Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 17, Number 1, (2022) pp. 319-330

http://www.ripublication.com/adsa.htm

Analysis of Fractional Order Lane-Emden Type Differential Equations

C.B. Shrungare
School of Mathematical Sciences,
Swami Ramanand Teerth Marathwada University, Nanded-431606, India

S.M. Jogdand
Shri Sant Gadge Maharaj College, Loha, Nanded

D. D. Pawar
School of Mathematical Sciences,
Swami Ramanand Teerth Marathwada University, Nanded-431606, India

W. D. Patil
Department of Applied Mathematics,
A. C. Patil College of Engineering,Navi Mumbai-410210, India

Abstract

The purpose of this literature is to tackle Lane-Emden type fractional order linear and non linear partial differential equations with initial and boundary conditions by applying Riemann - Leivoulli fractional integral. Fractional order homotopy perturbation method, proposed by D. D. Pawar has handled the problems very easily and precisely yield the approximate series solutions. Further the properties of Lane-Emden type fractional non linear differential equations with initial and boundary conditions have been interpreted in the form of two dimensional plots by using Matlab.

Keywords : Time fractional order partial differential equations, Lane-Emden type time fractional order differential equations, Fractional order homotopy perturbation method [FOHPM]

1. INTRODUCTION

In the view of recent challenges in the emerging fields of science and technology, researchers are looking forward through fractional order mathematical models [1], [2] to study the complex concepts of the physical phenomenon. It has been observed that fractional calculus has made it possible to analyse the physical and chemical properties more widely. At the same time, theory of fractional calculus have developed the confrontation regarding it's solutions. Fractional calculus has made tremendous evolution in the field of economics and finance [3], physics[4], hydraulics, geology and fluid dynamics [5], biomedical and biotechnology [6]-[8], control systems [9], signals and systems, communication theory [10], image processing [11] and so on. On account of memory effect of fractional calculus, the researchers and scientists have fruitfully utilised this branch of calculus to enhance the day to day human life applications.

To analyse the fractional order differential equations, we need to solve them more accurately. Researchers and scientists have demonstrated some of the analytic and numerical methods which freely handles mathematical models of fractional order derivatives to get their solutions . The various types of perturbative and non perturbative techniques like finite difference method [12], variation iteration method [13], Adomain decomposition method [14], modified Adomian decomposition method [15], fractional variational iteration method [16], homotopy analysis method [17], Ji Huan He [18] - [20] proposed homotopy perturbation method. D. D. Pawar et al. [21] have extended homotopy perturbation method to fractional order homotopy perturbation method.

Fractional order homotopy perturbation method has been applied to solve fractional order Emden- Fowler type differential equations, fractional order Klein-Gorden type differential non linear wave equations and fractional order evolution type differential equations by D. D. Pawar et al. [21].

In this paper , we have illustrated some of the time fractional Lane-Emden type differential equations by using fractional order homotopy perturbation method to get it's series solution and analysed it appropriately. In the next section, we have introduced fractional order homotopy perterbation method.

1.1. Fractional order homotopy perturbation method [FHPM]

In this section, we have briefly explained fractional order homotopy perturbation method to solve system of 'r' number of time fractional ordinary differential equations with initial conditions. The general form of system of time fractional order partial differential equations can be considered as follows

$$\begin{aligned} D^{\alpha_j} u_j(x_1, x_2, x_3, \ldots x_{r-1}, t) + N_j(x_1, x_2, x_3, \ldots x_{r-1}, t, u_1, u_2, \ldots, u_r) \\ = g_j(x_1, x_2, x_3, \ldots x_{r-1}, t) \end{aligned} \tag{1}$$

where $\alpha_j \in \mathbb{R}^+$ and $j = 1, 2, 3, \ldots r$
With the following initial conditions

$$u_{i,0}(x_1, x_2, x_3, ... x_{r-1}, t_0) = f_i(x_1, x_2, x_3, ... x_{r-1}) \tag{2}$$

where $i = 1, 2, 3, \ldots r$ all $N_j's$ are non linear operator and $f_i's$ are functions of $x_i's$ and t .
By taking 'p' as an embedded parameter, we construct homotopy for each of the differential equation as follows

$$(1-p)\left(D^{\alpha_j} u_j - u_{j,0}\right) + p(D^{\alpha_j} u_j + N_j(x_1, x_2, ... x_{r-1}, t, u_1, u_2, \ldots, u_r) \tag{3}$$
$$- g_j(x_1, x_2, \ldots x_{r-1}, t)) = 0 \tag{4}$$

$$\therefore\ D^{\alpha_j} u_j = u_{j,0} - p\left(u_{j,0} + N_j(x_1, x_2, x_3, ... x_{r-1}, t, u_1, u_2, \ldots, u_r)\right.$$
$$\left. + g_j(x_1, x_2, x_3, \ldots x_{r-1}, t)\right) \tag{5}$$

Applying inverse operator, ${}_{t_0}J_t^{\alpha_j}$ to both sides of 5,

$$U_j(x_1, x_2, x_3, ... x_{r-1}, t) = {}_{t_0}J_t^{\alpha_j} u_{j,0} - p\, {}_{t_0}J_t^{\alpha_j} [u_{j,0}$$
$$+ N_j(x_1, x_2, x_3, ... x_{r-1}, t, U_1, U_2, \ldots, U_r)$$
$$- g_j(x_1, x_2, x_3, \ldots x_{r-1}, t)] \tag{6}$$

where $j = 1, 2, 3, ...$ and $U_j(x_1, x_2, x_3, \ldots x_{r-1}, t_0) = u_j(x_1, x_2, x_3, \ldots x_{r-1}, t_0)$
We get the series for the system 1.1 which is given by equating the coefficients of power of p's in 4.

$$U_j(x_1, x_2, x_3, ... x_{r-1}, t) = U_{j,0} + p\, U_{j,1} + p^2\, U_{j,2} + \ldots . = \sum_{i=0}^{\infty} p^i U_{j,i} \tag{7}$$

where all $U_{j,i}$ are functions of $x_1, x_2, x_3, ... x_{r-1}$ and t. The approximate series solution for the system 1.1 yields by taking $p \to 1$ in 7 as

$$U_j(x_1, x_2, x_3, ... x_{r-1}, t) = U_{j,0} + U_{j,1} + U_{j,2} + \ldots . = \sum_{i=0}^{\infty} U_{j,i} \tag{8}$$

It is necessary to note that the major advantage of fractional order homotopy perturbation method [FHPM] is that it gives solution in the form of perturbation series which can freely give solution and it may be convergence in all sense which has been explained independently.

1.2. Fractional Integral and Fractional Derivative

Definition 1 *[1]-[2] A real function $h(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exist a real number $p\,(> \mu\,)$ such that $h(\,t\,) = t^p h_1(t)$ where $h_1(t) \in C[\,0,\infty\,]$ and it is said to be in the space C_μ^n if and only if $h^n \in C_\mu$, $n \in N$*

Definition 2 *[1]-[2]*

Riemann-Liouville fractional integral

Riemann-Liouville fractional order integral operator (J_t^α) of order $\alpha > 0$ of a function $h \in C_\mu$, $\mu \geq -1$ is defined as

$$ {}_{\cdot 0}J_t^\alpha\, h(t) = \frac{1}{\Gamma\,\alpha} \int_0^t (t-\tau)^{\alpha-1}.h\,(\tau)\,d\tau \qquad (\alpha > 0) $$

where $t \geq 0$ and $\Gamma\,(.)$ is a well known gamma function.

Some of the properties of Riemann -Liouville fractional integral operator have been explained. For $h(t) \in C_\mu$, $\mu \in R$, $\mu > -1$, $a \geq 0$ $\quad \alpha,\,\beta > 0$ and $\nu \geq -1$

1. ${}_{\cdot 0}J_t^\alpha\, h(t)\, {}_{\cdot 0}J_t^\beta\, h(t) = {}_{\cdot 0}J_t^{\alpha+\beta}\, h(t)$

2. ${}_{\cdot 0}J_t^\alpha\, h(t)\, {}_{\cdot 0}J_t^\beta\, h(t) = {}_{\cdot 0}J_t^\beta\, h(t)\, {}_{\cdot 0}J_t^\alpha\, h(t)$

3. ${}_{\cdot 0}J_t^\alpha\,(t-a)^\nu = \frac{\Gamma\,(\nu+1)}{\Gamma\,(\alpha+\nu+1)}(t-a)^{(\alpha+\nu)}$

Definition 3 *[1]-[2]*

Riemann-Liouville fractional order derivative

Let α be non negative real number. Let $h(t)$ be piecewise continuous on $(0,\,\infty)$ and integrable on any finite subinterval of $[0,\infty]$.

For $t > 0$, Riemann-Liouville fractional derivative of $h(t)$ of order α.

$$ {}_0D_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} h(\tau) d\tau, \qquad \alpha > 0. \tag{9} $$

where n is a positive integer such that $(n-1) < \alpha < n$.

Definition 4 *[1]-[2]*

Caputo sense fractional order derivative

Let α be non negative real number. Let $h(t)$ be piecewise continuous on $(0,\,\infty)$ and

integrable on any finite subinterval of $[0, \infty]$.
The Caputo sense fractional order derivative $\left({}_{\cdot 0}^{C}D_t^{\alpha}\right)$ *of* $h(t)$ *is defined as*

$$ {}_{\cdot 0}^{C}D_t^{\alpha}\, h(t) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}\left(\tau\right) d\tau $$

For $n-1 < \alpha \leq n \quad n \in \mathbb{N} \qquad t \geq 0$ *and* $h\left(t\right) \in C_{-1}^{n}$

Definition 5 *[1]-[2]*
Grunwald-Letnikov Fractional Derivatives
Grunwald-Letnikov definition of fractional derivative of a function generalize the notion of backward difference quotient of integer order. Grunwald-Letnikov fractional derivative of order α *of the function* $f(t)$ *is defined as*

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \frac{\Gamma((\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} f(t-jk) \tag{10} $$

where $\frac{t-a}{h}$ *is integer and* $\alpha \in C$.
If $\alpha = -1$, *we have a Riemann sum which is the first integral.*

2. TIME FRACTIONAL LANE-EMDEN TYPE DIFFERENTIAL EQUATIONS

In 1870, J. Homer L.A., et al. [22] have proposed the renewal of heat radiation from the sun by means of the mechanical power of the sun's mass and how it goes descending towards it's center. They have also focused on the temperatures and densities corresponding to assumed volume of sun . Further, it was proved that some of the known gases like hydrogen, carbon or supposing a mixture of gases are present inside the sun layers. It is hypothesised that pure hydrogen would give the lowest temperature of all known substances and all the equations have been analysed systematically.
Lane-Emden type non linear differential equations handles equilibrium density distribution in self-gravitative sphere of polytrophic isothermal gas, the thermal history of a spherical cloud of gas, isothermal gas spheres and thermionic currents. The equation bears great importance in the area of radiative cooling. In astrophysics, it forms the modelling of clusters of galaxies. It is to be noted that Lane-Emden type differential equation has a singularity at the origin [23]-[25]. In view of importance of Lane-Emden type differential equations, fractional order Lane-Emden differential equation's initial value problem have been solved by using fractional order

homotopy perterbation method which gives their approximate solution. As Lane-Emden differential equations have singularity behaviour at origin in this regards it has become more interesting to get solution of the equation more precisely.
In this section we have studied two types of time fractional order Lane-Emden type differential equations with initial conditions and the approximate solution emerged in the series form have been analysed significantly.

2.1. Example

Let us take time fractional non-linear Lane-Emden type differential equation

$$D_t^{\alpha} u(t) + \frac{2}{t} u_t(t) - 2(2t^2+3)u(t) = 0 \tag{11}$$

Where $0 \leq \alpha \leq 2$.
with initial condition $u(0) = 1$ and $u_t(0) = 0$ According to the homotopy perturbation method, we may construct linear operator as $L[u] = D_t^{\alpha} u(t)$ and non-linear operator as
$N[u(t)] = D_t^{\alpha} u(t) + \frac{2}{t} u_t(t) - 2(2t^2+3)u(t)$
Now homotopy have been constructed as

$$\begin{aligned} H\ (u,p) &= (1-p)\left[D_t^{\alpha} u(t) - u_0(t)\right] \\ &+ p\left(D_t^{\alpha} u(t) + \frac{2}{t} u_t(t) - 2(2t^2+3)u(t)\right) = 0 \end{aligned} \tag{12}$$

where $p \in [0,\ 1]$
Taking initial guess $u_0(t) = 1$

Equating coefficients of 'p' in equation 7

$$\begin{aligned} u_1(t) &= \frac{2^2\, t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2.3.t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2(t) &= \left(2^2 3^2 - \frac{2^3}{(\alpha+1)}\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{3.2^2}{(\alpha-1)} \cdot \frac{t^{(2\alpha-2)}}{\Gamma(2\alpha-1)} \\ &+ 2^3 3 \left(1 + \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right) \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} \\ &+ \frac{2^4 \cdot \Gamma(\alpha+5)}{\Gamma(\alpha+3)} \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} \end{aligned}$$

$$
\begin{aligned}
u_3(t) = &\left[\frac{2^6\Gamma(\alpha+5)\Gamma(2\alpha+7)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}\right]\frac{t^{3\alpha+6}}{\Gamma(3\alpha+7)}\\
&+\left[\frac{2^5.3.\Gamma(\alpha+5)}{\Gamma(\alpha+3)}+2^5.3\left(1+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right)\frac{\Gamma(2\alpha+5)}{\Gamma(2\alpha+3)}\right]\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\\
&+\left[\left(2^4 3^2-\frac{2^5}{(\alpha+1)}\right)\frac{\Gamma(2\alpha+3)}{\Gamma(2\alpha+1)}+2^4.3^2\left(1+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right)-\frac{2^4\Gamma(\alpha+5)}{(2\alpha+3)\Gamma(\alpha+3)}\right]\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\\
&+\left[2^3.3^3-\frac{3.2^4}{(\alpha+1)}-\frac{3.2^4\Gamma(2\alpha+1)}{(\alpha-1)\Gamma(2\alpha-1)}-2^4.3\left(1+\frac{\Gamma(\alpha+3)}{(2\alpha+1)\Gamma(\alpha+1)}\right)\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&-\left[\frac{2^3.3^2}{(\alpha-1)}+\left(2^3 3^2-\frac{2^4}{(\alpha_1)}\right)\frac{1}{2\alpha-1)}\right]\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}\\
&+\left[\frac{3.2^3}{(\alpha-1)(2\alpha-3)}\right]\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)}
\end{aligned}
$$

Using equation 7 , the approximate solution in the form of series is given by putting $p=1$ as

$$
\begin{aligned}
u(t) = &1+\frac{2^2\,t^{\alpha+2}}{\Gamma(\alpha+3)}+6\frac{t^{\alpha}}{\Gamma\,\alpha+1}+\left(2^2 3^3-\frac{2^3}{\alpha+1}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{3.2^2}{\alpha-1}\cdot\frac{t^{2\,\alpha-2}}{\Gamma(2\alpha-1)}\\
&+2^3 3\left(1+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{2^4\cdot\Gamma(\alpha+5)}{\Gamma(\alpha+3)}\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)}\\
&+\left[\frac{2^6\Gamma(\alpha+5)\Gamma(2\alpha+7)}{\Gamma(\alpha+3)\Gamma(2\alpha+5)}\right]\frac{t^{3\alpha+6}}{\Gamma(3\alpha+7)}\\
&+\left[\frac{2^5.3.\Gamma(\alpha+5)}{\Gamma(\alpha+3)}+2^5.3\left(1+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right)\frac{\Gamma(2\alpha+5)}{\Gamma(2\alpha+3)}\right]\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\\
&+\left[\left(2^4 3^2-\frac{2^5}{(\alpha+1)}\right)\frac{\Gamma(2\alpha+3)}{\Gamma(2\alpha+1)}+2^4.3^2\left(1+\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\right)-\frac{2^4\Gamma(\alpha+5)}{(2\alpha+3)\Gamma(\alpha+3)}\right]\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\\
&+\left[2^3.3^3-\frac{3.2^4}{(\alpha+1)}-\frac{3.2^4\Gamma(2\alpha+1)}{(\alpha-1)\Gamma(2\alpha-1)}-2^4.3\left(1+\frac{\Gamma(\alpha+3)}{(2\alpha+1)\Gamma(\alpha+1)}\right)\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&-\left[\frac{2^3.3^2}{(\alpha-1)}+\left(2^3 3^2-\frac{2^4}{(\alpha_1)}\right)\frac{1}{2\alpha-1)}\right]\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}+\left[\frac{3.2^3}{(\alpha-1)(2\alpha-3)}\right]\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)}
\end{aligned}
$$

The above equation is approximate series solution for the example 2.1. Lane-Emden type fractional order non- linear differential equation for various fractional order have been represented graphically as following figure 1.

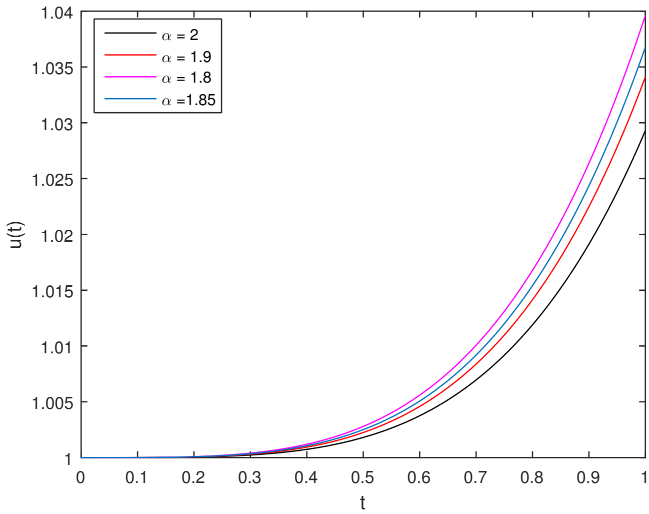

Figure 1: 2D plot represent approximate solution $u(t)$ vs x of Lane-Emden type time fractional differential equation 2.1 for $\alpha = 2, \alpha = 1.9, \alpha = 1.85, \alpha = 1.8$

2.2. Example

Let us take time fractional non linear homogeneous Lane-Emden type differential equation

$$D_t^\alpha u(t) + \frac{2}{t} u_t(t) + u^n(t) = 0$$

Where $0 < \alpha \leq 2$.
with initial condition $u(0) = 1$ and $u_t(0) = 0$.

According to fractional order homotopy perturbation method, we may construct linear operator as $L[u(t)] = D_t^\alpha u(t) - u_0(t)$ and non-linear operator as
$N[u(t)] = D_t^\alpha u(t) + \frac{2}{t} u_t(t) + u^n(t)$
Now homotopy can be constructed as

$$\begin{aligned} H\ (u,\, p) = (1-\, p)\left[D_t^\alpha u(t) - u_0(t)\right] \\ +\ p\left(D_t^\alpha u(t) + \frac{2}{t} u_t(t) + u^n(t)\right) = 0 \end{aligned} \tag{13}$$

Let's take $n = 1$
where $p \in [0,\, 1]$
Taking initial guess $u_0(t) = 1$
Equating coefficients of 'p' in equation 13

$$u_1(t) = -\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(t) = \left(\frac{2}{\alpha-1}\right)\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(t) = -\left(\frac{2^2}{(\alpha-1)(2\alpha-3)}\right)\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)} - \left(\frac{6\alpha-4}{(2\alpha-1)(\alpha-1)}\right)\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_4(t) = \left[\left(\frac{2^3}{(\alpha-1)(2\alpha-3)(3\alpha-5)}\right)\right]\frac{t^{4\alpha-6}}{\Gamma(4\alpha-5)} + \left(\frac{2^4}{(2\alpha-1)(2\alpha-3)}\right)\frac{t^{4\alpha-4}}{\Gamma(4\alpha-3)} + \left[\frac{2}{(3\alpha-1)(3\alpha-2)}\right]\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$\begin{aligned} u_5(t) = &\left(\frac{2^4}{(\alpha-1)(2\alpha-3)(3\alpha-5)(4\alpha-7)}\right)\frac{t^{5\alpha-8}}{\Gamma(5\alpha-7)} \\ &+ \left(\frac{2^2}{(2\alpha-1)} - \frac{2^2}{(2\alpha-3)}\right)\frac{2}{(\alpha-1)(4\alpha-5)}\frac{t^{5\alpha-6}}{\Gamma(\alpha-5)} \\ &- \left[\frac{2}{3\alpha-1} - \frac{2(3\alpha-2)}{(\alpha-1)(2\alpha-1)}\right]\frac{2}{(4\alpha-3)}\frac{t^{5\alpha-4}}{\Gamma(5\alpha-3)} \\ &- \frac{2}{(4\alpha-1)}\frac{t^{5\alpha-2}}{\Gamma(5\alpha-1)} - \frac{2^3}{(\alpha-1)(2\alpha-3)(3\alpha-5)}\frac{t^{5\alpha-6}}{\Gamma(5\alpha-5)} \\ &+ \left[\left(\frac{2^2}{(2\alpha-1)} - \frac{2^2}{(2\alpha-3)}\right)\frac{1}{\alpha-1}\right]\frac{t^{5\alpha-4}}{\Gamma(5\alpha-3)} \\ &- \left[\frac{2}{(3\alpha-1)} - \left(\frac{2(3\alpha-2)}{(\alpha-1)(2\alpha-1)}\right)\right]\frac{t^{5\alpha-2}}{\Gamma(5\alpha-1)} - \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \end{aligned}$$

and so on. Using equation 7, the approximate solution in the form of series is given by putting $p = 1$ as

$$\begin{aligned} u(x,t) = 1 &- \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2}{\alpha-1}\right)\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2^2}{(\alpha-1)(2\alpha-3)}\right)\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)} \\ &- \left(\frac{6\alpha-4}{(2\alpha-1)(\alpha-1)}\right)\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left[\left(\frac{2^2}{(\alpha-1)(2\alpha-3)}\right)\frac{2}{\Gamma(3\alpha-5)}\right]\frac{t^{4\alpha-6}}{\Gamma(4\alpha-5)} \\ &- \left[\left(\frac{6\alpha-4}{(\alpha-1)(2\alpha-1)}\right)\frac{2}{(3\alpha-2)} - \left(\frac{2^2}{(\alpha-1)(2\alpha-3)}\right)\right]\frac{t^{4\alpha-4}}{\Gamma(4\alpha-3)} \\ &+ \left[\frac{2}{(3\alpha-1)} - \frac{6\alpha-4}{(\alpha-1)(2\alpha-1)}\right]\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ &+ \left[\frac{2^4}{(\alpha-1)(2\alpha-3)(3\alpha-5)(4\alpha-7)}\right]\frac{t^{5\alpha-8}}{\Gamma(5\alpha-7)} + \left[\frac{2^2}{(2\alpha-1)} - \frac{2^2}{(2\alpha-3)}\right] \\ &\left(\frac{2}{(\alpha-1)(4\alpha-5)}\right)\frac{t^{5\alpha-6}}{\Gamma(5\alpha-5)} - \left[\frac{2}{3\alpha-1} - \frac{2(3\alpha-2)}{(\alpha-1)(2\alpha-1)}\right]\frac{2}{(4\alpha-3)}\frac{t^{5\alpha-4}}{\Gamma(5\alpha-3)} \\ &- \frac{2}{(4\alpha-1)}\frac{t^{5\alpha-2}}{\Gamma(5\alpha-1)} - \frac{2^3}{(\alpha-1)(2\alpha-3)(3\alpha-5)}\frac{t^{5\alpha-6}}{\Gamma(5\alpha-5)} \\ &+ \left[\left(\frac{2^2}{(2\alpha-1)} - \frac{2^2}{(2\alpha-3)}\right)\frac{1}{\alpha-1}\right]\frac{t^{5\alpha-4}}{\Gamma(5\alpha-3)} \end{aligned}$$

$$-\left[\frac{2}{(3\alpha-1)}-\left(\frac{2(3\alpha-2)}{(\alpha-1)(2\alpha-1)}\right)\right]\frac{t^{5\alpha-2}}{\Gamma(5\alpha-1)}-\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}+\ldots$$

The analysis for various fractional order have been represented graphically as following figure 2.

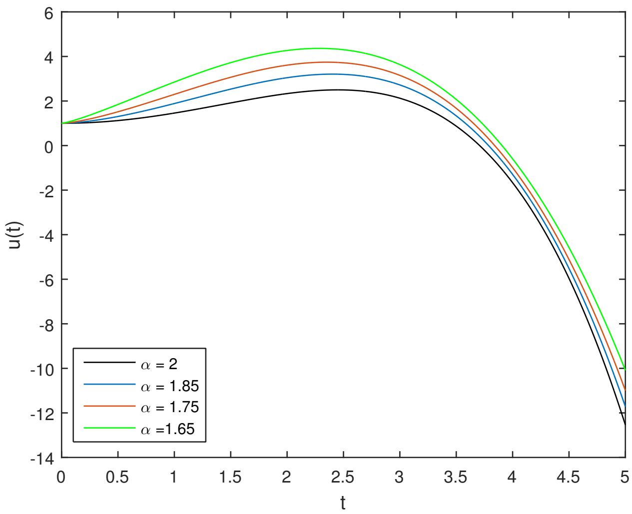

Figure 2: 2D plot represent approximate solution $u(t)$ vs t of Lane-Emden type time fractional differential equation 2.2 for $\alpha = 2$, $\alpha = 1.85$, $\alpha = 1.75$, $\alpha = 1.65$

3. CONCLUSIONS

The approximate series solutions have been obtained upto fourth term in first example and upto sixth term in second example for fractional Lane–Emden type differential equations with the sense of Riemann–Liouville derivative. It is being observed that fractional Lane–Emden type equation is useful to model many phenomena in mathematical physics and astrophysics. The proposed solution yields the reliable results of the model. The graphical results demonstrates the nature of the solution of fractional Lane–Emden type differential equations for various fractional orders. It is ascertained that the solution is more suitable and effective to analyse the complexity of fractional Lane-Emden type differential equations in both the examples.

REFERENCES

[1] S. G. Samko, et. al.; *Fractional Integrals and Derivatives Theory and Applications*, Gorden and Breach, New York , 1993.

[2] I. Podlubny; *Fractional Differential Equation*, Academic Press, New York, 1999.

[3] Dousseh, Paul Yaovi, et al., Adaptive Control of a New Chaotic Financial System with Integer Order and Fractional Order and Its Identical Adaptive Synchronization, *Mathematical problems in engineering*, 2021, 2021.

[4] M. Alquran, M. Ali, et al., Promoted residual power series technique with Laplace transform to solve some time-fractional problems arising in physics, *Results in Physics*, 19, 103667, 2020.

[5] Tian, Yuqiang, et al., A state estimator–based non-linear predictive control for a fractional-order Francis hydraulic turbine governing system, *Journal of vibration and control*, 26, 11-12, 1068-1080, 2020.

[6] Pawar, D. D., et al., Numerical solution of fractional order mathematical model of drug resistant tuberculosis with two line treatment, *Journal of Mathematical and Computational Sciences*, 10, 2, 262-276, 2019.

[7] Pawar, D.D., et al., Analysis of malaria dynamics using its fractional order mathematical model, *Journal of applied mathematics and informatics* 39, 1-2, 197–214, 2021.

[8] Pawar, D. D., et al., Fractional-order mathematical model for analysing impact of quarantine on transmission of COVID-19 in India, *Mathematical Modelling and Computing*, 2021, 253-266, 2021.

[9] D. Baleanu, et.al.; Fractional dynamics and control, *Springer, New York Dordrecht, London* ,2012.

[10] H. Sheng, Y. Chen et. al.; Fractional processes and fractional order signal processing: techniques and applications, signals and communication technology *Springer New York, NY, USA*, 2012.

[11] Ferdi, Youcef, et al., Some applications of fractional order calculus to design digital filters for biomedical signal processing, *Journal of mechanics in medicine and biology*, 12, 02, 2012.

[12] Zhang, Rongpei et al. Stable finite difference method for fractional reaction–diffusion equations by compact implicit integration factor methods, Advances in Difference Equations 2021, 1, 1-12, 2021.

[13] A.A. Soliman, et al., Numerical solutions of non-linear evolution equations using variational iteration method *Journal of computational and applied mathematics*, 207, 1, 111-120, 2007.

[14] Lu, Tzon-Tzer, et al., Adomian decomposition method for first order PDEs with unprescribed data, *Alexandria Engineering Journal*, 60, 2, 2563-2572, 2021.

[15] Y.Q. Hasan, et. al.; Solving singular boundary value problems of higher-order ordinary differential equations by modified Adomian decomposition method ,*Commun. Nonlinear Sci. Numer. Simul.*, 14, 2592–2596, 2009.

[16] Singh, Brajesh Kumar, et al., Fractional variational iteration method for solving fractional partial differential equations with proportional delay, *International journal of differential equations*, 2017, 2017.

[17] Shijun Liao, Notes on the homotopy analysis method: Some definitions and theorems,Communication Non-linear Science Numerical Simulation, Vol. 14, 983–997, 2009.

[18] J. H. He, Homotopy perturbation technique, *Comp. Meth. Appl. Mech. Eng.*, 178, 257-262, 1999.

[19] J. H. He, A Review on some new recently developed non linear analytical techniques, *International journal of non-linear sciences and numerical simulation*, 1, 1, 51-70, 2000.

[20] Pawar D. D., et al., Analysis of fractional order soil moisture diffusion equation for heterogeneous and homogeneous system, *Advances in Mathematics: Scientific Journal*, 9, 4779-4785, 2020.

[21] D. D. Pawar, et al., Numerical solution of time fractional order partial differential equations, *Journal of Mathematics and Computational Sciences*, 10.4, 1066-1082, 2020.

[22] Lane, Homer J., On the theoretical temperature of the sun, under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment, *American Journal of Science* 2.148, 57-74, 1870.

[23] O. P. Singh, et al., An analytic algorithm of Lane-Emden type equations arising in astrophysics using modified Homotopy analysis method, Comput. Phys. Commun., 180, (2009) 1116–1124.

[24] Mukherjee Supriya, et al., Solution of Lane-Emden equation by differential transform method, *International Journal of Nonlinear Science* 12.4, 478-484, 2011.

[25] Parand, K., et al.; An approximation algorithm for the solution of the non-linear Lane–Emden type equations arising in astrophysics using Hermite functions collocation method, *Computer Physics Communications*, 181, 6, 1096-1108, 2010.