Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 17, Number 1, (2022) pp. 375-406

http://www.ripublication.com/adsa.htm

Identification of the Diffusion Parameter in the 1D Heat Equation Model

D. V. Pongui Ngoma[*1], **G. Nguimbi**[1], **V. D. Mabonzo**[2] **and B. B. Bamvi Madzou**[1]

[1]*Ecole Nationale Supérieure Polytechnique, Marien Ngouabi University, Congo.*

[2]*Ecole Normale Supérieure, Marien Ngouabi University, Congo,*

Abstract

This work falls within the framework of mathematical modeling and that of numerical analysis and scientific calculation. This article aims to identify the diffusion parameter μ in the model of the heat equation of solution $u = u(x,t)$. In this work, we solved the heat equation numerically by Explicit Euler , Implicit Euler and Cranck-Nicholson methods. We studied the analytical stability in $L^\infty([0,1])$ and $L^2([0,1])$ while calculating the truncation errors of these methods. We also studied the convergence of these methods. Then, we compared numerically in norm $L^2([0,1])$ the solutions obtained during the resolution of said equation by these different methods. The differentiability of the solution u with respect to the parameter μ has been demonstrated. Finally we defined an optimization problem of least square type and then used the gradient descent algorithm with fixed steps to identify the thermal conductivity parameter μ of the heat equation. These numerical resolution technics were implemented in $Scilab$.

Keywords: Identification, explicit Euler method, implicite Euler method, Cranck-Nicholson method, Optimisation.
MSC 2020 No.: 65M06, 65M12, 65K05

[*]Corresponding Author

1. INTRODUCTION

Parameter identification techniques have been the subject of numerous studies due to their frequent occurrences in several applications in physics, biology, medicine, engineering, signal processing, control theory, finance and other different sciences (see [5], [1], [6], [7], [10], [14] and [13]). These techniques are usually read through mathematical models that are often represented by partial derivatives equations, ordinary differential equations or system of equations. The right hand side of these models is not always continuous. The result is less regular solutions and difficulties in adjusting model parameters, including using differentiable optimal control and optimization methods. For example, to adjust the thermal scattering parameter in the heat equation. This article discusses numerical methods for optimally adjusting the thermal conduction parameter of a parabolic model. We will illustrate the method of the heat equation model (see [13], [15] and [10]), which is a model that often represents the non-transient evolution of irreversible phenomena associated with diffusion processes. This model allows you to study the main features of the system, namely the evolution of the solution $u(x,t)$ according to time, the diffusion setting μ, the conduction speed c, etc. There is a relationship between the model's parameter μ and the main features of the solution,$u(x,t)$, but the adjustment of the parameter μ based on an asymptomatic formula is not yet automatic (see [4], and [10]). We present the problem of optimal control, especially where the functional cost is related to the characteristics of conduction speed. In this problem, the state variable is the solution of partial derivatives equations of the heat equation model. Numerical methods are used to solve this model, including Euler's explicit, implicit and Crank-Nicholson methods, in the loop of a differentiable optimization method. We showed that the model's solution is very sensitive to the parameter μ by comparing the numerical errors of the different solutions of that model. We will show that our differentiated optimization method (the constant step gradient method) makes it easy to retrieve the model solution by identifying the values of the parameter μ of the heat equation model. The method is general and can easily be applied to other parabolic models

2. MATHEMATICAL FORMULATION OF THE PROBLEM

In the context of this work, we want to compare three numerical methods including the explicit Euler, implicit Euler and Cranck-Nicholson methods respectively, to analytically and numerically solve the heat equation (1) of solution $u = u(x,t)$

$$\frac{\partial u}{\partial t}(x,t) - \mu\frac{\partial^2 u}{\partial x^2}(x,t) = f(x), \quad \text{where} \quad f(x) \in L^2([0,1]), \tag{1}$$

with homogeneous Dirichlet conditions

$$u(0,t) = u(1,t) = 0, \tag{2}$$

starting from the initial condition

$$u(x,0) = u_0(x) = 4x(1-x). \tag{3}$$

These different methods will help to describe the evolution of the solution u of the heat equation, while comparing the errors in norm L^2 of the solutions obtained. In this work, we will also define an optimization problem in order to identify the heat diffusion coefficient μ of the equation (1) by defining an objective function $\psi = \psi(\mu)$ which depends on the square of the difference of the numerical and experimental celerities of the heat propagation. The objective function $\psi = \psi(\mu)$ will be defined as follows,

$$\psi(\mu) = \frac{1}{2}(\hat{c} - c^*)^2, \quad \hat{c} = \hat{c}(\mu) \tag{4}$$

We will look for μ^* minimizing $\psi = \psi(\mu)$ where u is solution of (1) with boundary conditions and initial $(2) - (3)$, (see [13] and [5]).
All these numerical methods will be implemented with the *Scilab* software.

3. NUMERICAL RESOLUTION OF THE HEAT EQUATION

In this section, we solve the heat equation (1) of solution $u = u(x,t)$ by three different numerical methods including the explicit Euler, implicit Euler and Cranck Nicholson methods, (see [3], [11] and [9]). For two integers M and N, we discretize in a uniform way the intervals of space $\Omega = [0,1]$ and of time $[0,T]$ by introducing the points,

$$x_j = j\Delta x, \quad j = 0, 1, ..., N+1, \tag{5}$$

$$t_n = n\Delta t, \quad n = 0, 1, ..., M, \tag{6}$$

where Δx is the discretization step in space given by $\Delta x = \dfrac{1}{N+1}$ and Δt the discretization step in time with $\Delta t = \dfrac{T}{M}$. For $j = 0, 1, ..., N+1$ and $n = 0, 1, ..., M$, which define a mesh of the spatio-temporal domain $[0,1] \times [0,T]$, we then seek an approximation $u_j^n \backsimeq u(x_j, t_n)$ of the exact solution at nodes x_j by discretizing the derivative in space and at time t_n by discretizing the derivative in time.

3.1. Explicit Euler scheme

By applying Taylor's approximation, we obtain

$$\frac{\partial u}{\partial t}(x_j, t_n) \backsimeq \frac{u_j^{n+1} - u_j^n}{\Delta t}, \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) \simeq \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \tag{8}$$

replacing (7) and (8) in (1) we obtain

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \mu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = f(x_j), \tag{9}$$

such that $f(x_j) = f_j$, which allows to obtain

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \mu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + f_j,$$

$$u_j^{n+1} - u_j^n = \mu \frac{\Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \Delta t f_j. \tag{10}$$

by posing $\lambda = \mu \frac{\Delta t}{\Delta x^2}$, the equation (10) becomes

$$u_j^{n+1} = (1 - 2\lambda)u_j^n + \lambda u_{j+1}^n + \lambda u_{j-1}^n + \Delta t f_j. \tag{11}$$

the problem (1) is then written in discretized form by

$$\begin{cases} u_j^{n+1} = (1 - 2\lambda)u_j^n + \lambda u_{j+1}^n + \lambda u_{j-1}^n + \Delta t f_j \quad \text{pour} \quad j = 1, \cdots, N \\ u_{N+1}^n = u_0^n = 0 \\ u_j^0 = u^0(x_j) = 4x_j(1 - x_j), \end{cases} \tag{12}$$

and in the matrix form by

$$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & 0 & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \lambda \\ 0 & 0 & \cdots & & \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix} + \Delta t \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}.$$

Posing $u_j^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix}$, $u_j^n = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix}$, $F_j = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}$, and

$$A_\lambda = \begin{pmatrix} 1-2\lambda & \lambda & 0 & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \lambda \\ 0 & 0 & \cdots & & \lambda & 1-2\lambda \end{pmatrix}.$$

The system is written in linear form

$$u_j^{n+1} = A_\lambda u_j^n + \Delta t F_j \tag{13}$$

3.2. Implicit Euler scheme

We use a backward scheme of order 1 to evaluate the temporal derivative and a centered scheme of order 2 to evaluate the second derivative in space, after calculations we obtain

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \mu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} = f_j \tag{14}$$

what allows to obtain

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \mu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + f_j,$$

that is to say

$$u_j^n = (1+2\lambda)u_j^{n+1} - \lambda u_{j+1}^{n+1} - \lambda u_{j-1}^{n+1} - \Delta t f_j \quad \text{with} \quad \lambda = \mu \frac{\Delta t}{\Delta x^2}. \tag{15}$$

We note that the unknowns with iteration $n+1$ are linked together by an implicit relation (hence the name of the method).
By varying the index $j = 1, ..., N$ of the equation (15), we have

$$\begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix} = \begin{pmatrix} 1+2\lambda & -\lambda & 0 & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & 1+2\lambda & -\lambda & \cdots & 0 \\ \vdots & & \ddots & \ddots & & -\lambda \\ 0 & 0 & \cdots & & -\lambda & 1+2\lambda \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} - \Delta t \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}.$$

by posing $u_j^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix}, \quad u_j^n = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_N^n \end{pmatrix}, \quad F_j = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}$ and

$$B_\lambda = \begin{pmatrix} 1+2\lambda & -\lambda & 0 & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & 1+2\lambda & -\lambda & \cdots & 0 \\ \vdots & & \ddots & \ddots & & -\lambda \\ 0 & 0 & \cdots & & -\lambda & 1+2\lambda \end{pmatrix}.$$

The system is written in the linear form by

$$u_j^n = B_\lambda u_j^{n+1} - \Delta t F_j. \tag{16}$$

3.3. Cranck-Nickolson scheme

The Crank-Nicholson scheme is based on the previous two. We evaluate the diffusion term by taking the average of the explicit and implicit writing, which gives

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\mu}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}\right) = f_j. \tag{17}$$

After calculations, we get

$$(1+\lambda)u_j^{n+1} - \frac{\lambda}{2}(u_{j+1}^{n+1} + u_{j-1}^{n+1}) = (1-\lambda)u_j^n + \frac{\lambda}{2}(u_{j+1}^n + u_{j-1}^n) + \Delta t f_j \quad \text{with} \quad \lambda = \mu\frac{\Delta t}{\Delta x^2}. \tag{18}$$

This is the Crank-Nicholson numerical scheme. By varying the index $j = 1, \cdots, N$ the equation (18), we have

$$C_\lambda u_j^{n+1} = D_\lambda u_j^n + \Delta t F_j$$

this allows to obtain in matrix form

$$u_j^{n+1} = C_\lambda^{-1} D_\lambda u_j^n + C_\lambda^{-1} \Delta t F_j, \tag{19}$$

with

$$C_\lambda = \begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ 0 & -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & 0 & \cdots & & -\frac{\lambda}{2} & 1+\lambda \end{pmatrix}$$

and

$$D_\lambda = \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & 0 & 0 & \cdots & 0 \\ \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & 0 & \cdots & & \frac{\lambda}{2} & 1-\lambda \end{pmatrix}.$$

4. STUDY OF THE ANALYTICAL STABILITY IN $L^\infty([0,1])$ AND $L^2([0,1])$ OF THE EXPLICIT EULER, IMPLICIT EULER AND CRANCK-NICHOLSON METHODS

In this section, we study the analytical stability in norm $L^\infty([0,1])$ and in norm $L^2([0,1])$ of explicit Euler , Implicit Euler and Cranck-Nicholson methods for the heat equation.

4.1. Analytical stability in $L^\infty([0,1])$ and in $L^2([0,1])$ for the explicit Euler method

This paragraph serves to study the analytical stability in $L^\infty([0,1])$ and in $L^2([0,1]$ of the explicit Euler method. To do this, we first establish the Von-Neumann-Fourier stability criterion that we use thereafter to establish the stability of said method.

Von-Neumann-Fourier criterion

In this criterion, one does not take into account the edge effects of discretization (limiting conditions) and one analyzes only the equation (11). This comes to consider the problem (12) no longer in a bounded interval but in the whole $\mathbb{R}$ and to ignore the boundary conditions. We also take $f \equiv 0$. In this case, the exact solution of (12) is bounded. We then want to find this property on the approximations u_j^n. We seek a solution in the following particular form, with $\xi \in \mathbb{R}$

$$u_j^n = \xi^n e^{ik\pi j\Delta x} \quad \text{for} \quad k \quad \text{fixed.} \tag{20}$$

Since $|u_j^n| = |\xi^n|$, we impose, for any mode k, the condition

$$|\xi| \leq 1, \tag{21}$$

so that the approximate solution is bounded, for all n. It is the condition of stability corresponding to the criterion of Von-Neumann. The parameter ξ is called *amplification factor* associated with the mode k.

For all $i \in \mathbb{C}$ and $x_j = j\Delta x$ where Δx is the constant step of discretization in space, we have

$$u_j^n = \xi^n e^{ik\pi x_j} \simeq \xi^n e^{ik\pi j\Delta x} \quad (22)$$

$$u_j^{n+1} = \xi^{n+1} e^{ik\pi x_j} \simeq \xi(\xi^n e^{ik\pi j\Delta x}) \quad (23)$$

$$u_{j-1}^n = \xi^n e^{ik\pi x_{j-1}} = \xi^n e^{ik\pi(j-1)\Delta x} \simeq \xi^n e^{ik\pi j\Delta}.e^{-ik\pi\Delta x} \quad (24)$$

$$u_{j+1}^n = \xi^n e^{ik\pi x_{j+1}} = \xi^n e^{ik\pi(j+1)\Delta x} \simeq \xi^n e^{ik\pi j\Delta}.e^{ik\pi\Delta x} \quad (25)$$

by replacing the expressions (22), (23), (24) and (25) in the relation (11) for $f \equiv 0$, we obtain

$$\xi(\xi^n e^{ik\pi j\Delta x}) = (1-2\lambda)\xi^n e^{ik\pi j\Delta x} + \lambda(\xi^n e^{ik\pi j\Delta}.e^{ik\pi\Delta x} + \xi^n e^{ik\pi j\Delta}.e^{-ik\pi\Delta x}) \quad (26)$$

Dividing (26) by $\xi^n e^{ik\pi j\Delta x}$, we find

$$\begin{aligned}\xi &= (1-2\lambda) + \lambda(e^{ik\pi j\Delta x} + e^{-ik\pi j\Delta x}),\\ &= 1 - 2\lambda + 2\lambda\cos(k\pi\Delta x),\\ &= 1 + 2\lambda[\cos(k\pi\Delta x) - 1],\end{aligned}$$

since $\cos(k\pi\Delta x) - 1 = -2\sin^2\left(\frac{k\pi\Delta x}{2}\right)$ therefore, by replacing this equality in the previous relation, we obtain

$$\xi = 1 - 4\lambda\sin^2\left(\frac{k\pi\Delta x}{2}\right) \quad (27)$$

with the relation (27), the stability condition (21) becomes

$$-1 \le 1 - \sin^2\left(\frac{k\pi\Delta x}{2}\right) \le 1,$$

For all k, that to say

$$4\lambda\sin^2\left(\frac{k\pi\Delta x}{2}\right) \le 2.$$

For this last inequality to be verified whatever k, we impose $4\lambda \le 2$ that is to say

$$\lambda \le \frac{1}{2} \quad \text{that is to say} \quad \mu\frac{\Delta t}{\Delta x^2} \le 1. \quad (28)$$

The condition (28) is a condition of stability (according to the criterion of Von-Neumann-Fourier) which relates the step in time to the step of space.

Stability in $L^\infty([0,1])$

In this subsection, we study the analytical stability of the solution u_j^n of the heat equation of the explicit Euler scheme.

let us consider the relation (13) that is to say

$$u_j^{n+1} = A_\lambda u_j^n + \Delta t F_j \quad \text{with} \quad A_\lambda = I_d - \lambda A.$$

By taking the semi-norm and the $\sup$ member to member, we have

$$\begin{aligned} |u_j^{n+1}| &= |A_\lambda u_j^n + \Delta t F_j| \\ &\leq |A_\lambda u_j^n| + |\Delta t F_j| \\ \sup_{j=1,\cdots,N} |u_j^{n+1}| &\leq \sup_{j=1,\cdots,N} |A_\lambda||u_j^n| + \Delta t \sup_{j=1,\cdots,N} |F_j| \\ ||u^{n+1}||_\infty &\leq ||A_\lambda||_\infty ||u^n||_\infty + \Delta t ||f||_{L^\infty([0,1])} \end{aligned}$$

Now, since $\lambda \leq \frac{1}{2}$, we deduce that

$$||A_\lambda||_\infty = \max_{1\leq i\leq N} \sum_{j=1}^{N} |m_{ij}| = |1-2\lambda| + 2\lambda = 1$$

which implies

$$||u^{n+1}||_\infty \leq ||u_j^n||_\infty + \Delta t ||f||_{L^\infty([0,1])} \tag{29}$$

By simple induction, we have

$$\begin{aligned} &\text{For} \quad n=0, && ||u^1||_\infty \leq ||u^0||_\infty + \Delta t ||f||_{L^\infty([0,1])} \\ &\text{For} \quad n=1, && ||u^2||_\infty \leq ||u^1||_\infty + \Delta t ||f||_{L^\infty([0,1])} \leq ||u^0||_\infty + 2\Delta t ||f||_{L^\infty([0,1])} \\ &\text{for} \quad n=2, && ||u^3||_\infty \leq ||u^2||_\infty + \Delta t ||f||_\infty \leq ||u^0||_\infty + 3\Delta t ||f||_{L^\infty([0,1])} \\ &\quad\vdots && \\ &\text{at rank} \quad n, && ||u^{n+1}||_\infty \leq ||u^n||_\infty + \Delta t ||f||_\infty \leq ||u^0||_\infty + T ||f||_{L^\infty([0,1])} \end{aligned}$$

$$\text{that is to say,} \quad ||u^{n+1}||_\infty \leq ||u^0||_\infty + T||f||_{L^\infty([0,1])} \quad \text{hence the result.}$$

Consequently the explicit Euler scheme is stable in $L^\infty([0,1])$.

Stability in $L^2([0,1])$

Here, we study the analytical stability of the solution u_j^n of the heat equation of the explicit Euler scheme.

Indeed, consider the Euclidean norm $||u||_2 = \left(\sum_{i=1}^{N} u_i^2 \right)^{\frac{1}{2}}$.

By taking member by member the previous norm in the relation (13), we obtain

$$||u^{n+1}||_2 \leq ||A_\lambda||_2 ||u^n||_2 + \Delta t ||f||_2 \tag{30}$$

The matrix A_λ being symmetrical, we have

$$||A_\lambda||_2 = \rho(A_\lambda) = \max_k |\lambda_k|,$$

where λ_k denotes the eigenvalues of $A_\lambda = I_d - \lambda A$.

$$\text{We have} \quad \lambda_k = 1 - 4\lambda \sin^2 \left[\frac{k\pi}{2(N+1)} \right], \quad \text{for} \quad k = 0, \cdots, N.$$

Therefore, we see that λ_k is exactly the amplification factor ξ associated with the mode k of the Von-Neumann criterion. Thus, with $\lambda \leq \frac{1}{2}$ we have $|\lambda_k| \leq 1$ and therefore $||A_\lambda||_2 \leq 1$.

thus , the equation (30) becomes

$$||u^{n+1}||_2 \leq ||u^n||_2 + \Delta t ||f||_2 \tag{31}$$

by simple induction, we have

$$\begin{array}{lll} \text{For} & n = 0, & ||u^1||_2 \leq ||u^0||_2 + \Delta t ||f||_2 \\ \text{For} & n = 1, & ||u^2||_2 \leq ||u^1||_2 + \Delta t ||f||_2 \leq ||u^0||_2 + 2\Delta t ||f||_2 \\ \text{For} & n = 2, & ||u^3||_2 \leq ||u^2||_2 + \Delta t ||f||_2 \leq ||u^0||_2 + 3\Delta t ||f||_2 \\ & \vdots & \\ \text{at rank} & n, & ||u^{n+1}||_2 \leq ||u^n||_2 + \Delta t ||f||_2 \leq ||u^0||_2 + T \max ||f||_2 \end{array}$$

$$\text{that is to say} \quad ||u^{n+1}||_2 \leq ||u^0||_2 + T \max ||f||_2 \tag{32}$$

Hence the stability in $L^2([0,1])$ of the explicit Euler scheme.

4.2. Analytical stability in $L^\infty([0,1])$ and in $L^2([0,1])$ for the implicit Euler method

In this section, we study the analytical stability in $L^\infty([0,1])$ and in $L^2([0,1]$ of the implicit Euler method. To do so, we first establish the Von-Neumann-Fourier stability criterion that we use to establish the stability of said method.

Von-Neumann-Fourier criterion

With $f \equiv 0$, we are looking for the solution of the relation (15).
let $u_j^n = \xi^n e^{ik\pi x}$, $i \in \mathbb{C}$ and $x_j = j\Delta x$ where Δx is the constant step of discretization in space,
thus

$$u_j^n = \xi^n e^{ik\pi x_j} \simeq \xi^n e^{ik\pi j\Delta x} \tag{33}$$
$$u_j^{n+1} = \xi^{n+1} e^{ik\pi x_j} \simeq \xi(\xi^n e^{ik\pi j\Delta x}) \tag{34}$$
$$u_{j-1}^{n+1} = \xi^{n+1} e^{ik\pi x_{j-1}} = \xi(\xi^n e^{ik\pi(j-1)\Delta x}) \simeq \xi(\xi^n e^{ik\pi j\Delta}.e^{-ik\pi\Delta x}) \tag{35}$$
$$u_{j+1}^{n+1} = \xi^{n+1} e^{ik\pi x_{j+1}} = \xi(\xi^n e^{ik\pi(j+1)\Delta x}) \simeq \xi(\xi^n e^{ik\pi j\Delta}.e^{ik\pi\Delta x}) \tag{36}$$

By replacing expressions (33), (34), (35) and (36) in the relation (15) and by posing $f_j = 0$, we obtain

$$\xi^n e^{ik\pi j\Delta x} = (1+2\lambda)\xi(\xi^n e^{ik\pi j\Delta x}) - \lambda\xi(\xi^n e^{ik\pi j\Delta}.e^{ik\pi\Delta x}) - \lambda\xi(\xi^n e^{ik\pi j\Delta}.e^{-ik\pi\Delta x}) \tag{37}$$

dividing (30) by $\xi^n e^{ik\pi j\Delta x}$, we find

$$1 = (1+2\lambda)\xi - \lambda\xi e^{ik\pi\Delta x} - \lambda\xi e^{-ik\pi\Delta x} \tag{38}$$

which implies

$$\begin{aligned}(1+2\lambda)\xi - \lambda\xi(e^{ik\pi\Delta x} + e^{-ik\pi\Delta x}) &= 1,\\ (1+2\lambda)\xi - 2\lambda\xi\cos(k\pi\Delta x) &= 1,\\ \xi[1+2\lambda-2\lambda\cos(k\pi\Delta x)] &= 1\end{aligned}$$

$$\xi[1+2\lambda(1-\cos(k\pi\Delta x))] = 1. \tag{39}$$

Since

$$1-\cos(k\pi\Delta x) = 2\sin^2(\frac{k\pi\Delta x}{2}),$$

then by replacing this equality in the relationship (32), we obtain

$$\xi[1+4\lambda\sin^2(\frac{k\pi\Delta x}{2})] = 1$$

which implies

$$\xi = \frac{1}{1+4\lambda\sin^2(\frac{k\pi\Delta x}{2})}$$

For any k, we clearly have $0 \leq \xi \leq 1$ and consequently the implicit Euler scheme is *unconditionally stable* within the meaning of the Von-Neumann criterion.

4.2.1 Analytical stability in $L^\infty([0,1])$ for the implicit Euler method

In this subsection, we study the analytical stability in $L^\infty([0,1])$ of the solution u_j^n of the heat equation from the implicit Euler scheme. Indeed, we use the stability criterion of Von-Neumann-Fourier obtained previously and the principle of the maximum. Consider the relation (15)

$$u_j^n = (1+2\lambda)u_j^{n+1} - \lambda u_{j+1}^{n+1} - \lambda u_{j-1}^{n+1} - \Delta t f_j$$

which implies

$$(1+2\lambda)u_j^{n+1} = u_j^n + \lambda u_{j+1}^{n+1} + \lambda u_{j-1}^{n+1} + \Delta t f_j$$

By taking the semi-norm member to member, we have

$$(1+2\lambda)|u_j^{n+1}| = |u_j^n + \lambda u_{j+1}^{n+1} + \lambda u_{j-1}^{n+1} + \Delta t f_j|$$

Using the triangular inequality, we get

$$(1+2\lambda)|u_j^{n+1}| \leq |u_j^n| + \lambda|u_{j+1}^{n+1}| + \lambda|u_{j-1}^{n+1}| + |\Delta t f_j|$$

We deduce, for $j = 1, \cdots, N$

$$(1+2\lambda)||u_j^{n+1}||_\infty \leq ||u_j^n||_\infty + \lambda||u_{j+1}^{n+1}||_\infty + \lambda||u_{j-1}^{n+1}||_\infty + \Delta t||f||_{L^\infty([0,1])}$$

After calculations, we get

$$||u^{n+1}||_\infty \leq ||u^n||_\infty + \Delta t||f||_{L^\infty([0,1])} \tag{40}$$

By varying n, we get

$$||u^{n+1}||_\infty \leq ||u^0||_\infty + T||f||_{L^\infty([0,1])} \tag{41}$$

Hence the stability in $L^\infty([0,1])$ for the implicit Euler method.

4.2.2 Analytical stability in $L^2([0,1])$ for the implicit Euler method

In this subsection, we study the analytical stability in $L^2([0,1])$ of said method. Indeed,with homogeneous Dirichlet conditions, the linear system (16) becomes

$$u_j^{n+1} = B_\lambda^{-1}(u_j^n + \Delta t F_j), \quad \text{avec} \quad B_\lambda^{-1} = I_d + \lambda A, \tag{42}$$

We deduce that

$$||u^{n+1}||_2 \leq ||B_\lambda^{-1}||_2(||u^n||_2 + \Delta t||f||_2). \tag{43}$$

since

$$B_\lambda^{-1} = \frac{1}{\min_k |\mu_k|},$$

where μ_k are the eigenvalues of $B_\lambda = I_d + \lambda A$. So we have $\mu_k = 1 + \lambda\beta_k$ where $\beta_k = 4\sin^2\left[\frac{k\pi}{2(N+1)}\right]$ are the eigenvalues of A. We thus obtain

$$\mu_k = 1 + 4\lambda\sin^2\left[\frac{k\pi}{2(N+1)}\right] > 1, \quad \text{for} \quad k = 1\cdots, N \tag{44}$$

therefore $||B_\lambda^{-1}||_2 \leq 1$.

The relation (43) becomes

$$||u^{n+1}||_2 \leq ||u^n||_2 + \Delta t||f||_2 \tag{45}$$

By varying n, we obtain

$$||u^{n+1}||_2 \leq ||u^0||_2 + T\max||f||_2 \tag{46}$$

From where the analytical stability of the implicit Euler method in $L^2([0,1])$.

4.3. Analytical stability in $L^\infty([0,1])$ and in $L^2([0,1])$ for the Cranck-Nicholson method

At this step of the work, we study the analytical stability in $L^\infty([0,1])$ and in $L^2([0,1])$ of the solution u_j^n from the Cranck-Nicholson scheme. We first establish the Von-Neumann-Fourier stability criterion that we use for the stability of said method.

Von-Neumann-Fourier criterion

Let $u_j^n = \xi^n e^{ik\pi x}$, $i \in \mathbb{C}$ and $x_j = j\Delta x$ where Δx is the constant step of discretization in space,
thus:

$$u_j^n = \xi^n e^{ik\pi x_j} \simeq \xi^n e^{ik\pi j\Delta x} \tag{47}$$

$$u_j^{n+1} = \xi^{n+1} e^{ik\pi x_j} \simeq \xi(\xi^n e^{ik\pi j\Delta x}) \tag{48}$$

$$u_{j-1}^n = \xi^n e^{ik\pi x_{j-1}} = \xi^n e^{ik\pi(j-1)\Delta x} \simeq \xi^n e^{ik\pi j\Delta}.e^{-ik\pi\Delta x} \tag{49}$$

$$u_{j+1}^n = \xi^n e^{ik\pi x_{j+1}} = \xi^n e^{ik\pi(j+1)\Delta x} \simeq \xi^n e^{ik\pi j\Delta}.e^{ik\pi\Delta x} \tag{50}$$

By replacing expressions (47), (48) , (49) and (50) in the relation (18) and by posing $f_j \equiv 0$, we obtain

$$\begin{aligned}(1+\lambda)\xi^{n+1}e^{ik\pi j\Delta x} - \frac{\lambda}{2}(\xi^{n+1}e^{ik\pi(j+1)\Delta x} + \xi^{n+1}e^{ik\pi(j-1)\Delta x})\\ = (1-\lambda)\xi^n e^{ik\pi j\Delta x} + \frac{\lambda}{2}(\xi^n e^{ik\pi(j+1)\Delta x} + \xi^n e^{ik\pi(j-1)\Delta x})\end{aligned} \tag{51}$$

by dividing (51) par $\xi^n e^{ik\pi j\Delta x}$, we find

$$\begin{aligned}
(1+\lambda)\xi - \frac{\lambda}{2}(\xi e^{ik\pi\Delta x} + \xi e^{-ik\pi\Delta x}) &= 1-\lambda+\frac{\lambda}{2}(e^{ik\pi\Delta x}+e^{-ik\pi\Delta x}) \\
\xi[(1+\lambda) - \frac{\lambda}{2}(e^{ik\pi\Delta x}+e^{-ik\pi\Delta x})] & \\
&= 1-\lambda+\frac{\lambda}{2}(e^{ik\pi\Delta x}+e^{-ik\pi\Delta x}) \\
\xi[1+\lambda(1-\cos(k\pi\Delta x))] &= 1+\lambda[\cos(k\pi\Delta x)-1] \\
\xi[1+2\lambda\sin^2(\frac{k\pi\Delta x}{2})] &= 1-2\lambda\sin^2(\frac{k\pi\Delta x}{2})
\end{aligned}$$

Which implies

$$\xi = \frac{1-2\lambda\sin^2(\frac{k\pi\Delta x}{2})}{1+2\lambda\sin^2(\frac{k\pi\Delta x}{2})} \tag{52}$$

We have $|\xi| \leq 1$. The Crank-Nicholson scheme is *unconditionally stable in the sense of Von-Neumann.*

4.3.1 Analytical stability in $L^\infty([0,1])$ for the Cranck-Nicholson method

We study the analytical stability in $L^\infty([0,1])$ of the Cranck-Nicholson method used for the heat equation problem. To do this, we use the Von-Neumann-Fourier condition obtained previously and the maximum principle.

Consider linear interpolation (18) between u_{j-1}^n, u_j^n, u_j^{n+1}, u_{j-1}^{n+1} and u_{j+1}^{n+1}

$$(1+\lambda)u_j^{n+1} = \frac{\lambda}{2}(u_{j+1}^{n+1}+u_{j-1}^{n+1}) + (1-\lambda)u_j^n + \frac{\lambda}{2}(u_{j+1}^n+u_{j-1}^n) + \Delta t f_j$$

Using the maximum principle, (see [9]), we obtain

$$(1+\lambda)||u^{n+1}||_\infty \leq \lambda||u^{n+1}||_\infty + (\lambda+|1-\lambda|)||u^n||_\infty + \Delta t||f||_{L^\infty([0,1])}$$

Or $\lambda \leq 1$ therefore $\lambda+|1-\lambda| = \lambda+1-\lambda = 1$,

which allows to obtain after calculations, the relation

$$||u^{n+1}||_\infty \leq ||u^n||_\infty + \Delta t||f||_{L^\infty([0,1])} \tag{53}$$

By varying n, we obtain

$$||u^{n+1}||_\infty \leq ||u^0||_\infty + T||f||_{L^\infty([0,1])} \tag{54}$$

Hence the analytical stability in $L^\infty([0,1])$ for the Cranck-Nicholson method.

4.3.2 Analytical stability in $L^2([0,1])$ for the Cranck-Nicholson method

We want to study the analytical stability in $L^2([0,1])$ of the Cranck-Nicholson method used for solving the problem of the heat equation.
Let us now study the stability $L^2([0,1])$ of the Cranck-Nicholson scheme, using the lemma 4.1.5 (see [9]). Considering the homogeneous Dirichlet conditions, the system (19) is written

$$u_j^{n+1} = Pu^n + \frac{\Delta t}{2}\left(I_d + \frac{\lambda}{2}A\right)^{-1} F_j \quad \text{with} \quad P = \left(I_d + \frac{\lambda}{2}A\right)^{-1}\left(I_d - \frac{\lambda}{2}A\right). \tag{55}$$

we deduce that

$$||u^{n-1}||_2 \leq ||P||_2||u^n||_2 + \frac{\Delta t}{2}||\left(I_d + \frac{\lambda}{2}A\right)^{-1}||_2 \;\; ||f||_2. \tag{56}$$

By property 3 of lemma 4.1.5, P is symmetric and $||P||_2 = \rho(P) \leq 1$.
Note now that $C_\lambda^{-1} = \left(I_d + \frac{\lambda}{2}A\right)^{-1}$. We have $||C_\lambda^{-1}||_2 = \rho(C_\lambda^{-1})$ because C_λ^{-1} is symmetric and the eigenvalues of C_λ^{-1} are given by $\phi_k = \dfrac{1}{1+\frac{\lambda}{2}\mu_k}$ where μ_k are the eigenvalues of A. Since $\mu_k > 0$, we have $\phi_k < 1$, $\forall k$ and therefore $||C_\lambda^{-1}||_2 \leq 1$. We thus deduce from (55),

$$||u^{n+1}||_2 \leq ||u^n||_2 + \Delta t||f||_2 \tag{57}$$

We therefore have by induction

$$||u^{n+1}||_2 \leq ||u^0||_2 + T\max||f||_2 \tag{58}$$

Therefore the Crank-Nicholson scheme is a time-precise scheme (order 2) which is unconditionally L^2 -stable under the condition $\lambda \leq 1$.

5. CALCULATION OF THE TRUNCATION ERROR OF THE HEAT EQUATION

In this section, we compute the truncation error of the explicit Euler, implicit Euler and Cranck-Nicholson methods that we denote respectively ζ_E, ζ_I and ζ_C. Therefore, we will consider in all calculations $f \equiv 0$.

5.1. Calculation of the truncation error of the heat equation from the explicit Euler method

Here we want to calculate the truncation error of the explicit Euler method for the problem of the heat equation in order to prove the convergence of the chosen method.

Let the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \mu\frac{\partial^2 u}{\partial x^2}(x,t) = 0 \tag{59}$$

Let us first use the Taylor expansion of order 3 in time
forward scheme

$$u(x, t + \Delta t) = u(x,t) + \Delta t\frac{\partial u}{\partial t}(x,t) + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}(x,t) + \mathcal{O}(\Delta t^3)$$

$$u(x, t + \Delta t) - u(x,t) = \Delta t\frac{\partial u}{\partial t}(x,t) + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}(x,t) + \mathcal{O}(\Delta t^3) \tag{60}$$

dividing (60) by Δt we obtain

$$\frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} = \frac{\partial u}{\partial t}(x,t) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x,t) + \mathcal{O}(\Delta t^2) \tag{61}$$

but

$$\frac{\partial u}{\partial t}(x,t) = \mu\frac{\partial^2 u}{\partial x^2}(x,t), \tag{62}$$

using (62), we prove that

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}(x,t)\right], \\
&= \mu\frac{\partial^2}{\partial x^2}\left[\frac{\partial u}{\partial t}(x,t)\right], \\
\frac{\partial^2 u}{\partial t^2}(x,t) &= \mu^2\frac{\partial^4 u}{\partial x^4}(x,t). \qquad (63)
\end{aligned}$$

Replacing (62) and (63) in (61) we obtain

$$\frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} = \mu\frac{\partial^2 u}{\partial x^2}(x,t) + \mu^2\frac{\Delta t}{2}\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta t^2) \tag{64}$$

In addition, let us use the Taylor expansion of order 5 in space.
Backward scheme

$$u(x-\Delta x, t) = u(x,t) - \Delta x\frac{\partial u}{\partial x}(x,t) + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3}(x,t) + \frac{\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^5) \tag{65}$$

$$u(x+\Delta x, t) = u(x,t) + \Delta x\frac{\partial u}{\partial x}(x,t) + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3}(x,t) + \frac{\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^5) \tag{66}$$

the sum member to member of (65) and (66) gives

$$u(x-\Delta x, t) + u(x+\Delta x, t) = 2u(x,t) + \Delta x^2\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\Delta x^4}{12}\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^5)$$

$$u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t) = \Delta x^2\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\Delta x^4}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^5) \quad (67)$$

dividing (67) by Δx^2, we obtain

$$\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^3) \quad (68)$$

Let us Multiply (68) by $-\mu$

$$-\mu\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2} = -\mu\frac{\partial^2 u}{\partial x^2}(x,t)-\mu\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^3) \quad (69)$$

The addition member to member of (64) and (69) gives

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} - \mu\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2}$$

$$= \mu\frac{\partial^2 u}{\partial x^2}(x,t)+\mu^2\frac{\Delta t}{2}\frac{\partial^4 u}{\partial x^4}(x,t)-\mu\frac{\partial^2 u}{\partial x^2}(x,t)$$

$$-\mu\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^3+\Delta t^2)$$

$$= \mu^2\frac{\Delta t}{2}\frac{\partial^4 u}{\partial x^4}(x,t)-\mu\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^3+\Delta t^2)$$

we obtain

$$\zeta_E = \frac{\mu}{2}\left(\mu\Delta t-\frac{\Delta x^2}{6}\right)\frac{\partial^4 u}{\partial x^4}(x,t)+\mathcal{O}(\Delta x^3+\Delta t^2) \quad (70)$$

From where the truncation error of order 1 in time and order 2 in space of the explicit Euler method for the heat equation.

5.2. Calculation of the truncation error of the heat equation from the implicit Euler method

Here, we want to calculate the truncation error of the implicit Euler method for the problem of the heat equation in order to prove the convergence of the chosen method. Let the heat equation

$$\frac{\partial u}{\partial t} - \mu\frac{\partial^2 u}{\partial t^2}(x,t) = 0$$

In the relation (68), replace t by $t+\Delta t$, we obtain after a Taylor expansion in (x_j,t_n) that

$$\frac{u(x+\Delta x,t+\Delta t)-2u(x,t+\Delta t)+u(x-\Delta x,t+\Delta t)}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t)+$$

$$+\Delta t\frac{\partial^3 u}{\partial t\partial x^2}(x,t)+\mathcal{O}(\Delta x^3+\Delta t^2)$$

$$= \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t) + \Delta t \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}(x,t)\right) + \mathcal{O}(\Delta x^3 + \Delta t^2) \quad (71)$$

Replacing (62) in (71), we find

$$\frac{u(x+\Delta x, t+\Delta t) - 2u(x, t+\Delta t) + u(x-\Delta x, t+\Delta t)}{\Delta x^2}$$
$$= \frac{\partial^2 u}{\partial x^2}(x,t) + \left(\frac{\Delta x^2}{12} + \mu\Delta t\right)\frac{\partial^4 u}{\partial x^4}(x,t) +$$
$$+ \mathcal{O}(\Delta x^3 + \Delta t^2) \quad (72)$$

Let us multiply (72) by $-\mu$

$$-\mu\frac{u(x+\Delta x, t+\Delta t) - 2u(x, t+\Delta t) + u(x-\Delta x, t+\Delta t)}{\Delta x^2} = -\mu\frac{\partial^2 u}{\partial x^2}(x,t) -$$
$$-\mu\left(\frac{\Delta x^2}{12} + \mu\Delta t\right)\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^3 + \Delta t^2) \quad (73)$$

By making the sum member to member of the relations (64) and (73) we obtain the truncation error of order 1 in time and order 2 in space of the implicit Euler scheme

$$\zeta_I = -\frac{\mu}{2}\left(\mu\Delta t + \frac{\Delta x^2}{6}\right)\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^3 + \Delta t^2) \quad (74)$$

5.3. Calculation of the truncation error of the heat equation from the Cranck-Nicholson method

The Crank-Nicholson scheme is based on the explicit and implicit Euler schemes. Thus the Cranck-Nicholson error is the average between the error of order 2 in space below

$$\zeta_C = -\mu\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^3 + \Delta t^2) \quad (75)$$

6. STUDY OF THE CONVERGENCE OF METHODS

In this section, we will study the convergence of the solution u_j^n of the heat equation of the explicit Euler, implicit Euler and Cranck-Nicholson schemes towards the exact solution $u(x,t)$,(see [8]). To do this, we can either try to study convergence directly, which can be complicated and not always possible, or use the consistency and stability of our scheme because it is generally much easier to study than its convergence.

6.1. Convergence for the explicit Euler method of the heat equation

To study the convergence of the solution u_j^n from the heat equation of the explicit Euler scheme to the exact solution $u(x;t)$, we first study the consistency of said scheme.

Study of the consistency

This scheme is said to be consistent if the truncation error tends towards zero when the step of time discretization Δt and the space discretization step Δx tend to zero independently.

According to our analysis, the truncation error is

$$\zeta_E = \frac{\mu}{2}\left(\mu\Delta t - \frac{\Delta x^2}{6}\right)\frac{\partial^4 u}{\partial x^4}(x,t) + \mathcal{O}(\Delta x^3 + \Delta t^2)$$

By taking the limit we have

$$\lim_{(\Delta x,\Delta t)\to(0,0)} \zeta_E = \lim_{(\Delta x,\Delta t)\to(0,0)} \frac{\mu}{2}\left(\mu\Delta t - \frac{\Delta x^2}{6}\right)\frac{\partial^4 u}{\partial x^4}(x,t) = 0 \qquad (76)$$

whith

$$\mathcal{O}(\Delta x^3 + \Delta t^2) \to 0 \quad \text{when} \quad (\Delta x, \Delta t) \to (0,0)$$

Hence the consistency of the explicit Euler scheme. The scheme being stable and consistent, then the numerical solution of u_j^n of the explicit Euler scheme for the heat equation is convergent.

6.2. Convergence for the implicit Euler method of the heat equation

To study the convergence of the solution u_j^n of the heat equation of the implicit Euler scheme to the exact solution $u(x;t)$, let us first study the consistency of the scheme.

Study of the Consistency

This scheme is said to be consistent if the truncation error tends towards zero when the step of time discretization Δt and the space discretization step Δx tend to zero independently. Indeed, let us calculate the limit of the truncation error (74).

$$\lim_{(\Delta x,\Delta t)\to(0,0)} \zeta_I = \lim_{(\Delta x,\Delta t)\to(0,0)} -\frac{\mu}{2}\left(\mu\Delta t + \frac{\Delta x^2}{6}\right)\frac{\partial^4 u}{\partial x^4}(x,t) = 0 \qquad (77)$$

with

$$\mathcal{O}(\Delta x^3 + \Delta t^2) \to 0 \quad \text{when} \quad (\Delta x, \Delta t) \to (0,0)$$

Hence the implicit Euler scheme is consistent. Therefore the numerical solution of u_j^n of the implicit Euler scheme for the heat equation is convergent.

6.3. Convergence for the method of the Crank-Nicholson method of the heat equation

To study the convergence of the solution u_j^n of the heat equation of the Cranck-Nicholson scheme towards the exact solution $u(x;t)$, let us first study the consistency.

Study of the consistency

This scheme is said to be consistent if the truncation error tends towards zero when the step of time discretization Δt and the space discretization step Δx tend to zero independently. For that, let us compute the limit of the error of truncation (75) of the scheme of Cranck-Nicholson.

$$\lim_{(\Delta x,\Delta t)\to(0,0)} \zeta_C = \lim_{(\Delta x,\Delta t)\to(0,0)} -\mu\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(x,t) = 0 \tag{78}$$

whith

$$\mathcal{O}(\Delta x^3 + \Delta t^2) \to 0 \quad \text{when} \quad (\Delta x, \Delta t) \to (0,0)$$

So the Crank-Nicholson scheme is consistent. Therefore the numerical solution of u_j^n from the Cranck-Nicholson scheme for the heat equation is convergent.

7. NUMERICAL COMPARISON OF THE ERRORS IN NORM $L^2(\Omega)$ OF THE SOLUTIONS OBTAINED

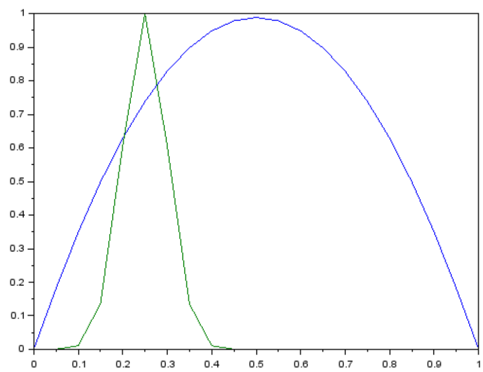
In this section, we want to calculate the different errors in norm $L^2(\Omega)$ of the solutions obtained by solving the heat equation by the numerical methods of explicit Euler, implicit Euler and Cranck-Nicholson respectively . These errors are calculated by the formula:

$$||e||_{L^2} = \sqrt{\sum_{i=1}^{N}(u_i - u_i^*)^2} \tag{79}$$

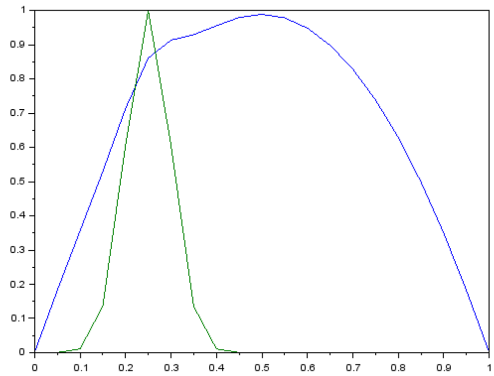
where u_i and u_i^* represent the different values (or components) of the numerical solutions obtained by the methods of explicit Euler, implicit Euler and Cranck-Nicholson in the resolution of the heat equation (1)-(3) for N points.

In order to calculate and compare the errors between the different solutions by the formula (79) of the three numerical methods stated above, we set the number of steps in space to 19 ($N = 19$) and the number step in time to 99 ($T = 99$) in all the calculations. Then, we seek to vary the thermal diffusion coefficient μ and the time step Δt of the various numerical methods especially explicit Euler, implicit Euler and Cranck-Nicholson methods.

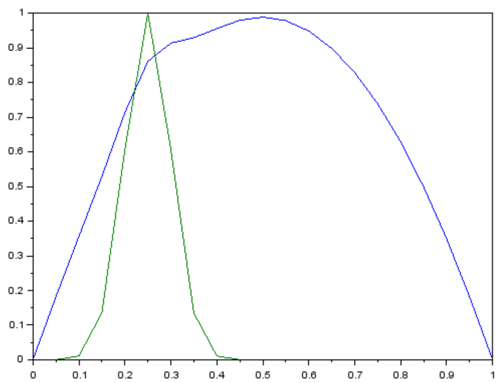
• By taking $\mu = 0.01$ and $\Delta t = 0.002$, the Table 1 and the Figure 1 give and display respectively the components and solutions of the numerical resolution of the heat equation (1) - (3) by the explicit Euler, implicit Euler and Cranck-Nicholson methods.



Explicit Euler



Implicit Euler



Cranck-Nicholson

Figure 1: Representation of the solution of the problem of the heat equation by the explicit Euler, implicit Euler and Cranck-Nicholson methods for $\mu = 0.01$, $\Delta t = 0.002$, $N = 19, \quad T = 99$ and $f(x) = e^{-200(x-\frac{1}{4})^2}$.

Table 1: Different values (or components) of the solution of the heat equation by three methods: explicit Euler, implicit Euler and Cranck-Nicholson for $\mu = 0.01$, $N = 19$,$\Delta t = 0.002$, $T = 99$ and $f(x) = e^{-200(x-\frac{1}{4})^2}$

Explicit Euler	**Implicit Euler**	**Cranck-Nicholson**
0.1782263	0.1806513	0.1805966
0.3449844	0.3567639	0.3566755
0.4942958	0.5384747	0.5384150
0.6241786	0.7355206	0.73560
0.7341622	0.8900153	0.8902163
0.8241602	0.9354999	0.9355897
0.8941600	0.9383305	0.9382754
0.94416	0.9559227	0.9558462
0.97416	0.9766143	0.9765736
0.98416	0.9845888	0.9845749
0.97416	0.9742244	0.9742209
0.94416	0.9441685	0.9441678
0.8941600	0.8941610	0.8941609
0.8241602	0.8241604	0.8241603
0.7341622	0.7341627	0.7341624
0.6241786	0.6241811	0.6241798
0.4942958	0.4943053	0.4943005
0.3449844	0.3450097	0.3449971
0.1782263	0.1782614	0.1782439

Error between explicit and implicit Euler solutions

Here, we make a numerical application of the formula (79) to calculate the numerical error between the explicit and implicit Euler solutions obtained in solving the heat equation (see Table 1). By considering u and u^* the components (or values) of solutions (Table 1) by explicit and implicit Euler respectively, we obtain

$$||e||_{L^2} = 0.22969063977 \tag{80}$$

Error between explicit Euler and Crank-Nicholson solutions

In this section, we apply the formula (79) calculating the numerical error between the explicit Euler and Cranck-Nicholson solutions obtained in solving the heat equation (see Table 1). Considering u and u^* the components of the solutions by explicit Euler and Cranck-Nicholson respectively, we obtain

$$||e||_{L^2} = 0.231005922 \tag{81}$$

Error between implicit Euler and Crank-Nicholson solutions

In this section, we calculate the numerical error between the implicit Euler and Cranck-Nicholson solutions obtained in the resolution of the heat equation by making the numerical application of the formula (79) of which u and u^* represent respectively the values of the solutions obtained by implicit Euler and Cranck-Nicholson (see Table 1). After calculations, we obtain

$$||e||_{L^2} = 0.000286277066 \tag{82}$$

• by taking $\mu = 0.3$ and $\Delta t = 0.0005$, the Table 2 and the Figure 2 give and display respectively the components and solutions of the numerical resolution of the heat equation (1) - (3) by the methods of explicit Euler, implicit Euler and Cranck-Nicholson.

Error between explicit and implicit Euler solutions

Here, we make a numerical application of the formula (79) to calculate the numerical error between the explicit and implicit Euler solutions obtained in the resolution of the heat equation (see Table 2). By considering u and u^* the components (or values) of the solutions (Table 2) by explicit and implicit Euler respectively, we obtain

$$||e||_{L^2} = 0.0418595628 \tag{83}$$

Error between explicit Euler method and Crank-Nicholson

In this section, we apply the formula (79) calculating the numerical error between the explicit Euler and Cranck-Nicholson solutions obtained in solving the heat equation (see Table 2). By considering u and u^* the components of the explicit and implicit Euler solutions respectively, we obtain

$$||e||_{L^2} = 0.081322329098 \tag{84}$$

Table 2: Different values (or components) of the solution of the heat equation by three methods: explicit Euler, implicit Euler and Cranck-Nicholson for $\mu = 0.3$, $N = 19$,$\Delta t = 0.0005$, $T = 99$ and $f(x) = e^{-200(x-\frac{1}{4})^2}$

Euler implicit	Euler implicit	Cranck-Nicholson
0.1443489	0.1479206	0.1478626
0.2841340	0.2919131	0.2918145
0.4151620	0.4282953	0.4281824
0.5338909	0.5530032	0.5528996
0.6375685	0.6597231	0.6596417
0.7242252	0.7433730	0.7433147
0.7925616	0.8058830	0.8058414
0.8417872	0.8501543	0.8501209
0.8714628	0.8764271	0.8763955
0.8813755	0.8841823	0.8841479
0.8714628	0.8729912	0.8729501
0.8417872	0.8426154	0.8425621
0.7925616	0.7930460	0.7929739
0.7242252	0.7245709	0.724474
0.6375685	0.6378791	0.6377560
0.5338909	0.5342010	0.5340589
0.4151620	0.4154582	0.4153149
0.2841340	0.2843742	0.2842557
0.1443489	0.1444848	0.1444172

Error between implicit Euler solutions and Crank-Nicholson

In this section, we calculate the numerical error between the implicit Euler and Cranck-Nicholson solutions obtained in the resolution of the heat equation by making the numerical application of the formula (79) of which u and u^* respectively represent the values of the solutions obtained by implicit Euler and Cranck-Nicholson (see Table 2). After calculations, we obtain

$$||e||_{L^2} = 0.0000001452229 \tag{85}$$

Discussion In section 7, we calculated the errors in norm $L^2(\Omega)$ of the solutions obtained by solving the heat equation by the numerical methods of explicit Euler,

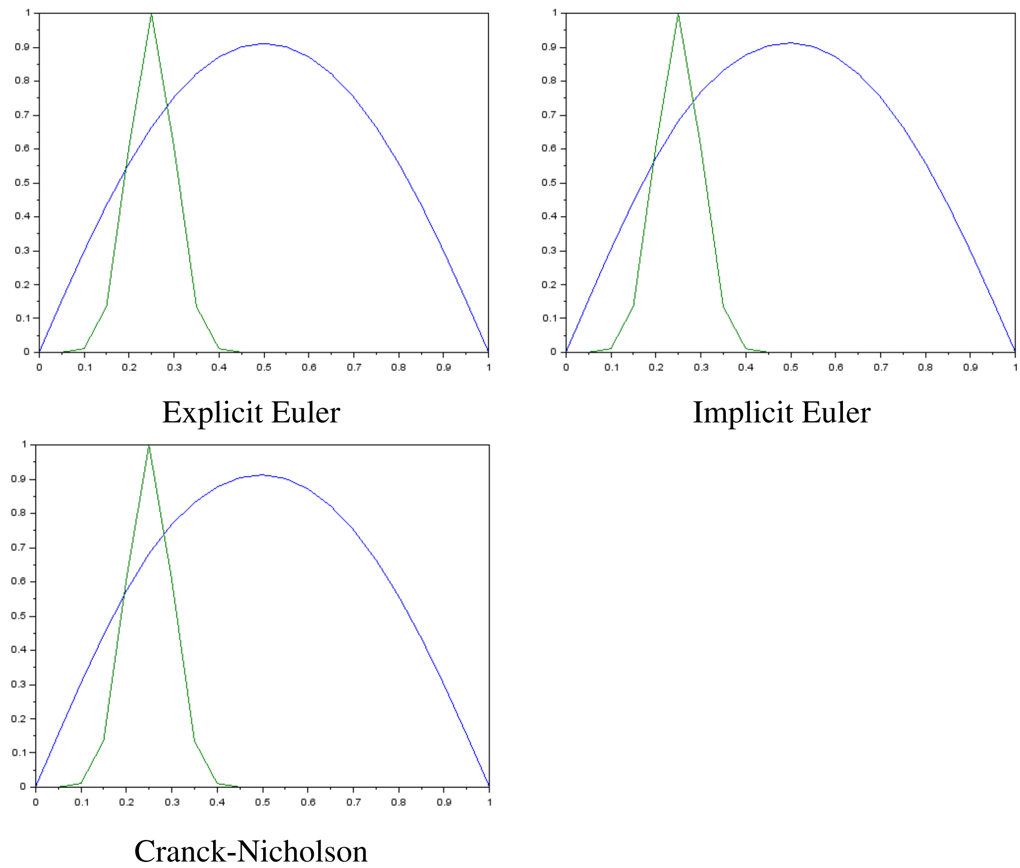


Figure 2: Representation of the solution of the problem of the heat equation by explicit Euler, implicit Euler and Cranck-Nicholson method respectively, for $\mu = 0.3$, $\Delta t = 0.0005$, $N = 19, T = 99$, and $f(x) = e^{-200(x-\frac{1}{4})^2}$.

implicit Euler and Cranck-Nicholson. We found that the errors (82) and (85) between the implicit Euler solutions and Cranck-Nicholson calculated for the values of: $\mu = 0.01, \quad \Delta t = 0.002$ and $\mu = 0.3, \quad \Delta t = 0.0005$ respectively are very close to each other. This leads us to choose the solution obtained by the Cranck-Nicholson method with $\mu = 0.3$ and $\Delta = 0.0005$ to solve the problem in the section 8.

8. IDENTIFICATION PROBLEM OF THE DIFFUSION COEFFICIENT μ IN THE THE HEAT EQUATION MODEL

To identify the diffusion coefficient in the model (2.1), we have defined an optimization problem of the least square type $\psi = \psi(\mu)$ which depends on the square of the difference of the numerical and experimental celerities of heat propagation, (see [14],

[6]). The objective function $\psi = \psi(\mu)$ is defined as follows:

$$\psi(\mu) = \frac{1}{2}(\widehat{c} - c^*)^2, \quad \text{avec} \quad \widehat{c} = \widehat{c}(\mu). \tag{86}$$

$\widehat{c}$ is the numerical speed and c^* is the experimental speed.

We will look for μ^* minimizing $\psi = \psi(\mu)$ where u is the solution of (1) with the boundary conditions (2) and initial (3).

8.1. Differentiability of u of the heat equation model with respect to μ

We want to minimize the function $\psi = \psi(\mu)$. For this, we want to check if the function ψ is differentiable with respect to the heat diffusion coefficient μ, that is to say $\mu \mapsto u_\mu \mapsto c(u_\mu) \mapsto \psi(c)$. The chain derivation requires that the function relating μ to ψ be differentiable, in particular u_μ with respect to μ. We want to check if the solution of the equation (1) is differentiable with respect to the heat diffusion coefficient μ.
To do this, denote by $\delta u = \delta u(x,t)$ the directional derivative of the solution of the model of the heat equation and $\delta\mu$ the increment of the heat diffusion coefficient, (see [13]). We will admit that the source term $f(x) = 0$. The equation (1) becomes

$$\frac{\partial u}{\partial t} - \mu\frac{\partial^2 u}{\partial x^2} = 0. \tag{87}$$

Let u_μ be the solution of

$$\frac{\partial u_\mu}{\partial t} - \mu\frac{\partial^2 u_\mu}{\partial x^2} = 0. \tag{88}$$

we consider $u + \delta u$ the solution of

$$\frac{\partial(u_\mu + \delta u)}{\partial t} - (\mu + \delta\mu)\frac{\partial^2(u_\mu + \delta u)}{\partial x^2} = 0. \tag{89}$$

combining equations (88) and (89), we obtain

$$\frac{\partial\delta u}{\partial t} - (\mu + \delta\mu)\frac{\partial^2\delta u}{\delta x^2} = \delta\mu\frac{\partial^2 u_\mu}{\partial x^2}, \tag{90}$$

which is equivalent to the system

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta u\\ \\ 0\end{pmatrix} - \underbrace{\frac{\partial^2}{\partial x^2}\left[\begin{pmatrix}\mu + \delta\mu\\ \\ 0\end{pmatrix}\begin{pmatrix}\delta u\\ \\ 0\end{pmatrix}\right]}_{\text{Diffusion term}} = \underbrace{\delta\mu\begin{pmatrix}\frac{\partial^2 u_\mu}{\partial x^2}\\ \\ 0\end{pmatrix}}_{\text{Source term}}, \tag{91}$$

which leads to

$$(\delta u)_t = (\delta u)_{xx} + (\delta u_\mu)_{xx}. \tag{92}$$

We use the following initial condition because not being interested in studying the variation of the solution u compared to the initial condition

$$\delta u(x,0) = \delta u_0(x) \quad \text{with} \quad u_0(x) = 4x(1-x) \tag{93}$$

The system (92) proves that the solution $u = u(x,t)$, $x \in [0,1]$, $t \in [0,T]$ is differentiable with respect to μ.

9. NUMERICAL OPTIMIZATION METHOD USED

We now present the numerical method used to minimize the function $\psi = \psi(\mu)$. Since this function is a composite function $\mu \mapsto u_\mu \mapsto c(u_\mu) \mapsto \psi(c)$, with a complex dependency and differentiable with respect to μ. We therefore used a differentiable optimization method, in particular gradient descent method with fixed steps to identify the heat diffusion coefficient μ of the heat equation model.

9.1. Gradient descent method with fixed steps

Gradient descent method with fixed steps allows to calculate numerically μ^* minimising $\psi = \psi(\mu)$ such that

$$\psi(\mu^*) = \min_{\mu \in \mathbb{R}} \psi(\mu)$$

The principle is to build an iterative algorithm of the form,(see [9], and [2])

$$\mu_{j+1} = \mu_j - \rho \nabla \psi(\mu_j) \; \forall j \in \mathbb{N}$$

where $-\nabla\psi(\mu_j)$ is the direction of strongest decay of ψ, $\rho \in \mathbb{R}$ is the step of the method and $\mu_0 \in \mathbb{R}$ the initial value. For $j \in \mathbb{N}$, the iterated μ_{j+1} is calculated from μ_j by

$$\begin{cases} \mu_0 \text{ given in } \mathbb{R} \\ \mu_{j+1} = \mu_j - \rho \nabla \psi(\mu_j) \\ \rho \in \mathbb{R} \quad \text{fixed} \end{cases} \tag{94}$$

This method requires the calculation of the gradient of $\psi(\mu_j)$, that is to say$\nabla\psi(\mu_j)$

9.1.1 Calculation of the gradient

The idea here is to increment the parameters to calculate an approximation by the finite difference method of the partial derivative of $\psi(\mu_j)$ with respect to μ

$$\text{that is to say} \quad \nabla\psi(\mu_j) = \frac{\partial \psi(\mu_j)}{\partial \mu}. \tag{95}$$

Indeed, the forward finite difference approximation before the previous partial derivative following the step h gives

$$\frac{\partial \psi(\mu_j)}{\partial \mu} \simeq \frac{\psi(\mu_j + h) - \psi(u_j)}{h} \tag{96}$$

The choice of h is made by numerical test to ensure the good precision on the computation of the gradient, (see [9]).

10. NUMERICAL VALIDATION TEST

To validate our approach, we want to retrieve, using the Gradient descent algorithm with fixed steps, the value of the parameter $\mu^* = 0.3$ which is such that $\psi(\mu^*) = 0$. We have studied the behavior of the method according to the tolerance (δ) for a solution μ_0 next to μ^*. The convergence criterion requires that

$$||\mu_{k+1} - \mu_k|| \leq \delta.$$

For two successive iterations μ_k and μ_{k+1}, $k \leq k_{\max}$. We display the parameter μ_{final} obtained when the convergence criterion is reached. Table 3 displays the parameter μ_{final} obtained when the convergence criterion is reached for a tolerance of the order of $\delta = 10^{-5}$ with different choices of μ_0 to initialize the method and $h = 10^{-2}$ in the finite difference method and $c^* = 0.3653898m/s$ the numerical speed.

Table 3: Statistic of the constant-step gradient algorithm.

μ_0	k	μ_{final}	$\psi(\mu_{\text{final}})$	$\|\|\mu_{\text{final}} - \mu_0\|\|_{L^2(\Omega)}$
0.2	32	0.300038	7.127D-12	0.280038
0.35	39	0.299990	1.178D-14	0.05001
0.39	39	0.3000005	8.970D-16	0.0899995
0.4	40	0.300012	3.355D-17	0.099988
0.5	40	0.300007	4.331D-18	0.199993

We read Each line of Table 3 as follows, for example for the first line: starting from $\mu_0 = 0.2$ close to the minimizer $\mu^* = 0.3$, we performed 32 iterations until convergence is reached $\delta < 10^{-5}$, which gives $\mu_{\text{final}} = 0.300038$, then we evaluate the function $\psi = \psi(\mu)$ at point μ_{final} with which the function ψ is very small and finally, we calculate the norm in $L^2(\Omega)$ between the final μ_{final} and initial μ_0 solutions.

For the test case of Table 3, we were able to reach the optimal solution with a precision close to the order of 10^{-5} for any initial solution μ_0 close to $\mu^* = 0.3$ the gradient descent algorithm with fixed step requires a good choice of μ_0 close to μ^* to guarantee convergence towards a global minimum. The fact that $\psi(\mu_{\text{final}}) = 0$ at most to 10^{-14} confirms that this numerical method allows to identify the thermal conductivity term of the heat equation model by correctly choosing μ_0.

Discussion

The work proposed in this paper allowed us to highlight the explicit Euler, implicit Euler and Cranck-Nicholson methods, but also to discover the importance of these methods in the numerical resolution of the heat equation problem. We carried out a numerical resolution of the heat equation problem by the explicit Euler, implicit Euler and Cranck-Nicholson methods and we obtained linear systems whose matrices are tridiagonal and symmetrical then we used the condition of Von -Neumann-Fourier and the maximum principle to prove the analytical stability in $L^\infty([0,1])$ and $L^2([0,1])$ respectively. We have proved the convergence of the solution u_j^n by using the truncation errors of the methods. A numerical resolution of the heat equation was carried out by fixing the number of steps in space at 19 ($N = 19$) and the number of steps in time at 99 ($T = 99$) then by varying the heat diffusion coefficient μ and the time step Δt. By using the solutions obtained by these methods, we made a comparative study of the errors in norm L^2 between these different solutions. We used this numerical comparison to choose the solution obtained by the Cranck-Nicholson method with $\mu = 0.3$ and $\Delta t = 0.0005$ to solve identification problem of the diffusion coefficient μ in the heat equation model.

11. CONCLUSION ET PERSPECTIVES

To identify the diffusion coefficient μ, we have defined an optimization problem of the least square type depending on the numerical and experimental speeds of the propagation of heat. Then, we proved the differentiability of the solution u of the heat equation model with respect to the diffusion coefficient μ. Finally, we used the gradient descent algorithm with fixed steps to identify the thermal conductivity term μ.
For future work, we plan to develop a technique for identifying the thermal conductivity parameter of the 2D heat equation.

REFERENCES

[1] F. A. Aliev, N. S. Hajieva, A. A. Namazov and N. A. Safarova, *Identification Problem for Determining the Parameters of a Discrete Dynamic System,* International Applied Mechanics, Springer, Vol. 55, No. 3, pp.110 -116, 2019

[2] G. Allaire and S. M. Kaber, *Numerical linear algebra: lessons and exercises*, Editions ellipses, Paris, 2002.

[3] Marc Ethier et Y. Bourgault, *Semi-implicit time-discretization schemes for the bidomaine model.* SIAM .J. Numer. Anal. 46: 2443-2468, 2008.

[4] L. Edsberg and P. Wedin, *Numerical tools for parameter estimation in ODE systems*, J. Math. Anal. Appl., 360, 548-560, 2009

[5] R. R. Ferdinand, *Parameter Estimation in Nonlinear Coupled Advection-Diffusion Equation, Applications and Applied Mathematics*. An International Journal (AAM), Vol. 2, No. 1, pp. 1-13, 2007.

[6] E. Karlo Nyarkoa and R. Scitovski, *Solving the parameter identification problem of mathematical models using genetic algorithms*,Applied Mathematics and Computation, Elsevier, Vol. 153, No. 3, pp. 651-658, 2004.

[7] I. Knowles, *Parameter identification for elliptic problems*, Journal of Computational and Applied Mathematics, Elsevier, Vol. 131, No. 2, pp. 175 - 194, 2001.

[8] J. Liu , B. Wang and Z. Liu, *Determination of a source term in a heat equation,* International Journal of Computer Mathematics, Vol. 87, No. 5, pp. 969-975, 2010.

[9] B. Lucquin, *Partial differential equations and their approximations*, Ellipses, 2004.

[10] J. Pearce-Lance, M. Pop and Y. Bourgault, *A Numerical Method for the Optimal Adjustment of Parameters in Ionic Models Accounting for Restitution Properties*, Functional Imaging and Modelling of the Heart, Lecture Notes in Computer Science, Springer, Cham, Vol. 11504, pp. 46-54, 2019.

[11] D. V. Pongui Ngoma, G. Nguimbi, V. D. Mabonzo and N. Batangouna, *Mathematical and numerical resolution of the convection-diffusion problem*, Pioneer Journal of Mathematics and Mathematical Sciences, Vol. 28, No. 2, pp. 129-144, 2020.

[12] D. V. Pongui Ngoma, G. Nguimbi, V. D. Mabonzo and N. Batangouna, *Numerical methods for advection problem, European Journal of Pure and Applied mathematics*, Vol. 13, No. 1, pp. 144-157, 2020.

[13] D. V. Pongui Ngoma, Y. Bourgaul, M. Pop and H. Nkounkou, *Adjustment of Parameters in ionic models using optimal control problems*, Functional Imaging and Modelling of the Heart, Lecture Notes in Computer Science, Springer, Cham, Vol. 10263, pp. 322-332, 2017.

[14] D. V. Pongui Ngoma, Y. Bourgaul and H. Nkounkou, *Parameter Identification for a non -differentiable ionic model used in cardiac electrophysiology*, Applied Mathematical Sciences, Vol. 150, No. 9, pp. 7483-7507, 2015.

[15] M. Rioux and Y. Bourgault, *A predictive method allowing the use of a single ionic model in numerical cardiac electrophysiology*, ESAIM: Math. Modell. Numer. Anal., Vol. 47, No. 8, pp.987-1016, 2013.