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Continuity of the Extinction Time of Some Stochastic Differential Equations

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Abstract

This paper concerns the study of the following stochastic differential equation

$$\left\{ \begin{array}{l} dX = -f(X)dt + \sigma(X)_0 dW, \\ X(0) = x_0 > 0, \end{array} \right\}$$

where $f : [0, \infty) \to [0, \infty)$ is an increasing function $\sigma \in C^1(\mathbb{R})$, W is a (one dimensional) Wiener process defined on a given probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration $\{\mathfrak{F}\}_{t\geq 0}$ satisfying the usual conditions. Under some conditions, we show that any solution of the above problem extincts in a finite time and its extinction time as a function of the initial datum is continuous. We also extend the above results to other classes of extinction problems.

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1. INTRODUCTION

In this paper, we address the following stochastic differential equation (SDE)

$$dX = -f(X)dt + \sigma(X)_0 dW, \quad (1)$$
$$X(0) = x_0 > 0, \quad (2)$$

where $f : [0, \infty) \to [0, \infty)$ is a C^1 increasing function for positive values, $\sigma \in C^1(\mathbb{R})$, W is a (one dimensional) Wiener process defined on a given probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration $\{\mathfrak{F}\}_{t\geq 0}$ satisfying the usual conditions, namely, it is right continuous and $\{\mathfrak{F}\}_{t\geq 0}$ contains all $\mathbb{P}$-null sets (see, [14]). Let us notice that our stochastic differential equation is given in Stratonovich form. It is well known that if a SDE is given in Itô form, then it may be rewritten in Stratonovich form. In fact, if $X(t)$ solves $dX = -f(X)dt + g(X)dW$, where the SDE is given in Itô form, then $X(t)$ solves

$$dX = -f(X)dt + b(X)_0 dW$$

with

$$b(s) = f(s) + \frac{1}{2}g'(s)g(s).$$

The first SDE which dates back to 1930 has been written by Uhlenbeck and Ornstein (see, [21]). This SDE has been used as a model for the Brownian motion (irregular motion of a particle suspended in a fluid first observed on the microscope by the botanist Brown in the XIX century). A mathematical study of SDEs is due to Itô half a century age and they have extensively used in practically all branches of science and technology from physics to biology (see [1], [6], [11], [14], [15], [17], [19], [20], [21], and the references cited therein). We know that a solution $X(t)$ of the SDE in (1)-(2) may extinct in a finite time, namely, there exists a finite time T such that $X(t) > 0$ for $t \in [0, T)$, but $X(t) = 0$ for $t \geq T$. The time T is called the extinction time of $X(t)$. In [16], Nabongo and Boni have considered the SDE defined in (1)-(2) in the case where $\sigma(X)$ is replaced by $\sigma(X, \varepsilon)$, ε being a positive parameter. Under some conditions, they have showed that any solution $X(t)$ of (1)-(2) extincts in a finite time, and its extinction time goes to that of the solution $\alpha(t)$ of the following ordinary differential equation (ODE)

$$\alpha'(t) = f(\alpha(t)), \quad t > 0, \quad \alpha(0) = x_0,$$

as ε goes to zero with total probability. Motivated by their work, in the present paper, we are interested in the continuity of the extinction time as a function of the initial datum. We consider the following SDE

$$dX^{\varepsilon} = -f(X^{\varepsilon})dt + \sigma(X^{\varepsilon})_0 dW, \quad (3)$$
$$X^{\varepsilon}(0) = x_0^{\varepsilon} > 0, \quad (4)$$

where $x_0^\varepsilon > x_0$, and $\lim_{\varepsilon\to 0} x_0^\varepsilon = x_0$. Under some assumptions, we show that, any solution $X(t)$ of (1)-(2) extincts in a finite time. In addition, we prove that any solution $X^\varepsilon(t)$ of (3)-(4) also extincts in a finite time, and its extinction time goes to that of $X(t)$ with total probability as ε goes to zero. The remainder of the paper is written in the following manner. In the next section, we give a result about stability of ODEs. In the third section, under some assumptions, we show that any solution $X(t)$ of the SDE defined in (1)-(2) extincts in a finite time and its extinction time as a function of the initial datum is continuous. Finally, in the last section, we reveal the possibility of extension for the results of Section 3 to other classes of extinction problems.

2. STABILITY OF ODES

In this section, we prove a result about stability of ODEs. Consider the solution $y(t)$ of the following ODE

$$\begin{aligned} y'(t) &= -H(t, y(t)), \quad t > 0, \\ y(0) &= \alpha > 0, \end{aligned}$$

where $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, is a function, $H \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}_+)$. Let $z(t)$ be the solution of the ODE below

$$\begin{aligned} z'(t) &= -H(t, z(t)), \quad t > 0, \\ z(0) &= \alpha^\varepsilon > 0, \end{aligned}$$

where $\alpha^\varepsilon \geq \alpha$ and $\lim_{\varepsilon\to 0} \alpha^\varepsilon = \alpha$.
Our result about stability of ODEs is stated in the following theorem.

Theorem 2.1. *Let T be a positive fixed time, and suppose that $y \in C^1([0,T])$ such that $y(t) \geq \rho > 0$ for $t \in [0,T]$. Then, we have*

$$\sup_{t\in[0,T]} |y(t) - z(t)| = O(|\alpha - \alpha^\varepsilon|, \quad \textit{as} \quad \varepsilon \to 0.$$

Proof. Let $t(\varepsilon)$ be the first $t \in (0,T)$ such that

$$|y(t) - z(t)| \leq \frac{\rho}{2} \quad \text{for} \quad t \in (0, t(\varepsilon)). \tag{5}$$

It is clear that $|y(0) - z(0)| = |\alpha - \alpha^\varepsilon|$ goes to zero as ε tends to zero. This implies that $t(\varepsilon) > 0$ when ε is sufficiently small. An application of the triangle inequality gives

$$z(t) \geq y(t) - |y(t) - z(t)| \geq \rho - \frac{\rho}{2} = \frac{\rho}{2} \quad \text{for} \quad t \in (0, t(\varepsilon)).$$

Introduce the error

$$e(t) = y(t) - z(t) \quad \text{for} \quad t \in (0, t(\varepsilon)).$$

Invoking the mean value theorem, we easily see that

$$e'(t) = -H_y(t, \xi(t))e(t) \quad \text{for} \quad t \in (0, t(\varepsilon)), \quad e(0) = y(0) - z(0),$$

where $\xi(t)$ is an intermediate value between $y(t)$ and $z(t)$, and H_y is the partial derivative of H with respect to the second variable. Let

$$L = \sup_{t \in [0,T], y \geq \frac{\rho}{2}} |H_y(t, y(t))|$$

and introduce the function

$$w(t) = e^{Lt}|y(0) - z(0)|, \quad \text{for} \quad t \in [0, T].$$

A direct calculation renders

$$w'(t) \geq -H_y(t, \xi(t))w(t) \quad \text{for} \quad t \in (0, t(\varepsilon)), \quad w(0) \geq e(0).$$

With the help of the maximum principle, we get $w(t) \geq e(t)$ for $t \in (0, t(\varepsilon))$. In the same way, we also prove that $w(t) \geq -e(t)$ for $t \in (0, t(\varepsilon))$, which implies that $|e(t)| \leq w(t)$ for $t \in (0, t(\varepsilon))$, or equivalently

$$|y(t) - z(t)| \leq e^{Lt}|y(0) - z(0)|, \quad \text{for} \quad t \in (0, t(\varepsilon)). \tag{6}$$

Now, let us show that $t(\varepsilon) = T$. Suppose that $t(\varepsilon) < T$. Making use of (5) and (6), we see that

$$\frac{\rho}{2} = |y(t(\varepsilon)) - z(t(\varepsilon))| \leq e^{Lt}|y(0) - z(0)| = e^{Lt}|\alpha - \alpha^{\varepsilon}|.$$

Since the quantity on the right hand side of the second equality goes to zero as ε tends to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is impossible. Consequently, $t(\varepsilon) = T$ and the proof is complete. □

3. CONTINUITY OF EXTINCTION TIMES

In this section, under some conditions, we show that any solution $X(t)$ of (1)-(2) extincts in a finite time and its extinction time as a function of the initial datum is continuous. For the sake of simplicity, let us start with some examples concerning ODEs. Consider the following ODE

$$y'(t) = -y^p(t), \quad t > 0, \tag{7}$$

$$y(0) = M > 0, \tag{8}$$

where $p > 0$. An explicit solution of (7)-(8) is given by

$$y(t) = \begin{cases} \frac{1}{(M^{1-p}+(p-1)t)^{\frac{1}{p-1}}} & \text{if} \quad p > 1, \\ Me^{-t} & \text{if} \quad p = 1 \\ (M^{1-p} - (p-1)t)_+^{\frac{1}{p-1}} & \text{if} \quad 0 < p < 1, \end{cases}$$

where $(x)_+ = \max\{x, 0\}$. Thus, we see that if $p \geq 1$, then $0 < y(t) < M$ for $t \geq 0$ and $\lim_{t\to\infty} y(t) = 0$, but if $0 < p < 1$, then $0 < y(t) < M$ for $t \in [0, \frac{M^{1-p}}{1-p})$ and $y(t) = 0$ for $t \geq \frac{M^{1-p}}{1-p}$. In this case, we say that the solution $y(t)$ of (7)-(8) extincts in a finite time, and the time $T_0 = \frac{M^{1-p}}{1-p}$ is called the extinction time of the solution $y(t)$. Now, let $y_\varepsilon(t)$ be the solution of the ODE below

$$\begin{aligned} y'_\varepsilon(t) &= -y^p_\varepsilon(t), \quad t > 0, \\ y_\varepsilon(0) &= M^\varepsilon > 0, \end{aligned}$$

where $M^\varepsilon \geq M$ and $\lim_{\varepsilon\to 0} M^\varepsilon = M$. When $0 < p < 1$, then reasoning as above, we see that $y_\varepsilon(t)$ extincts in a finite time $T_\varepsilon = \frac{M_\varepsilon^{1-p}}{1-p}$. It is not hard to check that $\lim_{\varepsilon\to 0} T_\varepsilon = \frac{M^{1-p}}{1-p} = T_0$. More generally, consider the following ODE

$$\alpha'(t) = -f(\alpha(t)), \quad t > 0, \tag{9}$$

$$\alpha(0) = M > 0, \tag{10}$$

where $f \in C^1(\mathbb{R})$ is an increasing function for positive values. It is well known that, if the integral $\int_0^M \frac{ds}{f(s)}$ diverges, then the solution $\alpha(t)$ of (9)-(10) satisfies $0 < \alpha(t) < M$ and $\lim_{t\to\infty} \alpha(t) = 0$, but if the integral $\int_0^M \frac{ds}{f(s)}$ converges, then the solution of (9)-(10) extincts in a finite time and its extinction time T_0 is given explicitly by $T_0 = \int_0^M \frac{ds}{f(s)}$. Now, let $\alpha_\varepsilon(t)$ be the solution of the ODE below

$$\begin{aligned} \alpha'_\varepsilon(t) &= -f(\alpha_\varepsilon(t)), \quad t > 0, \\ \alpha_\varepsilon(0) &= M^\varepsilon > 0, \end{aligned}$$

where $M^\varepsilon \geq M$ and $\lim_{\varepsilon\to 0} M^\varepsilon = M$. Reasoning as previously, we note that $\alpha_\varepsilon(t)$ extincts in a finite time at the time $T_\varepsilon = \int_0^{M^\varepsilon} \frac{d\sigma}{f(\sigma)}$. This implies that $\lim_{\varepsilon\to 0} T_\varepsilon = T_0$. Thus, we see that for certain ODEs, the extinction times as functions of the initial datum are continuous. Now, let us consider the SDEs. Our first result is the following.

Theorem 3.1. *Suppose that $\sigma(X) = X$ and $\int_0^\alpha \frac{ds}{f(s)} < \infty$ for any positive real α. Then for almost every ω, any solution $X(t)$ of the SDE in (1)-(2) extincts in a finite time T^ω. In addition, for every $\varepsilon > 0$, any solution $X^\varepsilon(t)$ of the SDE in (3)-(4) extincts in a finite time T^ω_ε, and the following relation holds $\lim_{\varepsilon\to 0} T^\omega_\varepsilon = T^\omega$ with total probability.*

Proof. Since $\sigma(X) = X$, then the problem (1)-(2) becomes

$$dX = -f(X)dt + X_0 dW, \quad X(0) = x_0.$$

Setting $Z = Xe^{-W}$, it is not hard to see that $dZ = e^{-W}dX - e^{-W}X_0 dW$, which implies that
$dZ = -e^{-W}f(e^W Z)dt$. This gives a non-autonomous ODE for each ω such that $W(., \omega)$ is continuous, and Z satisfies

$$\dot{Z}_\omega(t) = -e^{-W(t,\omega)}f(e^{W(t,\omega)}Z_\omega(t))dt, \quad t > 0,$$

$$\dot{Z}_\omega(0) = x_0.$$

In the same manner, setting $Z^\varepsilon = X^\varepsilon e^{-W}$, we see that Z^ε obeys

$$\dot{Z}^\varepsilon_\omega(t) = -e^{-W(t,\omega)}f(e^{W(t,\omega)}Z^\varepsilon_\omega(t)), \quad t > 0, \tag{11}$$

$$\dot{Z}^\varepsilon_\omega(0) = x^\varepsilon_0. \tag{12}$$

In the above problems, ω is regarded as a parameter. Let $M > 0$, and define

$$A_M = \{\omega : W(\cdot, \omega) \quad \text{is continuous and} \quad \max_{0\le t\le T_*+1} |W(\cdot, \omega)| \le M\},$$

where $T_* = \int_0^{x_0} \frac{ds}{f(s)}$. Let Z_1 be the solution of the following ODE

$$\begin{aligned} Z_1'(t) &= -e^{-M}f(e^{-M}Z_1(t)), \quad t > 0, \\ Z_1(0) &= x^\varepsilon_0. \end{aligned}$$

If $\omega \in A_M$, then we observe that

$$\dot{Z}^\varepsilon_\omega(t) \le -e^{-M}f(e^M Z^\varepsilon_\omega(t)), \quad t > 0, \tag{13}$$

$$Z^\varepsilon_\omega(0) = x^\varepsilon_0. \tag{14}$$

It is not hard to see that $Z_1(t)$ extincts at the time $T_1 = e^{2M}\int_0^{e^{-M}x^\varepsilon_0} \frac{ds}{f(s)}$. By the maximum principle for ODE, we discover that

$$0 \le Z_\omega(t) \le Z^\varepsilon_\omega(t) \le Z_1(t) \quad \text{for} \quad t \ge 0, \quad \omega \in A_M.$$

Therefore, if $\omega \in A_M$, then $Z^\varepsilon_\omega(t)$ and $Z_\omega(t)$ extinct in finite times T^ω_ε and T^ω, respectively, such that $T^\omega \le T^\omega_\varepsilon \le T_1$. Let $0 < \varepsilon < \frac{T^\omega}{2}$. There exists $\rho \in (0, 1)$ such that

$$e^{2M}\int_0^{e^{-M}\rho} \frac{d\sigma}{f(\sigma)} \le \frac{\varepsilon}{2}. \tag{15}$$

Since $Z_\omega(t)$ extincts at the time T^ω, there exists a time $T_0 \in (T^\omega - \frac{\varepsilon}{2}, T^\omega)$ such that $0 < Z_\omega(t) < \frac{\rho}{2}$
for $t \in [T_0, T^\omega)$. Invoking Theorem 2.1, we easily see that the solution $Z_\omega^\varepsilon(t)$ of (11)-(12) verifies
$|Z_\omega^\varepsilon(t) - Z_\omega(t)| \leq \frac{\rho}{2}$ for $t \in [0, T_0]$. An application of the triangle inequality gives

$$Z_\omega^\varepsilon(T_0) \leq |Z_\omega^\varepsilon(T_0) - Z_\omega(T_0)| + Z_\omega(T_0) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{16}$$

The estimate (13) may be rewritten as follows

$$\frac{dZ_\omega^\varepsilon}{f(e^{-M} Z_\omega^\varepsilon)} \leq -e^{-M} dt, \quad t > 0.$$

Integrate the above inequality over $(T_0, T_\varepsilon^\omega)$ to obtain

$$T_\varepsilon^\omega - T_0 \leq e^{2M} \int_0^{e^{-M} Z_\omega^\varepsilon(T_0)} \frac{d\sigma}{f(\sigma)}. \tag{17}$$

We deduce from (15)-(17) that

$$0 \leq T_\varepsilon^\omega - T^\omega = T_\varepsilon^\omega - T_0 + T_0 - T^\omega \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Use the fact that $\mathbb{P}(\cup_{M=1}^\infty A_M) = 1$, $X = e^W Z$ and $X^\varepsilon = e^W Z^\varepsilon$ to complete the proof. □

Consider now the SDEs in Îto sense. Then, $X(t)$ solves

$$dX = -f(X)dt + XdW, \tag{18}$$
$$X(0) = x_0, \tag{19}$$

and $X^\varepsilon(t)$ solves

$$dX^\varepsilon = -f(X^\varepsilon)dt + X^\varepsilon dW, \tag{20}$$
$$X^\varepsilon(0) = x_0^\varepsilon. \tag{21}$$

Then, we have the following result.

Theorem 3.2. *Theorem 3.1 remains valid if $X(t)$ and $X^\varepsilon(t)$ solve (18)-(19) and (20)-(21), respectively.*

Proof. According to the introduction of the paper, we see that the SDE in (18)-(19) may be rewritten in Stratonovich sense in the following manner
$dX = -f(X) - \frac{X}{2} + X_0 dW$, $X(0) = x_0$.

Setting $Z = Xe^{-W}$, it is not hard to see that $dZ = e^{-W}dX - e^{-W}X_0dW$, which implies that
$dZ = -[e^W f(e^W Z) + \frac{1}{2}e^W Z]dt$. This gives a non-autonomous ODE for each ω such that $W(\cdot,\omega)$ is continuous, and Z satisfies

$$\begin{aligned} \dot{Z}_\omega(t) &= -[e^{W(t,\omega)} + f(e^{W(t,\omega)})Z_\omega(t) + \frac{1}{2}e^{W(t,\omega)}Z_\omega(t)], \quad t > 0, \\ Z_\omega(0) &= x_0. \end{aligned}$$

In the same way, setting $Z^\varepsilon = X^\varepsilon e^{-W}$, we find that

$$\dot{Z}^\varepsilon_\omega(t) = -[e^{W(t,\omega)} + f(e^{W(t,\omega)})Z^\varepsilon_\omega(t) + \frac{1}{2}e^{W(t,\omega)}Z^\varepsilon_\omega(t)], \quad t > 0, \tag{22}$$
$$Z^\varepsilon_\omega(0) = x^\varepsilon_0. \tag{23}$$

Consider $M > 0$, and define

$$A_M = \{\omega : W(\cdot,\omega) \quad \text{is continuous and} \quad \max_{0\le t\le T_*+1} |W(\cdot,\omega)| \le M\},$$

where $T_* = \int_0^{x_0} \frac{ds}{f(s)}$. Let Z_1 be the solution of the following ODE

$$\begin{aligned} \dot{Z}^\varepsilon_1(t) &= -[e^{W(t,\omega)}f(e^{W(t,\omega)})Z^\varepsilon_1(t) + \frac{1}{2}e^{W(t,\omega)}Z^\varepsilon_1(t)], \quad t > 0, \\ Z^\varepsilon_1(0) &= x^\varepsilon_0. \end{aligned}$$

It is not difficult to see that $Z_1(t)$ extincts at the time

$$T_1 = e^{2M}\int_0^{e^{-M}x^\varepsilon_0} \frac{d\sigma}{f(\sigma) + \frac{1}{2}e^M\sigma}.$$

Owing to the maximum principle for ODE, we obtain

$$0 \le Z_\omega(t) \le Z^\varepsilon_\omega(t) \le Z_1(t) \quad \text{for} \quad t \ge 0, \quad \omega \in A_M.$$

We deduce that, if $\omega \in A_M$, then $Z^\varepsilon_\omega(t)$ and $Z_\omega(t)$ extinct in finite times T^ω_ε and T^ω, respectively, such that $T^\omega \le T^\omega_\varepsilon \le T_1$. Again, we know that

$$\dot{Z}^\varepsilon_\omega(t) \le -[e^{-M}f(e^{-M})Z^\varepsilon_\omega(t) + \frac{1}{2}e^{-M}Z^\varepsilon_\omega(t)], \quad t > 0, \tag{24}$$
$$Z^\varepsilon_\omega(0) = x^\varepsilon_0. \tag{25}$$

Let $0 < \varepsilon < \frac{T^\omega}{2}$. There exists $\rho \in (0,1)$ such that

$$e^{2M}\int_0^{e^{-M}\rho} \frac{d\sigma}{f(\sigma) + \frac{1}{2}e^{-M}\sigma} \le \frac{\varepsilon}{2}. \tag{26}$$

Since $Z_\omega(t)$ extincts at the time T^ω, there exists a time $T_0 \in (T^\omega - \frac{\varepsilon}{2}, T^\omega)$ such that $0 < Z_\omega(t) < \frac{\rho}{2}$
for $t \in [T_0, T^\omega)$. Invoking Theorem 2.1, we easily see that the solution $Z^\varepsilon_\omega(t)$ of (22)-(23) verifies
$|Z^\varepsilon_\omega(t) - Z_\omega(t)| \le \frac{\rho}{2}$ for $t \in [0, T_0]$. An application of the triangle inequality gives

$$Z^\varepsilon_\omega(T_0) \le |Z^\varepsilon_\omega(T_0) - Z_\omega(T_0)| + Z_\omega(T_0) \le \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{27}$$

The inequality (24) may be rewritten as follows

$$\frac{dZ^\varepsilon_\omega}{e^{-M} f(e^{-M} Z^\varepsilon_\omega) + \frac{1}{2} e^{-M} Z^\varepsilon_\omega} \le -dt, \quad t > 0.$$

Integrate the above inequality over $(T_0, T^\omega_\varepsilon)$ to obtain

$$T^\omega_\varepsilon - T_0 \le e^{2M} \int_0^{e^{-M} Z^\varepsilon_\omega(T_0)} \frac{d\sigma}{f(\sigma) + \frac{1}{2} e^{-M}\sigma} \le \frac{\varepsilon}{2}. \tag{28}$$

It follows from (21)-(23) that

$$0 \le T^\omega_\varepsilon - T^\omega = T^\omega_\varepsilon - T_0 + T_0 - T^\omega \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Use the fact that $\mathbb{P}(\cup_{M=1}^\infty A_M) = 1$, $X = e^W Z$ and $X^\varepsilon = e^W Z^\varepsilon$ to complete the proof. □

Now, let us consider the general case. The statement of our main result is given in the theorem below.

Theorem 3.3. *Let $\phi(s,x)$ be the flux associated to the ODE $\dot{y} = \sigma(y)$, $y(0) = x$ such that $\phi(s,0) = 0$, $\phi(s,x) > 0$ for $x > 0$, and let $H(s,x) = \frac{f(\phi(s,x))\sigma(x)}{\sigma(\phi(s,x))}$.*
Suppose that

$$H(s,x) \ge H(t,x) \quad \textit{if} \quad s \ge t,$$

and there exists a function $k_s(x)$ such that

$$\frac{1}{H(s,x)} \le k_s(x) \in L^1([0, x_0]). \tag{29}$$

Then for almost every ω, any solution $X(t)$ of (1)-(2) extincts in a finite time T^ω. Moreover, for everey $\varepsilon > 0$, any solution $X^\varepsilon(t)$ of (3)-(4) extincts in a finie time T^ω_ε and the following relation holds $\lim_{\varepsilon \to 0} T^\omega_\varepsilon = T^\omega$ with total probability.

Proof. Exploiting the fact that $\phi(t,x)$ is the flux of the following ODE

$$\dot{y} = \sigma(y), \quad y(0) = x,$$

we derive the following equalities

$$\phi_t(t,x) = \sigma((\phi(t,x)), \quad \phi(0,x) = x. \tag{30}$$

Let $Z_\omega(t)$ be the solution of the Random differential equation

$$\dot{Z}_\omega(t) = -\frac{f\left(\phi(W(t,\omega), Z_\omega(t))\right)}{\phi_x(W(t,\omega), Z_\omega(t))}, \quad t > 0, \tag{31}$$

$$Z_\omega(0) = x_0. \tag{32}$$

Set $X(t,\omega) = \phi(W(t,\omega), Z_\omega(t))$. A straightforward computation reveals that

$$\begin{aligned} dX &= \phi_t(W, Z_\omega)dW + \phi_x(W, Z_\omega)dZ_\omega \\ &= \sigma\left(\phi(W, Z_\omega)\right)dW + \phi_x(W, Z_\omega)dZ_\omega \\ &= \sigma(X)dW - f(X)dZ_\omega. \end{aligned}$$

Therefore, we observe that X is a solution of the SDE defined in (1)-(2). On the other hand, the use of (30) leads us to $\frac{d\phi}{\sigma(\phi)}$, which implies that

$$\int_x^{\phi(t,x)} \frac{ds}{\sigma(s)} = t. \tag{33}$$

Taking the derivative in x of (33) we obtain $\frac{\phi_x(t,x)}{\sigma(\phi(t,x))} - \frac{1}{\sigma(x)} = 0$, or equivalently $\phi_x(t,x) = \frac{\sigma(\phi(t,x))}{\sigma(x)}$. We infer from (31) that

$$\dot{Z}_\omega(t) = -\frac{f\left(\phi(W(t,\omega), Z_\omega(t))\right)\sigma(Z_\omega(t))}{\sigma\left(\phi_x(W(t,\omega), Z_\omega(t))\right)}, \quad t > 0.$$

Take the expression of H to arrive at

$$\begin{aligned} \dot{Z}_\omega(t) &= -H\left(\phi(W(t,\omega), Z_\omega(t))\right), \quad t > 0, \\ Z_\omega(0) &= x_0. \end{aligned}$$

In the same way, we also show that

$$\dot{Z}^\varepsilon_\omega(t) = -H\left(\phi(W(t,\omega), Z^\varepsilon_\omega(t))\right), \quad t > 0, \tag{34}$$

$$Z^\varepsilon_\omega(0) = x^\varepsilon_0. \tag{35}$$

Consider $M > 0$, and define

$$A_M = \{\omega : W(\cdot,\omega) \quad \text{is continuous and} \quad \max_{0 \le t \le T_* + 1} |W(\cdot,\omega)| \le M\},$$

where $T_* = \int_0^{x_0} \frac{ds}{f(s)}$. If $\omega \in A_M$, then we note that

$$\dot{Z}_\omega^\varepsilon(t) \leq -H\left(\phi(W(t,\omega), Z_\omega^\varepsilon(t))\right), \quad t > 0, \tag{36}$$

$$Z_\omega^\varepsilon(0) = x_0^\varepsilon. \tag{37}$$

Let Z_1 be the solution of the following ODE

$$Z_1'(t) \leq -H\left(\phi(W(t,\omega), Z_1(t))\right), \quad t > 0, \tag{38}$$

$$Z_1(0) = x_0^\varepsilon. \tag{39}$$

Employing (29), we observe that the integral $\int_0^{x_0^\varepsilon} \frac{dx}{H(-M,x)}$ is finite. We deduce that the solution Z_1 extincts in a finite time $T_1 = \int_0^{x_0^\varepsilon} \frac{dx}{H(-M,x)}$. Making use of the maximum principle for ODE, we derive the following estimates

$$0 \leq Z_\omega(t) \leq Z_\omega^\varepsilon(t) \leq Z_1(t) \quad \text{for} \quad t \geq 0, \quad \omega \in A_M.$$

Therefore, if $\omega \in A_M$, then $Z_\omega^\varepsilon(t)$ extincts in a finite time T_ε^ω such that

$$0 < T^\omega < T_\varepsilon^\omega < T_1.$$

Let $0 < \varepsilon < \frac{T^\omega}{2}$. There exists $\rho \in (0,1)$ such that

$$\int_0^\rho \frac{dx}{H(-M,x)} \leq \frac{\varepsilon}{2}. \tag{40}$$

Since $Z_\omega(t)$ extincts at the time T^ω, there exists a time $T_0 \in (T^\omega - \frac{\varepsilon}{2}, T^\omega)$ such that $0 < Z_\omega(t) < \frac{\rho}{2}$
for $t \in [T_0, T^\omega)$. Invoking Theorem 2.1, we easily see that the solution $Z_\omega^\varepsilon(t)$ of (34)-(35) satisfies
$|Z_\omega^\varepsilon(t) - Z_\omega(t)| \leq \frac{\rho}{2}$ for $t \in [0, T_0]$. An application of the triangle inequality gives

$$Z_\omega^\varepsilon(T_0) \leq |Z_\omega^\varepsilon(T_0) - Z_\omega(T_0)| + Z_\omega(T_0) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{41}$$

The estimate (36) may be rewritten in the following manner

$$\frac{dZ_\omega^\varepsilon}{H(-M, Z_\omega^\varepsilon)} \leq -dt, \quad t > 0.$$

Integrate the above inequality over $(T_0, T_\varepsilon^\omega)$ to obtain

$$T_\varepsilon^\omega - T_0 \leq \int_0^{Z_\omega^\varepsilon(T_0)} \frac{d\sigma}{H(-\sigma, Z_\omega^\varepsilon)} \leq \frac{\varepsilon}{2}. \tag{42}$$

It follows from (40)-(42) that

$$0 \leq T_\varepsilon^\omega - T^\omega = T_\varepsilon^\omega - T_0 + T_0 - T^\omega \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Use the fact that $\mathbb{P}(\cup_{M=1}^\infty A_M) = 1$, $X = \phi(W, Z)$ and $X^\varepsilon = \phi(W, Z^\varepsilon)$ to complete the proof. □

Remark 3.1. *It is worth noting that, if $\sigma(x) = x$ and $f(s) = s^q$ with $0 < q < 1$, then*

$$\phi(s, x) = xe^s \quad \text{and} \quad H(s, x) = x^q e^{-(1-q)s}.$$

4. OTHER EXTINCTION TIMES

In this section, we show the possibility to extend the results of the previous section to another problem of extinction which is called problem of quenching. To illustrate our analysis, let us consider the following ODE

$$y'(t) = -(y(t))^{-p}, \ t > 0 \tag{43}$$

$$y(0) = M, \tag{44}$$

where $p > 0$. An explicit solution of (43)-(44) is given by

$$y(t) = \Big(M^{1+p} - (1+p)t\Big)^{\frac{1}{1+p}} \quad \text{for} \quad t \in [0, \frac{M^{1+p}}{1+p}).$$

Hence, we see that, if $t = \frac{M^{1+p}}{1+p}$, then $y(t)$ reaches the value zero which implies that $y'(t)$ exploses at the same time. In this case, we say that the solution $y(t)$ quenches in a finite time, and it is not hard to check that the quenching time as a function of the initial datum is continuous. More generally, consider the following ODE

$$y'(t) = -f(y(t)), \ t > 0 \tag{45}$$

$$y(0) = M, \tag{46}$$

where $f(s)$ is a positive, decreasing function for positive values of s, $\lim_{s\to 0^+} f(s) = \infty$, $\int_0^M \frac{ds}{f(s)} < \infty$. It is not hard to see that $M > y(t) > 0$ for $t \in [0, T_0)$, but $\lim_{T\to T_0} y(t) = 0$ where $T_0 = \int_0^M \frac{ds}{f(s)}$. Therefore, we discover that $y(t)$ quenches in a finite time, and the time T_0 is called the quenching time of $y(t)$. Let us also notice that the derivative in t of $y(t)$ exploses at the time T_0. Here again, we easily show that the quenching time as a function of the initial datum is continuous. Now, let us consider the following SDEs

$$dX = -f(X)dt + \sigma(X, \varepsilon)_0 dW, \tag{47}$$

$$X(0) = x_0 > 0, \tag{48}$$

and

$$dX^\varepsilon = -f(X^\varepsilon)dt + \sigma(X^\varepsilon)_0 dW, \tag{49}$$

$$X^\varepsilon(0) = x_0^\varepsilon > 0, \tag{50}$$

where $f(s)$ is positive, decreasing function for positive values of s, $\lim_{s\to 0^+} f(s) = \infty$, $\int_0^\alpha \frac{ds}{f(s)} < \infty$ for any positive real α. Using the methods developed in the previous section, it is not difficult to prove that the above theorems remain valid when $X(t)$ and $X^\varepsilon(t)$ solve (45)-(46) and (47)-(8), respectively.

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