}))^{\frac{1}{\rho-1}}]] = \exp\{F[\frac{n}{\rho}; \frac{1}{\overline{\sigma}}, \rho-1]\}$ we obtain

$$E_n^1 \leq \exp\{-nF + \frac{n}{\rho}F\}$$

or

$$\frac{\rho}{\rho-1}\log[E_n^1]^{-\frac{1}{n}} \geq \alpha^{-1}[(\frac{1}{\overline{\sigma}}\alpha(\frac{n}{\rho}))^{\frac{1}{\rho-1}}]$$

or

$$\overline{\sigma} = \sigma + \varepsilon \geq \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^1]^{-\frac{1}{n}}]\}^{\rho-1}}.$$

Proceeding to limits, we get

$$\sigma \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^1]^{-\frac{1}{n}}]\}^{\rho-1}}$$

or

$$\sigma \geq \delta_1. \tag{3.3}$$

Inequality (3.3) obviously holds when $\sigma = \infty$.
Next we will prove $\sigma \leq \delta_3$. Suppose that $\delta_3 < \sigma$. Then for every $\theta, \delta_3 < \theta < \sigma$,

$$\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^3]^{-\frac{1}{n}}]\}^{\rho-1}} \leq \theta$$

for sufficiently large n. Hence

$$E_n^3 \leq \exp\{-nF[\frac{n}{\rho}; \frac{1}{\theta}, \rho - 1] + \frac{n}{\rho}F[\frac{n}{\rho}; \frac{1}{\theta}, \rho - 1]\}.$$

Using Lemma 2.1, we get

$$\sigma \leq \theta$$

where $\sigma = \sigma(\alpha, g)$ is the generalized type of $g(z)$. Since θ has been chosen less than σ, we get a contradiction. Thus $\sigma \leq \delta_3$.

Now assume that f be a function defined and bounded on K and such that for $s = 1, 2, 3$,

$$\delta_s = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^s(f, K)]^{-\frac{1}{n}}]\}^{\rho-1}}.$$

For every $\theta_1 > \delta_s$ and for sufficiently large n, we have

$$\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log[E_n^s]^{-\frac{1}{n}}]\}^{\rho-1}} \leq \theta_1$$

or

$$E_n^s \leq \exp\{-nF[\frac{n}{\rho}; \frac{1}{\theta_1}, \rho - 1] + \frac{n}{\rho}F[\frac{n}{\rho}; \frac{1}{\theta_1}, \rho - 1]\}.$$

Proceeding to limits as $n \to \infty$, we get

$$\lim_{n\to\infty} [E_n^s]^{\frac{1}{n}} \leq 0.$$

Also, it is obvious that

$$\lim_{n\to\infty} [E_n^s]^{\frac{1}{n}} \geq 0.$$

Hence, it gives

$$\lim_{n\to\infty} [E_n^s]^{\frac{1}{n}} = 0.$$

Thus, following Janik ([6], Prop. 3.1), we conclude that the function f can be continuously extended to an entire function. Let us put

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1}),$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier.

Now we have to show that g is the required continuation of f and $\sigma(\alpha, g) = \delta_s$. For every $\theta_1 \geq \delta_3$ and for sufficiently large n, we have

$$E_n^3 \leq \exp\{-nF + \frac{n}{\rho}F\}$$

or

$$||l_n - l_{n-1}|| \leq \exp\{-nF + \frac{n}{\rho}F\}.$$

Now in view of Lemma 2.1, we obtain

$$\sigma(\alpha, g) \leq \theta_1.$$

Since $\theta_1 > \delta_3$ is arbitrary, it gives

$$\sigma(\alpha, g) \leq \delta_3.$$

Taking into account the inequalities (3.1), (3.2) with the proof of first part given above, we have $\sigma(\alpha, g) = \delta_3$ as required. Hence the proof is completed.

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