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The Improved Generalized Tanh-Coth Method for (2+1)-Dimensional Extension of The Benjamin – Ono Equation with Time-Dependent Coefficients

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Abstract

In this paper, the new improved generalized tanh-coth method have been devised for getting new exact soliton solutions of a (2+1)-dimensional extension of the Benjamin – Ono equation. This equation arises in the study of long internal gravity waves in deep stratified fluids. The solitons and other solutions achieved by this method can be categorized as a single and combo solitons . The results obtained confirm that proposed method is efficient techniques for analytic treatment of a wide variety of the nonlinear evolution equations.

Keywords: The improved generalized tanh-coth method, New exact soliton solutions, The (2+1)-dimensional extension of the Benjamin – Ono equation with time-dependent coefficients.

1. INTRODUCTION

This paper is introduced with the (2+1)-dimensional extension of the Benjamin – Ono equation with time-dependent coefficients

$$u_{tt} + \alpha(t)u_{xx}^2 + \beta(t)u_{xxxx} + \gamma(t)u_{yyyy} = 0, \tag{1}$$

where the $\alpha(t)$ cotrols the nonlinear and the characteristic speed of the long waves,the other $\beta(t)$ and $\gamma(t)$ are the fluid depth, $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real functions

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of variable t and $u(x, y, t)$ is the elevation of the free surface of the fluid. For $\alpha(t) = \alpha$,$\beta(t) = \beta$ and $\gamma(t) = 0$,the equation (1) gives the standard Benjamin – Ono equation [1-5].

Nonlinear evolution equations (NLEEs) have been extensively used to depict natural phenomena of science and engineering including electromagnetism, plasma physics, mathematical physics, fiber optics, mathematical biology, thermodynamic and vice versa. The study of wave propagation has attracted much attention that resulted to significant research findings during the past decades. Nonlinear evolution equations play a major role in a variety of scientific and engineering fields, such as ocean engineering, optical fiber communications, plasma physics and fluid dynamics, where useful results on solitions have been reported in the literature [4-11].

Another powerful method has been presented by Malfliet [12], who had customized the tanh technique and called the tanh method. In 2002 Fan and Hon [13], extended the tanh method which is called the extended tanh method. In 2006 Wazwaz [14], extended and improved this method which is called the tanh-coth method. In 2008 Salas and Gomez [15], improved and generalized this method which is called the improved generalized tanh-coth method [15-18].

The rest composition of the paper is devised as follows. In section 2, we briefly describe the extended and improved this method, In section 3, the improved generalized tanh-coth method is applied to the (2+1)-dimensional extension of the Benjamin – Ono equation with time-dependent coefficients. The last section is some conclusions are given.

2. THE IMPROVED GENERALIZED TANH-COTH METHOD

In this section, we present short description of the extended improved generalized tanh-coth method. Consider the general nonlinear PDE:

$$P(U, U_x, U_t, U_{xx}, U_{tt}, U_{xt}...) = 0, \tag{2}$$

where $u = u(x, t)$ is an unkhown function, P is a polynomial in $u(x, t)$ and the subscripts for the partial derivatives.
Step 1: The traveling wave transformation is given by

$$U(x, t) = u(\eta), \eta = x - \lambda t + \eta_0, \tag{3}$$

where λ is the wave speed. We can redure (3) to the ordinary differential equation

$$O(u, u', u'', u''', u'''', u'''''...) = 0. \tag{4}$$

Step 2: According to the improved generalized tanh-coth method,we seek the exact solution of (3) that can be expressed in the following from:

$$u(\eta) = \sum_{i=0}^{M} a_i(t)\phi(\eta)^i + \sum_{M+1}^{2M} a_i(t)\phi(\eta)^{M-1}, \tag{5}$$

where M is a positive integer integer that will be determined by balanving the highest order derivative term with the highest order nonlinear term. The coefficients $a_i(t)$ are variable funtion of t ($a_M(t) \neq 0$ and $a_{-M}(t) \neq 0$) that are determined later while the new variable $\phi(\eta)$ is the solution to the generalized Riccti equation

$$\phi'(\eta) = b_0(t) + b_1(t)\phi(\eta) + b_2(t)\phi(\eta)^2, \tag{6}$$

where $b_0(t), b_1(t)$ and $b_2(t)$ are variable funtion of t. The solutions of generalized Riccti equation by [15].

Case 1 (exponential function solutions) . When $b_0(t) = 0$,

$$\phi(\eta) = \frac{b_1(t)}{-b_2(t) + b_1(t)e^{-b_1(t)\eta}}. \tag{7}$$

Case 2 (trigonometric and hyperbolic function solutions). When $b_1(t) = 0$,

$$\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\tan(\sqrt{b_0(t)\, b_2(t)}\,\eta), \qquad b_0(t) > 0, b_2(t) > 0, \tag{8}$$

$$\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\tanh(\sqrt{b_0(t)\, b_2(t)}\,\eta), \qquad b_0(t) > 0, b_2(t) < 0, \tag{9}$$

$$\phi(\eta) = \frac{\sqrt{-b_0(t)b_2(t)}}{b_2(t)}\tanh(-\sqrt{-b_0(t)\, b_2(t)}\,\eta), \quad b_0(t) < 0, b_2(t) > 0, \tag{10}$$

$$\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\tan(-\sqrt{b_0(t)\, b_2(t)}\,\eta), \qquad b_0(t) < 0, b_2(t) < 0. \tag{11}$$

Case 3 (exponential function solutions). When $b_2(t) = 0$

$$\phi(\eta) = \frac{-b_0(t) + b_1(t)e^{b_1(t)\eta}}{b_1(t)}. \tag{12}$$

Case 4 (rational function solutions). When $b_0(t) = b_1(t) = 0$

$$\phi(\eta) = -\frac{1}{b_2(t)\,\eta}. \tag{13}$$

Case 5(trigonometric function solutions). When $b_1(t)^2 < 4b_0(t)b_2(t)$ and $b_2(t) \neq 0$,

$$\phi(\eta) = \frac{\sqrt{4b_0(t)b_2(t) - b_1(t)^2}\tan(\frac{1}{2}\sqrt{4b_0(t)b_2(t) - b_1(t)^2}\,\eta) - b_1(t)}{2b_2(t)}. \tag{14}$$

Case 6(hyperbolic function solutions). When $b_1(t)^2 > 4b_0(t)b_2(t)$ and $b_2(t) \neq 0$,

$$\phi(\eta) = \frac{\sqrt{b_1(t)^2 - 4b_0(t)b_2(t)}\,\tanh(\frac{1}{2}\sqrt{b_1(t)^2 - 4b_0(t)b_2(t)}\,\eta) - b_1(t)}{2b_2(t)}. \tag{15}$$

Case 7(rational function solutions). When $b_1(t)^2 \neq 0$ and $b_1(t)^2 = 4b_0(t)b_2(t)$,

$$\phi(\eta) = -\frac{2b_0(t)(b_1(t)\,\eta + 2)}{b_1(t)^2\,\eta}. \tag{16}$$

We substitute equation (5) into equation (6) and collect all terms with the same order of $\phi(\eta)^i$; we get a polynomial in $\phi(\eta)$. Equating each coefficient of the polynomial to zero, we will give a system of algebraic equations involving the parameters $b_0(t), b_1(t), b_2(t), \lambda, a_0, a_1, a_2, ..., a_M$.Solving the system, we can construct a variety of exact solutions of equation (3)

3. THE IMPROVED GENERALIZED TANH-COTH METHOD OF THE BENJAMIN-ONO EQUATION

We use the wave transformations $u = V(\eta)$ and $\eta = \alpha_1 x + \alpha_2 y + \alpha_3 t$, to reduce equation (1) to the following ODE:

$$\alpha_3^2 V(\eta)'' + \alpha(t)\alpha_1^2 (V(\eta)^2)'' + (\beta(t)\alpha_1^4 + \gamma(t)\alpha_2^4) V(\eta)''''. \tag{17}$$

Balancing the highest order linear term $V(\eta)''''$ with the highest order nonlinear term $(V(\eta)^2)''$ in equation (17), we get $M = 2$. Consequently, we set

$$V(\eta) = q_0(t) + q_1(t)\phi(\eta) + q_2(t)\phi(\eta)^2 + \frac{q_3(t)}{\phi(\eta)} + \frac{q_4(t)}{\phi(\eta)^2}. \tag{18}$$

Using equation (6) and equation (18) in equation (19) and equating all the coefficients of power of $\phi(\eta)$ to be zero, we obtain a system of algebraic equation in the unknowns $q_0(t), q_1(t), q_2(t), q_3(t)$ and $q_4(t)$.

$$\begin{aligned} \phi(\eta)^{-6}: \quad & 20\,\alpha\,{\alpha_1}^2\,(q_4\,(t))^2\,(b_0\,(t))^2 + 120\,q_4\,(t)\,\beta\,{\alpha_1}^4\,(b_0\,(t))^4 \\ & +120\,q_4\,(t)\,\gamma\,{\alpha_2}^4\,(b_0\,(t))^4 = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} \phi(\eta)^{-5}: \quad & 36\,\alpha\,{\alpha_1}^2\,(q_4\,(t))^2\,b_1\,(t)\,b_0\,(t) + 24\,\alpha\,{\alpha_1}^2 q_3\,(t)\,q_4\,(t)\,(b_0\,(t))^2 \\ & +336\,q_4\,(t)\,\beta\,{\alpha_1}^4 b_1\,(t)\,(b_0\,(t))^3 + 336\,q_4\,(t)\,\gamma\,{\alpha_2}^4 b_1\,(t)\,(b_0\,(t))^3 \\ & +24\,q_3\,(t)\,\beta\,{\alpha_1}^4\,(b_0\,(t))^4 + 24\,q_3\,(t)\,\gamma\,{\alpha_2}^4\,(b_0\,(t))^4 = 0, \end{aligned} \tag{20}$$

$$\begin{aligned}\phi(\eta)^{-4}: \quad & 42\,\alpha\,{\alpha_1}^2 q_3(t)\, q_4(t)\, b_1(t)\, b_0(t) + 32\,\alpha\,{\alpha_1}^2 (q_4(t))^2\, b_2(t)\, b_0(t) \\ & +12\,\alpha\,{\alpha_1}^2 q_0(t)\, q_4(t)\, (b_0(t))^2 + 60\, q_3(t)\,\beta\,{\alpha_1}^4 b_1(t)\,(b_0(t))^3 \\ & +60\, q_3(t)\,\gamma\,{\alpha_2}^4 b_1(t)\,(b_0(t))^3 + 240\, q_4(t)\,\beta\,{\alpha_1}^4 b_2(t)\,(b_0(t))^3 \\ & +330\, q_4(t)\,\beta\,{\alpha_1}^4 (b_1(t))^2 (b_0(t))^2 + 240\, q_4(t)\,\gamma\,{\alpha_2}^4 b_2(t)\,(b_0(t))^3 \\ & +330\, q_4(t)\,\gamma\,{\alpha_2}^4 (b_1(t))^2 (b_0(t))^2 + 6\,\alpha\,{\alpha_3}^2 q_4(t)\,(b_0(t))^2 \\ & +6\,\alpha\,{\alpha_1}^2 (q_3(t))^2 (b_0(t))^2 + 16\,\alpha\,{\alpha_1}^2 (q_4(t))^2 (b_1(t))^2 = 0, \end{aligned} \tag{21}$$

$$\begin{aligned}\phi(\eta)^{-3}: \quad & 28\,\alpha\,{\alpha_1}^2 (q_4(t))^2\, b_2(t)\, b_1(t) + 40\, q_3(t)\,\beta\,{\alpha_1}^4 b_2(t)\,(b_0(t))^3 \\ & +18\,\alpha\,{\alpha_1}^2 q_3(t)\, q_4(t)\,(b_1(t))^2 + 4\,\alpha\,{\alpha_1}^2 q_0(t)\, q_3(t)\,(b_0(t))^2 \\ & +4\,\alpha\,{\alpha_1}^2 q_1(t)\, q_4(t)\,(b_0(t))^2 + 50\, q_3(t)\,\beta\,{\alpha_1}^4 (b_1(t))^2 (b_0(t))^2 \\ & +40\, q_3(t)\,\gamma\,{\alpha_2}^4 b_2(t)\,(b_0(t))^3 + 50\, q_3(t)\,\gamma\,{\alpha_2}^4 (b_1(t))^2 (b_0(t))^2 \\ & +130\, q_4(t)\,\beta\,{\alpha_1}^4 (b_1(t))^3\, b_0(t) + 130\, q_4(t)\,\gamma\,{\alpha_2}^4 (b_1(t))^3\, b_0(t) \\ & +10\,\alpha\,{\alpha_1}^2 (q_3(t))^2\, b_1(t)\, b_0(t) + 440\, q_4(t)\,\beta\,{\alpha_1}^4 b_2(t)\, b_1(t)\,(b_0(t))^2 \\ & +440\, q_4(t)\,\gamma\,{\alpha_2}^4 b_2(t)\, b_1(t)\,(b_0(t))^2 + 36\,\alpha\,{\alpha_1}^2 q_3(t)\, q_4(t)\, b_2(t)\, b_0(t) \\ & +20\,\alpha\,{\alpha_1}^2 q_0(t)\, q_4(t)\, b_1(t)\, b_0(t) + 2\,{\alpha_3}^2 q_3(t)\,(b_0(t))^2 \\ & +10\,{\alpha_3}^2 q_4(t)\, b_1(t)\, b_0(t) = 0, \end{aligned} \tag{22}$$

$$\begin{aligned}\phi(\eta)^{-2}: \quad & 60\, q_3(t)\,\beta\,{\alpha_1}^4 b_2(t)\, b_1(t)\,(b_0(t))^2 + 60\, q_3(t)\,\gamma\,{\alpha_2}^4 b_2(t)\, b_1(t)\,(b_0(t))^2 \\ & +232\, q_4(t)\,\beta\,{\alpha_1}^4 b_2(t)\,(b_1(t))^2\, b_0(t) + 232\, q_4(t)\,\gamma\,{\alpha_2}^4 b_2(t)\,(b_1(t))^2\, b_0(t) \\ & +6\,\alpha\,{\alpha_1}^2 q_1(t)\, q_4(t)\, b_1(t)\, b_0(t) + 30\,\alpha\,{\alpha_1}^2 q_3(t)\, q_4(t)\, b_2(t)\, b_1(t) \\ & +6\,\alpha\,{\alpha_1}^2 q_0(t)\, q_3(t)\, b_1(t)\, b_0(t) + 16\,\alpha\,{\alpha_1}^2 q_0(t)\, q_4(t)\, b_2(t)\, b_0(t) \\ & +136\, q_4(t)\,\gamma\,{\alpha_2}^4 (b_2(t))^2 (b_0(t))^2 + 8\,\alpha\,{\alpha_1}^2 (q_3(t))^2\, b_2(t)\, b_0(t) \\ & +15\, q_3(t)\,\beta\,{\alpha_1}^4 (b_1(t))^3\, b_0(t) + 8\,\alpha\,{\alpha_1}^2 q_0(t)\, q_4(t)\,(b_1(t))^2 \\ & +15\, q_3(t)\,\gamma\,{\alpha_2}^4 (b_1(t))^3\, b_0(t) + 136\, q_4(t)\,\beta\,{\alpha_1}^4 (b_2(t))^2 (b_0(t))^2 \\ & +12\,\alpha\,{\alpha_1}^2 (q_4(t))^2 (b_2(t))^2 + 16\, q_4(t)\,\beta\,{\alpha_1}^4 (b_1(t))^4 + 16\, q_4(t)\,\gamma\,{\alpha_2}^4 (b_1(t))^4 \\ & +3\,{\alpha_3}^2 q_3(t)\, b_1(t)\, b_0(t) + 8\,{\alpha_3}^2 q_4(t)\, b_2(t)\, b_0(t) + 4\,\alpha\,{\alpha_1}^2 (q_3(t))^2 (b_1(t))^2 \\ & +4\,{\alpha_3}^2 q_4(t)\,(b_1(t))^2 = 0, \end{aligned} \tag{23}$$

$$\phi(\eta)^{-1} \quad : 22\, q_3\,(t)\,\beta\,\alpha_1{}^4 b_2\,(t)\,(b_1\,(t))^2\, b_0\,(t) + 22\, q_3\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)\,(b_1\,(t))^2\, b_0\,(t)$$
$$+120\, q_4\,(t)\,\beta\,\alpha_1{}^4\,(b_2\,(t))^2\, b_1\,(t)\, b_0\,(t) + 120\, q_4\,(t)\,\gamma\,\alpha_2{}^4\,(b_2\,(t))^2\, b_1\,(t)\, b_0\,(t)$$
$$+4\,\alpha\,\alpha_1{}^2 q_1\,(t)\, q_4\,(t)\, b_2\,(t)\, b_0\,(t) + 4\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_3\,(t)\, b_2\,(t)\, b_0\,(t)$$
$$+12\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_4\,(t)\, b_2\,(t)\, b_1\,(t) + 12\,\alpha\,\alpha_1{}^2 q_3\,(t)\, q_4\,(t)\,(b_2\,(t))^2$$
$$+2\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_3\,(t)\,(b_1\,(t))^2 + 2\,\alpha\,\alpha_1{}^2 q_1\,(t)\, q_4\,(t)\,(b_1\,(t))^2$$
$$+16\, q_3\,(t)\,\gamma\,\alpha_2{}^4\,(b_2\,(t))^2\,(b_0\,(t))^2 + 30\, q_4\,(t)\,\beta\,\alpha_1{}^4 b_2\,(t)\,(b_1\,(t))^3$$
$$+30\, q_4\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)\,(b_1\,(t))^3 + 6\,\alpha\,\alpha_1{}^2\,(q_3\,(t))^2\, b_2\,(t)\, b_1\,(t)$$
$$+16\, q_3\,(t)\,\beta\,\alpha_1{}^4\,(b_2\,(t))^2\,(b_0\,(t))^2 + q_3\,(t)\,\beta\,\alpha_1{}^4\,(b_1\,(t))^4$$
$$+q_3\,(t)\,\gamma\,\alpha_2{}^4\,(b_1\,(t))^4 + 2\,\alpha_3{}^2 q_3\,(t)\, b_2\,(t)\, b_0\,(t) + 6\,\alpha_3{}^2 q_4\,(t)\, b_2\,(t)\, b_1\,(t)$$
$$+\alpha_3{}^2 q_3\,(t)\,(b_1\,(t))^2 = 0, \tag{24}$$

$$\phi(\eta)^{0} \quad : q_1\,(t)\,\beta\,\alpha_1{}^4\,(b_1\,(t))^3\, b_0\,(t) + q_1\,(t)\,\gamma\,\alpha_2{}^4\,(b_1\,(t))^3\, b_0\,(t)$$
$$+q_3\,(t)\,\beta\,\alpha_1{}^4 b_2\,(t)\,(b_1\,(t))^3 + q_3\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)\,(b_1\,(t))^3$$
$$+2\,\alpha\,\alpha_1{}^2 q_1\,(t)\, q_4\,(t)\, b_2\,(t)\, b_1\,(t) + 2\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_3\,(t)\, b_2\,(t)\, b_1\,(t)$$
$$+2\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_1\,(t)\, b_1\,(t)\, b_0\,(t) + 2\,\alpha\,\alpha_1{}^2 q_2\,(t)\, q_3\,(t)\, b_1\,(t)\, b_0\,(t)$$
$$+8\, q_3\,(t)\,\beta\,\alpha_1{}^4\,(b_2\,(t))^2\, b_1\,(t)\, b_0\,(t) + 8\, q_3\,(t)\,\gamma\,\alpha_2{}^4\,(b_2\,(t))^2\, b_1\,(t)\, b_0\,(t)$$
$$+8\, q_1\,(t)\,\beta\,\alpha_1{}^4 b_2\,(t)\,(b_0\,(t))^2\, b_1\,(t) + 8\, q_1\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)\,(b_0\,(t))^2\, b_1\,(t)$$
$$+16\, q_4\,(t)\,\beta\,\alpha_1{}^4\,(b_2\,(t))^3\, b_0\,(t) + 14\, q_4\,(t)\,\beta\,\alpha_1{}^4\,(b_2\,(t))^2\,(b_1\,(t))^2$$
$$+16\, q_4\,(t)\,\gamma\,\alpha_2{}^4\,(b_2\,(t))^3\, b_0\,(t) + 4\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_2\,(t)\,(b_0\,(t))^2$$
$$+16\, q_2\,(t)\,\beta\,\alpha_1{}^4\,(b_0\,(t))^3\, b_2\,(t) + 16\, q_2\,(t)\,\gamma\,\alpha_2{}^4\,(b_0\,(t))^3\, b_2\,(t)$$
$$+14\, q_2\,(t)\,\beta\,\alpha_1{}^4\,(b_1\,(t))^2\,(b_0\,(t))^2 + 14\, q_2\,(t)\,\gamma\,\alpha_2{}^4\,(b_1\,(t))^2\,(b_0\,(t))^2$$
$$+14\, q_4\,(t)\,\gamma\,\alpha_2{}^4\,(b_2\,(t))^2\,(b_1\,(t))^2 + 4\,\alpha\,\alpha_1{}^2 q_0\,(t)\, q_4\,(t)\,(b_2\,(t))^2$$
$$+\alpha_3{}^2 q_1\,(t)\, b_1\,(t)\, b_0\,(t) + \alpha_3{}^2 q_3\,(t)\, b_2\,(t)\, b_1\,(t) + 2\,\alpha\,\alpha_1{}^2\,(q_1\,(t))^2\,(b_0\,(t))^2$$
$$+2\,\alpha\,\alpha_1{}^2\,(q_3\,(t))^2\,(b_2\,(t))^2 + 2\,\alpha_3{}^2 q_2\,(t)\,(b_0\,(t))^2$$
$$+2\,\alpha_3{}^2 q_4\,(t)\,(b_2\,(t))^2 = 0, \tag{25}$$

$$\begin{aligned}
\phi(\eta)^1: \quad & q_1(t)\,\beta\,{\alpha_1}^4\,(b_1(t))^4 + q_1(t)\,\gamma\,{\alpha_2}^4\,(b_1(t))^4 \\
& +6\,{\alpha_3}^2 q_2(t)\,b_1(t)\,b_0(t) + 6\,\alpha\,{\alpha_1}^2\,(q_1(t))^2\,b_1(t)\,b_0(t) \\
& +16\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^2\,(b_0(t))^2 + 16\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^2\,(b_0(t))^2 \\
& +120\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_0(t))^2\,b_2(t)\,b_1(t) + 12\,\alpha\,{\alpha_1}^2 q_0(t)\,q_2(t)\,b_1(t)\,b_0(t) \\
& +22\,q_1(t)\,\beta\,{\alpha_1}^4 b_2(t)\,(b_1(t))^2\,b_0(t) + 22\,q_1(t)\,\gamma\,{\alpha_2}^4 b_2(t)\,(b_1(t))^2\,b_0(t) \\
& +120\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_0(t))^2\,b_2(t)\,b_1(t) \\
& +{\alpha_3}^2 q_1(t)\,(b_1(t))^2 + 2\,{\alpha_3}^2 q_1(t)\,b_2(t)\,b_0(t) \\
& +12\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,(b_0(t))^2 + 2\,\alpha\,{\alpha_1}^2 q_0(t)\,q_1(t)\,(b_1(t))^2 \\
& +2\,\alpha\,{\alpha_1}^2 q_2(t)\,q_3(t)\,(b_1(t))^2 + 30\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_1(t))^3\,b_0(t) \\
& +30\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_1(t))^3\,b_0(t) + 4\,\alpha\,{\alpha_1}^2 q_2(t)\,q_3(t)\,b_2(t)\,b_0(t) \\
& +4\,\alpha\,{\alpha_1}^2 q_0(t)\,q_1(t)\,b_2(t)\,b_0(t) = 0, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\phi(\eta)^2: \quad & 136\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_0(t))^2\,(b_2(t))^2 + 136\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_0(t))^2\,(b_2(t))^2 \\
& +15\,q_1(t)\,\beta\,{\alpha_1}^4 b_2(t)\,(b_1(t))^3 + 4\,{\alpha_3}^2 q_2(t)\,(b_1(t))^2 \\
& +60\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^2\,b_1(t)\,b_0(t) + 232\,q_2(t)\,\gamma\,{\alpha_2}^4 b_0(t)\,b_2(t)\,(b_1(t))^2 \\
& +4\,\alpha\,{\alpha_1}^2\,(q_1(t))^2\,(b_1(t))^2 + 12\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,(b_0(t))^2 \\
& +3\,{\alpha_3}^2 q_1(t)\,b_1(t)\,b_2(t) \\
& +16\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_1(t))^4 + 16\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_1(t))^4 + 8\,{\alpha_3}^2 q_2(t)\,b_2(t)\,b_0(t) \\
& +30\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,b_1(t)\,b_0(t) + 6\,\alpha\,{\alpha_1}^2 q_0(t)\,q_1(t)\,b_1(t)\,b_2(t) \\
& +6\,\alpha\,{\alpha_1}^2 q_2(t)\,q_3(t)\,b_1(t)\,b_2(t) + 232\,q_2(t)\,\beta\,{\alpha_1}^4 b_0(t)\,b_2(t)\,(b_1(t))^2 \\
& +16\,\alpha\,{\alpha_1}^2 q_0(t)\,q_2(t)\,b_2(t)\,b_0(t) + 60\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^2\,b_1(t)\,b_0(t) \\
& +8\,\alpha\,{\alpha_1}^2\,(q_1(t))^2\,b_2(t)\,b_0(t) + 8\,\alpha\,{\alpha_1}^2 q_0(t)\,q_2(t)\,(b_1(t))^2 \\
& +15\,q_1(t)\,\gamma\,{\alpha_2}^4 b_2(t)\,(b_1(t))^3 = 0, \qquad (27)
\end{aligned}$$

$$\begin{aligned}
\phi(\eta)^3: \quad & 440\,q_2(t)\,\beta\,{\alpha_1}^4 b_0(t)\,(b_2(t))^2\,b_1(t) + 440\,q_2(t)\,\gamma\,{\alpha_2}^4 b_0(t)\,(b_2(t))^2\,b_1(t) \\
& +20\,\alpha\,{\alpha_1}^2 q_0(t)\,q_2(t)\,b_2(t)\,b_1(t) + 36\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,b_2(t)\,b_0(t) \\
& +10\,{\alpha_3}^2 q_2(t)\,b_2(t)\,b_1(t) + 2\,{\alpha_3}^2 q_1(t)\,(b_2(t))^2 + 18\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,(b_1(t))^2 \\
& +28\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,b_1(t)\,b_0(t) + 10\,\alpha\,{\alpha_1}^2\,(q_1(t))^2\,b_2(t)\,b_1(t) \\
& +4\,\alpha\,{\alpha_1}^2 q_0(t)\,q_1(t)\,(b_2(t))^2 + 4\,\alpha\,{\alpha_1}^2 q_2(t)\,q_3(t)\,(b_2(t))^2 \\
& +40\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^3\,b_0(t) + 50\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^2\,(b_1(t))^2 \\
& +40\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^3\,b_0(t) + 50\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^2\,(b_1(t))^2 \\
& +130\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_1(t))^3\,b_2(t) + 130\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_1(t))^3\,b_2(t) = 0, \qquad (28)
\end{aligned}$$

$$\phi(\eta)^4: \quad 42\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,b_2(t)\,b_1(t) + 32\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,b_2(t)\,b_0(t)$$
$$+12\,\alpha\,{\alpha_1}^2 q_0(t)\,q_2(t)\,(b_2(t))^2 + 240\,q_2(t)\,\beta\,{\alpha_1}^4 b_0(t)\,(b_2(t))^3$$
$$+240\,q_2(t)\,\gamma\,{\alpha_2}^4 b_0(t)\,(b_2(t))^3 + 330\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_1(t))^2\,(b_2(t))^2$$
$$+330\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_1(t))^2\,(b_2(t))^2 + 60\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^3\,b_1(t)$$
$$+60\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^3\,b_1(t) + 16\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,(b_1(t))^2$$
$$+6\,\alpha\,{\alpha_1}^2\,(q_1(t))^2\,(b_2(t))^2 + 6\,{\alpha_3}^2 q_2(t)\,(b_2(t))^2 = 0, \tag{29}$$

$$\phi(\eta)^5: \quad 36\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,b_2(t)\,b_1(t) + 24\,\alpha\,{\alpha_1}^2 q_1(t)\,q_2(t)\,(b_2(t))^2$$
$$+336\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^3\,b_1(t) + 336\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^3\,b_1(t)$$
$$+24\,q_1(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^4 + 24\,q_1(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^4 = 0, \tag{30}$$

$$\phi(\eta)^6: \quad 120\,q_2(t)\,\gamma\,{\alpha_2}^4\,(b_2(t))^4 + 120\,q_2(t)\,\beta\,{\alpha_1}^4\,(b_2(t))^4$$
$$+20\,\alpha\,{\alpha_1}^2\,(q_2(t))^2\,(b_2(t))^2 = 0. \tag{31}$$

Solving the system of algebraic equations with the aid of Maple, using equation (19)-(31), we obtain the following results.

First Set

$$q_0(t) = -\frac{1}{2}\,\frac{8\,\beta\,b_0(t)\,b_2(t)\,{\alpha_1}^4 + \beta\,(b_1(t))^2\,{\alpha_1}^4 + 8\,\gamma\,b_0(t)\,b_2(t)\,{\alpha_2}^4 + \gamma\,(b_1(t))^2\,{\alpha_2}^4 + {\alpha_3}^2}{\alpha\,{\alpha_1}^2},$$
$$q_1(t) = 0, q_2(t) = 0, q_3(t) = -6\frac{(\beta\,{\alpha_1}^4 + \gamma\,{\alpha_2}^4)\,b_1(t)\,b_0(t)}{\alpha\,{\alpha_1}^2},$$
$$q_4(t) = -6\,\frac{(b_0(t))^2\,(\beta\,{\alpha_1}^4 + \gamma\,{\alpha_2}^4)}{\alpha\,{\alpha_1}^2}.$$

Case 1. When $b_0(t) = 0$ and $\phi(\eta) = \frac{b_1(t)}{-b_2(t)+b_1(t)e^{-b_1(t)\eta}}$,

$$u_1(x,y,t) = -\frac{1}{2}\,(\frac{8\,b_0(t)\,\gamma\,{\alpha_2}^4 b_2(t) + 8\,b_0(t)\,{\alpha_1}^4\beta\,b_2(t) + \gamma\,{\alpha_2}^4\,(b_1(t))^2 + {\alpha_1}^4\beta\,(b_1(t))^2}{\alpha\,{\alpha_1}^2})$$
$$-\frac{1}{2}(\frac{{\alpha_3}^2}{\alpha\,{\alpha_1}^2}) - \frac{6\,(\beta\,{\alpha_1}^4 + \gamma\,{\alpha_2}^4)\,b_0(t)\,\left(-b_2(t) + b_1(t)\,\mathrm{e}^{-b_1(t)(t\alpha_3 + x\alpha_1 + y\alpha_2)}\right)}{\alpha\,{\alpha_1}^2}$$
$$-\frac{6\,(b_0(t))^2\,(\beta\,{\alpha_1}^4 + \gamma\,{\alpha_2}^4)\,\left(-b_2(t) + b_1(t)\,\mathrm{e}^{-b_1(t)(t\alpha_3 + x\alpha_1 + y\alpha_2)}\right)^2}{\alpha\,{\alpha_1}^2\,(b_1(t))^2}. \tag{32}$$

Case 2. When $b_1(t)=0$,

1. $\phi(\eta)=\frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\,tan(\sqrt{b_0(t)\,b_2(t)}\,\eta),\ \ b_0(t)>0, b_2(t)>0,$

$$\begin{aligned} u_2(x,y,t) = \quad & -\frac{1}{2}\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)+8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t)+\gamma\,\alpha_2{}^4\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2} \\ & -\frac{1}{2}\,\frac{\alpha_1{}^4\beta\,\,(b_1\,(t))^2+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}-\frac{6\,b_0\,(t)\,(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\left(\tan\left(\sqrt{b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)\right)^2} \\ & \frac{6\,b_0\,(t)\,(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\left(\tan\left(\sqrt{b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)\right)^2}. \end{aligned} \tag{33}$$

2. $\phi(\eta)=\frac{\sqrt{-b_0(t)b_2(t)}}{b_2(t)}\,tanh\,(-\sqrt{-b_0(t)\,b_2(t)}\,\eta),\ \ b_0(t)<0, b_2(t)>0,$

$$\begin{aligned} u_3(x,y,t) = \quad & -\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)+8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t)}{2\alpha\,\alpha_1{}^2} \\ & -\frac{\gamma\,\alpha_2{}^4\,(b_1\,(t))^2+\alpha_1{}^4\beta\,\,(b_1\,(t))^2+\alpha_3{}^2}{2\alpha\,\alpha_1{}^2} \\ & +6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1\,(t)\,b_0\,(t)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\sqrt{-b_0\,(t)\,b_2\,(t)}\tanh\left(\sqrt{-b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)} \\ & +6\,\frac{b_0\,(t)\,(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\left(\tanh\left(\sqrt{-b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)\right)^2}. \end{aligned} \tag{34}$$

3. $\phi(\eta)=\frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\,tan(-\sqrt{b_0(t)\,b_2(t)}\,\eta),\ \ \ b_0(t)<0, b_2(t)<0,$

$$\begin{aligned} u_4(x,y,t) = \quad & -\frac{1}{2}\,\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)+8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t)+\gamma\,\alpha_2{}^4\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2} \\ & -\frac{1}{2}\,\frac{\alpha_1{}^4\beta\,\,(b_1\,(t))^2+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}-6\,\frac{b_0\,(t)\,(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\left(\tan\left(\sqrt{b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)\right)^2} \\ & +6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1\,(t)\,b_0\,(t)\,b_2\,(t)}{\alpha\,\alpha_1{}^2\sqrt{b_0\,(t)\,b_2\,(t)}\tan\left(\sqrt{b_0\,(t)\,b_2\,(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)}. \end{aligned} \tag{35}$$

Case 3. When $b_2(t)=0$ and $\phi(\eta)=\frac{-b_0(t)+b_1(t)e^{b_1(t)\eta}}{b_1(t)}$,

$$\begin{aligned} u_5(x,y,t) = \quad & -\frac{1}{2}\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t)+8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t)+\gamma\,\alpha_2{}^4\,(b_1\,(t))^2+\alpha_1{}^4\beta\,\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2} \\ & -\frac{1}{2}\frac{\alpha_3{}^2}{\alpha\,\alpha_1{}^2}-6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,(b_1\,(t))^2\,b_0\,(t)}{\alpha\,\alpha_1{}^2\,(-b_0\,(t)+b_1\,(t)\,\mathrm{e}^{b_1(t)(t\alpha_3+x\alpha_1+y\alpha_2)})} \\ & -6\,\frac{(b_0\,(t))^2\,(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2\,(-b_0\,(t)+b_1\,(t)\,\mathrm{e}^{b_1(t)(t\alpha_3+x\alpha_1+y\alpha_2)})^2}. \end{aligned} \tag{36}$$

Case 4. When $b_0(t) = b_1(t) = 0$ and $\phi(\eta) = -\frac{1}{b_2(t)\ \eta}$,

$$u_6(x,y,t) = -6\,\frac{(b_0(t))^2(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)(b_2(t))^2(t\alpha_3+x\alpha_1+y\alpha_2)^2}{\alpha\,\alpha_1{}^2} -4\,\frac{b_0(t)\,\gamma\,\alpha_2{}^4 b_2(t)}{\alpha\,\alpha_1{}^2}+6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1(t)\,b_0(t)\,b_2(t)\,(t\alpha_3+x\alpha_1+y\alpha_2)}{\alpha\,\alpha_1{}^2} -\frac{1}{2}\frac{8\,b_0(t)\,\alpha_1{}^4\beta\,b_2(t)+\gamma\,\alpha_2{}^4(b_1(t))^2+\alpha_1{}^4\beta\,(b_1(t))^2+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}. \quad (37)$$

Case 5. When $b_1(t)^2 < 4b_0(t)b_2(t),\ b_2(t) \neq 0$
and $\phi(\eta) = \frac{\sqrt{4b_0(t)b_2(t)-b_1(t)^2}tan(\frac{1}{2}\sqrt{4b_0(t)b_2(t)-b_1(t)^2}\ \eta)-b_1(t)}{2b_2(t)}$,

$$u_7(x,y,t) = -\frac{1}{2}\frac{8\,b_0(t)\,\gamma\,\alpha_2{}^4 b_2(t)+8\,b_0(t)\,\alpha_1{}^4\beta\,b_2(t)+\gamma\,\alpha_2{}^4(b_1(t))^2}{\alpha\,\alpha_1{}^2} -\frac{1}{2}\frac{\alpha_1{}^4\beta\,(b_1(t))^2+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}-(12\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1(t)\,b_0(t)\,b_2(t)}{\alpha\,\alpha_1{}^2\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}}) \cdot\left(\tan\left(\frac{1}{2}\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)-b_1(t)\right)^{-1} -(24\,\frac{(b_0(t))^2(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)}{\alpha\,\alpha_1{}^2})\cdot (\frac{(b_2(t))^2}{\left(\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}\tanh\left(1/2\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}\eta\right)-b_1(t)\right)^2} \quad (38)$$

Case 6. When $b_1(t)^2 > 4b_0(t)b_2(t)$,$b_2(t) \neq 0$ and
$\phi(\eta) = \frac{\sqrt{b_1(t)^2-4b_0(t)b_2(t)}\tanh(\frac{1}{2}\sqrt{b_1(t)^2-4b_0(t)b_2(t)}\ \eta)-b_1(t)}{2b_2(t)}$,

$$u_8(x,y,t) = -\frac{1}{2}\frac{8\,b_0(t)\,\gamma\,\alpha_2{}^4 b_2(t)+8\,b_0(t)\,\alpha_1{}^4\beta\,b_2(t)+\gamma\,\alpha_2{}^4(b_1(t))^2}{\alpha\,\alpha_1{}^2} -\frac{1}{2}\,\frac{\alpha_1{}^4\beta\,(b_1(t))^2+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}-(12\frac{\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}}) \cdot(\frac{b_1(t)\,b_0(t)\,b_2(t)}{\tanh\left(\frac{1}{den}\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}\,(t\alpha_3+x\alpha_1+y\alpha_2)\right)-b_1(t)}) (-24\,\frac{(b_0(t))^2(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)}{\alpha\,\alpha_1{}^2})\cdot (\frac{(b_2(t))^2}{\left(\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}\tanh\left(\frac{1}{2}\sqrt{(b_1(t))^2-4\,b_0(t)\,b_2(t)}\eta\right)-b_1(t)\right)^2} \quad (39)$$

Case 7. When $b_1(t)^2 \neq 0$,$b_1(t)^2 = 4b_0(t)b_2(t)$ and
$\phi(\eta) = -\frac{2b_0(t)(b_1(t)\ \eta+2)}{b_1(t)^2\ \eta}$,

$$\begin{aligned} u_9(x,y,t) = \quad & -\frac{1}{2}\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t) + 8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t) + \gamma\,\alpha_2{}^4\,(b_1\,(t))^2 + \alpha_1{}^4\beta\,\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2} \\ & -\frac{1}{2}\frac{\alpha_3{}^2}{\alpha\,\alpha_1{}^2} + 3\,\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,(b_1\,(t))^3\,(t\alpha_3 + x\alpha_1 + y\alpha_2)}{\alpha\,\alpha_1{}^2\,(b_1\,(t)\,(t\alpha_3 + x\alpha_1 + y\alpha_2) + 2)} \\ & -\frac{3}{2}\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,(b_1\,(t))^4\,(t\alpha_3 + x\alpha_1 + y\alpha_2)^2}{\alpha\,\alpha_1{}^2\,(b_1\,(t)\,(t\alpha_3 + x\alpha_1 + y\alpha_2) + 2)^2}. \end{aligned} \tag{40}$$

Second Set

$$\begin{aligned} & q_0\,(t) = -\frac{1}{2}\frac{8\,b_0\,(t)\,\gamma\,\alpha_2{}^4 b_2\,(t) + 8\,b_0\,(t)\,\alpha_1{}^4\beta\,b_2\,(t) + \gamma\,\alpha_2{}^4\,(b_1\,(t))^2 + \alpha_1{}^4\beta\,\,(b_1\,(t))^2 + \alpha_3{}^2}{\alpha\,\alpha_1{}^2}, \\ & q_1(t) = -6\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,b_1\,(t)\,b_2\,(t)}{\alpha\,\alpha_1{}^2},\;\; q_2(t) = -6\,\frac{(b_2\,(t))^2\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)}{\alpha\,\alpha_1{}^2}, \\ & q_3(t) = q_4\,(t) = 0. \end{aligned} \tag{41}$$

Case 1. When $b_0(t) = 0$ and $\phi(\eta) = \frac{b_1(t)}{-b_2(t)+b_1(t)e^{-b_1(t)\eta}}$,

$$\begin{aligned} u_{10}(x,y,t) = \quad & -\frac{1}{2}\,\frac{\gamma\,\,(b_1\,(t))^2\,\alpha_2{}^4 + \alpha_3{}^2}{\alpha\,\alpha_1{}^2} - 6\,\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,(b_1\,(t))^2\,b_2\,(t)}{\alpha\,\alpha_1{}^2\,(-b_2\,(t) + b_1\,(t)\,\mathrm{e}^{-b_1(t)(t\alpha_3 + x\alpha_1 + y\alpha_2)})} \\ & -\frac{1}{2}\,\frac{8\,\beta\,b_0\,(t)\,b_2\,(t)\,\alpha_1{}^4 + \beta\,\,(b_1\,(t))^2\,\alpha_1{}^4 + 8\,\gamma\,b_0\,(t)\,b_2\,(t)\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\ & -6\,\frac{(b_2\,(t))^2\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,(b_1\,(t))^2}{\alpha\,\alpha_1{}^2\,(-b_2\,(t) + b_1\,(t)\,\mathrm{e}^{-b_1(t)(t\alpha_3 + x\alpha_1 + y\alpha_2)})^2}. \end{aligned} \tag{42}$$

Case 2. When $b_1(t) = 0$
1. $\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\ tan(\sqrt{b_0(t)\ b_2(t)}\ \eta)$, $\;b_0(t) > 0, b_2(t) > 0$,

$$\begin{aligned} u_{11}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0\,(t)\,b_2\,(t)\,\alpha_1{}^4 + \beta\,\,(b_1\,(t))^2\,\alpha_1{}^4 + 8\,\gamma\,b_0\,(t)\,b_2\,(t)\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\ & -\frac{1}{2}\frac{\gamma\,\,(b_1\,(t))^2\,\alpha_2{}^4 + \alpha_3{}^2}{\alpha\,\alpha_1{}^2} - (6\,\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,b_1\,(t)\,\sqrt{b_0\,(t)\,b_2\,(t)}}{\alpha\,\alpha_1{}^2}) \\ & \cdot(\tan\left(\sqrt{b_0\,(t)\,b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)) \\ & 6\,\frac{b_2\,(t)\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\,b_0\,(t)\,\left(\tan\left(\sqrt{-b_0\,(t)\,b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)\right)^2}{\alpha\,\alpha_1{}^2} \end{aligned} \tag{43}$$

2. $\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\, tanh(\sqrt{b_0(t)\, b_2(t)}\,\eta), \quad b_0(t) > 0, b_2(t) < 0,$

$$
\begin{aligned}
u_{12}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\, b_0\,(t)\, b_2\,(t)\,\alpha_1{}^4 + \beta\,\,(b_1\,(t))^2\,\alpha_1{}^4 + \gamma\,\,(b_1\,(t))^2\,\alpha_2{}^4 + \alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -4\,\frac{\gamma\, b_0\,(t)\, b_2\,(t)\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} - [6\,\frac{\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\
& \cdot b_1\,(t)\,\sqrt{b_0\,(t)\, b_2\,(t)}\tanh\left(\sqrt{b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)] \\
& -[6\,\frac{b_2\,(t)\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)}{\alpha\,\alpha_1{}^2} \\
& \left(\tanh\left(\sqrt{b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)\right)^2]. \qquad (44)
\end{aligned}
$$

3. $\phi(\eta) = \frac{\sqrt{-b_0(t)b_2(t)}}{b_2(t)}\, tanh\,(-\sqrt{-b_0(t)\, b_2(t)}\,\eta), \;\; b_0(t) < 0, b_2(t) > 0,$

$$
\begin{aligned}
u_{13}(x,y,t) = \quad & \frac{-1}{2}\,\frac{8\,\beta\, b_0\,(t)\, b_2\,(t)\,\alpha_1{}^4 + \beta\,\,(b_1\,(t))^2\,\alpha_1{}^4 + 8\,\gamma\, b_0\,(t)\, b_2\,(t)\,\alpha_2{}^4 + \alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -\frac{\gamma\,\,(b_1\,(t))^2\,\alpha_2{}^4}{2\alpha\,\alpha_1{}^2} + (6\,\frac{(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\, b_1\,(t)\,\sqrt{-b_0\,(t)\, b_2\,(t)}}{\alpha\,\alpha_1{}^2}) \\
& (\tanh\left(\sqrt{-b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)) \\
& +6\,\frac{b_2\,(t)\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\, b_0\,(t)\left(\tanh\left(\sqrt{-b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)\right)^2}{\alpha\,\alpha_1{}^2} \qquad (45)
\end{aligned}
$$

4. $\phi(\eta) = \frac{\sqrt{b_0(t)b_2(t)}}{b_2(t)}\, tan(-\sqrt{b_0(t)\, b_2(t)}\,\eta), \quad b_0(t) < 0, b_2(t) < 0,$

$$
\begin{aligned}
u_{14}(x,y,t) = \quad & -\frac{8\,\beta\, b_0\,(t)\, b_2\,(t)\,\alpha_1{}^4 + \beta\,\,(b_1\,(t))^2\,\alpha_1{}^4 + 8\,\gamma\, b_0\,(t)\, b_2\,(t)\,\alpha_2{}^4 + \alpha_3{}^2}{2\alpha\,\alpha_1{}^2} \\
& -\frac{\gamma\,\,(b_1\,(t))^2\,\alpha_2{}^4}{2\alpha\,\alpha_1{}^2} + (6\,\frac{\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2}) \\
& (b_1\,(t)\,\sqrt{b_0\,(t)\, b_2\,(t)}\tan\left(\sqrt{b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)) \\
& -6\,\frac{b_2\,(t)\,(\beta\,\alpha_1{}^4 + \gamma\,\alpha_2{}^4)\, b_0\,(t)\left(\tan\left(\sqrt{b_0\,(t)\, b_2\,(t)}\,(t\alpha_3 + x\alpha_1 + y\alpha_2)\right)\right)^2}{\alpha\,\alpha_1{}^2} \qquad (46)
\end{aligned}
$$

Case 3. When $b_2(t)=0$ and $\phi(\eta)=\frac{-b_0(t)+b_1(t)e^{b_1(t)\eta}}{b_1(t)}$,

$$
\begin{aligned}
u_{15}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0(t)\,b_2(t)\,\alpha_1{}^4+\beta\,(b_1(t))^2\,\alpha_1{}^4+8\,\gamma\,b_0(t)\,b_2(t)\,\alpha_2{}^4+\alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2(t)\left(-b_0(t)+b_1(t)\,\mathrm{e}^{b_1(t)(t\alpha_3+x\alpha_1+y\alpha_2)}\right)}{\alpha\,\alpha_1{}^2} \\
& -6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\left(-b_0(t)+b_1(t)\,\mathrm{e}^{b_1(t)(t\alpha_3+x\alpha_1+y\alpha_2)}\right)^2}{\alpha\,\alpha_1{}^2\,(b_1(t))^2} \\
& -6\,\frac{(b_2(t))^2}{\alpha\,\alpha_1{}^2\,(b_1(t))^2}-\frac{1}{2}\frac{\gamma\,(b_1(t))^2\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2}. \qquad (47)
\end{aligned}
$$

Case 4. When $b_0(t)=b_1(t)=0$ and $\phi(\eta)=-\frac{1}{b_2(t)\,\eta}$,

$$
\begin{aligned}
u_{16}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0(t)\,b_2(t)\,\alpha_1{}^4+\beta\,(b_1(t))^2\,\alpha_1{}^4+8\,\gamma\,b_0(t)\,b_2(t)\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\
& -\frac{1}{2}\frac{\gamma\,(b_1(t))^2\,\alpha_2{}^4+\alpha_3{}^2}{\alpha\,\alpha_1{}^2}+6\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1(t)}{\alpha\,\alpha_1{}^2\,(t\alpha_3+x\alpha_1+y\alpha_2)} \\
& -6\,\frac{\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2\,(t\alpha_3+x\alpha_1+y\alpha_2)^2}. \qquad (48)
\end{aligned}
$$

Case 5. When $b_1(t)^2<4b_0(t)b_2(t)$, $b_2(t)\neq 0$
and $\phi(\eta)=\frac{\sqrt{4b_0(t)b_2(t)-b_1(t)^2}tan(\frac{1}{2}\sqrt{4b_0(t)b_2(t)-b_1(t)^2}\ \eta)-b_1(t)}{2b_2(t)}$,

$$
\begin{aligned}
u_{17}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0(t)\,b_2(t)\,\alpha_1{}^4+\beta\,(b_1(t))^2\,\alpha_1{}^4+8\,\gamma\,b_0(t)\,b_2(t)\,\alpha_2{}^4+\alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -\frac{1}{2}\frac{\gamma\,(b_1(t))^2\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2}-(3\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1(t)\,\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}}{\alpha\,\alpha_1{}^2}) \\
& (\tan\left(\frac{1}{2}\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}\eta\right)-b_1(t))-(\frac{3}{2}\,\frac{\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2}) \\
& \left(\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}\tan\left(\frac{1}{2}\sqrt{-(b_1(t))^2+4\,b_0(t)\,b_2(t)}\eta\right)-b_1(t)\right)^2 \qquad (49)
\end{aligned}
$$

Case 6. When $b_1(t)^2 > 4b_0(t)b_2(t)$, $b_2(t) \neq 0$ and
$\phi(\eta) = \frac{\sqrt{b_1(t)^2-4b_0(t)b_2(t)}tanh(\frac{1}{2}\sqrt{b_1(t)^2-4b_0(t)b_2(t)}\ \eta)-b_1(t)}{2b_2(t)}$,

$$
\begin{aligned}
u_{18}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0\,(t)\,b_2\,(t)\,\alpha_1{}^4+\beta\,\left(b_1\,(t)\right)^2\alpha_1{}^4+8\,\gamma\,b_0\,(t)\,b_2\,(t)\,\alpha_2{}^4+\alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -\frac{1}{2}\frac{\gamma\,\left(b_1\,(t)\right)^2\alpha_2{}^4}{\alpha\,\alpha_1{}^2} - [3\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_1\,(t)}{\alpha\,\alpha_1{}^2} \\
& \sqrt{\left(b_1\,(t)\right)^2-4\,b_0\,(t)\,b_2\,(t)}\tanh\left(\frac{1}{2}\sqrt{\left(b_1\,(t)\right)^2-4\,b_0\,(t)\,b_2\,(t)}\eta\right)-b_1\,(t)] \\
& -[\frac{3}{2}\frac{\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\
& \left(\sqrt{\left(b_1\,(t)\right)^2-4\,b_0\,(t)\,b_2\,(t)}\tanh\left(\frac{1}{2}\sqrt{\left(b_1\,(t)\right)^2-4\,b_0\,(t)\,b_2\,(t)}\eta\right)-b_1\,(t)\right)^2]
\end{aligned} \tag{50}
$$

Case 7. When $b_1(t)^2 \neq 0$,$b_1(t)^2 = 4b_0(t)b_2(t)$ and
$\phi(\eta) = -\frac{2b_0(t)(b_1(t)\ \eta+2)}{b_1(t)^2\ \eta}$,

$$
\begin{aligned}
u_{19}(x,y,t) = \quad & -\frac{1}{2}\frac{8\,\beta\,b_0\,(t)\,b_2\,(t)\,\alpha_1{}^4+8\,\gamma\,b_0\,(t)\,b_2\,(t)\,\alpha_2{}^4+\alpha_3{}^2}{\alpha\,\alpha_1{}^2} \\
& -\frac{1}{2}\frac{\beta\,\left(b_1\,(t)\right)^2\alpha_1{}^4+\gamma\,\left(b_1\,(t)\right)^2\alpha_2{}^4}{\alpha\,\alpha_1{}^2} \\
& +12\,\frac{(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\,b_2\,(t)\,b_0\,(t)\,(b_1\,(t)\,\eta+2)}{b_1\,(t)\,\alpha\,\alpha_1{}^2\,(t\alpha_3+x\alpha_1+y\alpha_2)} \\
& -24\,\frac{\left(b_2\,(t)\right)^2(\beta\,\alpha_1{}^4+\gamma\,\alpha_2{}^4)\left(b_0\,(t)\right)^2(b_1\,(t)\,\eta+2)^2}{\alpha\,\alpha_1{}^2\left(b_1\,(t)\right)^4(t\alpha_3+x\alpha_1+y\alpha_2)^2}.
\end{aligned} \tag{51}
$$

4. CONCLUSIONS

In summary, we have proposed the (2+1)-dimensional extension of the Benjamin – Ono equation with time-dependent coefficients. the extended improved generalized tanh-coth Method has been effectively applied to the (2+1)-dimensional extension of the Benjamin – Ono equation with time-dependent coefficients that describes inner waves of deep-stratified fluids. Our results revealed that the proposed method is effective for handling nonlinear evolution equations. Thus, we will extend the proposed method for some nonlinear fractional partial differential equations in a future work.

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REFERENCES

[1] T. Benjamin, “Internal waves of permanent form in fluids of great depth,” Journal of Fluid Mechanics, vol. 29(3), pp. 559–592, 1967

[2] Y. Matsuno, “Exact multi-soliton solution of the BenjaminOno equation,” Journal of Physics A: Mathematical and General, vol. 12(4), pp. 619–621, 1979.

[3] D. Ambrose and J. Wilkening, “Computation of timeperiodic solutions of the Benjamin–Ono equation,” Journal of Nonlinear Science, vol. 20(3), pp. 277–308, 2010.

[4] A. Wazwaz, “A (2 + 1)-dimensional extension of the Benjamin-Ono equation,” International Journal of Numerical Methods Heat Fluid and Flow, vol. 28(11), pp. 2681–2687, 2018.

[5] L. Wu, X. Zhang and J. Manafian, “On the Exact Solitary Wave Solutions to the New (2+1) and (3+1)-Dimensional Extensions of the Benjamin-Ono Equations,” Advances in Mathematical Physics, Vol. 2021, 6672819, 9 pages.

[6] R. Kumar and A. Kumar, “Dynamical behavior of similarity solutions of CKOEs with conservation law,” Applied Mathematics and Computations, vol. 20, 103766, 2021.

[7] M. Alshbool, O. Isik and I. Hashim, “Fractional Bernstein Series Solution of Fractional Diffusion Equations with Error Estimate,” Axioms, vol. 422(4):126976, 2022.

[8] A. Polyanin, V. Sorokin, “New exact solutions of nonlinear wave type PDEs with delay,” Applied Mathematics, vol. 108, 106512, 2020.

[9] F. Khodadad, F. Nazari, M. Eslami and H. Rezazadeh, Soliton solutions of the conformable fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity, Opt Quant Electron 49(384) (2017), 12 pages.

[10] M. Islam, M. Akbar and M. Azad, “Closed-form travelling wave solutions to the nonlinear space-time fractional coupled Burger’s equation,” Arab Journal of basic and applied sciences, vol. 26(1), pp. 1-11, 2019.

[11] U. Younas, M. Younis, A. Seadawy, S. Rizvi, S. Althobaiti and S. Sayed, “Diverse exact solutions for modified nonlinear Schrödinger equation with conformable fractional derivative,” Results in Physics, vol. 20, 103766, 2021.

[12] W. Malfliet, “Solitary wave solutions of nonlinear wave equation,” American Journal of Physics, Vol. 60(7), pp. 650-654, 1992.

[13] E. Fan and Y. Hona, “Generalized tanh method extended to special types of nonlinear equations,” Zeitshrift fur Naturforschung A, Vol. 57(8), pp. 692-700, 2002.

[14] A. M. Wazwaz, “The extended tanh method for new solutions for many forms of the fifth-order KdV equation,” Applied Mathematics and Computation, 84(2), pp.1002-1014, 2007.

[15] A. Salas and C. A. Gomez, "The Cole Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6)," Applied Mathematics and Computation, Vol. 204(15), pp. 957-962, 2008.
[16] G. Garzon and S. Gomez, "The generalized tanh-coth method applied to biological model referent to nano-solitons of ionic wave," Applied Mathematical Sciences, Vol. 9(138), pp. 6877-6882, 2015.
[17] M. Torvattanabun and S. Koonprasert, "Variational iteration method combined with the improved generalized tanh-coth method for the generalized$(1+1)$-dimensional and $(2+1)$-dimensional equations, Far East Journal of Mathematical Sciences, Vol. 8(100), pp. 1337-1355, 2016.
[18] M. Torvattanabun, J. Simmapim, D. Saennuad and T. Somaumchan, "The improved generalized tanh-coth method applied to sixth-order solitary wave equation, Journal of Mathematics, Vol. (2017), 11 pages, 2017.