Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 17, Number 1, (2022) pp. 285-298

http://www.ripublication.com/adsa.htm

AVD-Total-Coloring of Pan, Sunlet and Other Related Graphs

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Abstract

This article includes the study of variation of classic total coloring problem for graphs known as adjacent-vertex-distinguishing total coloring *AV-DTC* problem which provides minimum number of colors.With reference to the total chromatic number of a graph G, $\chi''(G)$is the smallest number of colors required for total coloring. Adjacent-vertex-distinguishing total coloring *AV-DTC* is a total coloring with an additional property that for any adjacent vertices u and v, the set of colors used on the edges incident on u including the color of u is different from the set of colors used on the edges incident on v including the color of v. Adjacent-vertex-distinguishing(AVD)-total chromatic number of G, where $\chi''_a(G))$ is the minimum number of colors required for a valid AVD-total coloring of G.It is conjectured that $\chi'' \leq \triangle(G) + 2$, it is called *total coloring conjuncture*. Here we restrict ourselves to some set of graphs and generalize few results.

Keywords: Adjacent vertex distinguishing total coloring (*AV- DTC*) adjacent vertex distinguishing total chromatic number, Pan graph Friendship graph Tadpole graph Sunlet graph Barbell bull graph Barbell butterfly graph.

2020 Mathematics Subject Classification : 05C15

1. INTRODUCTION

Coloring of graphs is one of the most important well-known and consciously studied sub-fields of graph theory [1, 2]. The graph colouring problem is one of the most fundamentally investigated due to its theoretical and practical significance hence the

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coloring problem of graph is one of the fundamentally studied field by various experts [5, 6] and scholars all over the world as some network problems can be converted to the adjacent-vertex distinguish total coloring and adjacent-vertex distinguish edge coloring [3, 4, 14]. One can see the domination parameters of line graph of sunlet graph in [15]. Moreover, for forbidden induced subgraphs for line graphs one can refer to [16]. Some of the results on Chromatic number can be found in [17, 18, 20] and with Chromatic index can be seen in [19]. The fundamental coloring problem of a graph is to determine the number of various kinds of colorings [7, 8, 9, 11]. AVD-Total-Coloring of Some Simple Graphs can be seen in [13]. Here we consider the graphs that are simple,finite and undirected graphs.Let $G = (V, E)$ be a graph with the vertex set V and the edge set E respectively. Let G be a finite simple graph with no component K_2.Let C be a finite set of colors and let $\phi : E(G) \to C$ be a proper edge coloring of G.The color set of a $v \in V(G)$ with respect to phi, is the set of edges incident with v. The coloring ϕ is adjacent vertex distinguishing(or neighbor distinguishing)if it distinguishes any two adjacent vertices by their color set. [13]. The total coloring of a graph G is a color assignment given to the vertices and the edges such that

(i.) no two adjacent vertices receive the same color.

(ii.) no two adjacent edges receive the same color.

(iii.) if an edge e intersects a vertex v, v and e receive separate colors.

Here we give sufficient condition for a **Adjacent-vertex-distinguishing total coloring** *AV-DTC*. Also inferred some results on Adjacent-vertex-distinguishing total coloring *AV-DTC* of *Pan graph,Sunlet graph,Friendship graph*, *Tadpole graph,Barbell bull graph,barbell Butterfly graph*

Definition 1.1. *([13]) Vertex coloring: Let $G = (V, E)$ be a graph and let V be the set of all vertices ofG and let $\{1, 2, 3, ...k\}$ denotes the set of all colors which are assigned to each vertex of G.A proper vertex coloring of a graph is a mapping $C : V(G)$ $\{1, 2, 3, ...k\}$ such that $C(u) \neq C(v)$ for all arbitrary adjacent vertices u,v ∈V.*

Definition 1.2. *([13]) Chromatic number: If G has a proper vertex coloring then the chromatic number of G is the minimum number of colors needed to color G.The chromatic number of G is denoted by $\chi(G)$.*

Definition 1.3. *([13]) Total coloring: A total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent nor incident vertices and edges of are assigned the same colors.*

Definition 1.4. *([13]) Total chromatic number: The total chromatic number is the minimum number of colors needed to total color G [5] and it is denoted by $\chi''(G)$.*

Definition 1.5. *([13]) AVD-total coloring: If G is a simple graph and ϕ is total coloring of $G.\phi$ is an AVD-total coloring if $\forall u, v \in V$ and uv adjacent, we have $C(u) \neq C(v)$. Here $C(u)$: Set of colors that occur in a vertex u.*

Example 1.1.

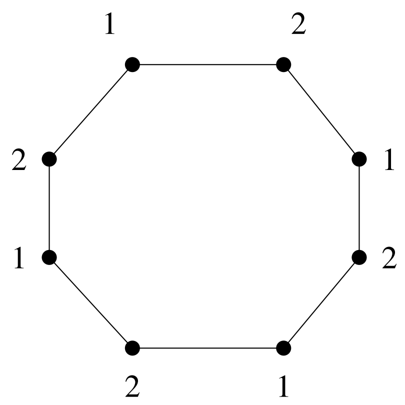

Figure 1: AVD-Total coloring of C_8

$$\rightarrow C(v_4) = \{2, 3, 4\}$$
$$\rightarrow C(v_5) = \{1, 3, 4\}$$
$$C(v_2) \neq C(v_3)$$

Definition 1.6. *Pan Graph:- $n-$pan graph is the graph obtained by joining a cyclic graph C_n to a singleton graph K_1 with a bridge. The n_1 pan graph is isomorphic with (n, 1)-tadpole graph. The 3-pan graph is often known as the paw graph and the 4-pan graph as the banner graph.These are the special cases of pan graph.*

Example 1.2.

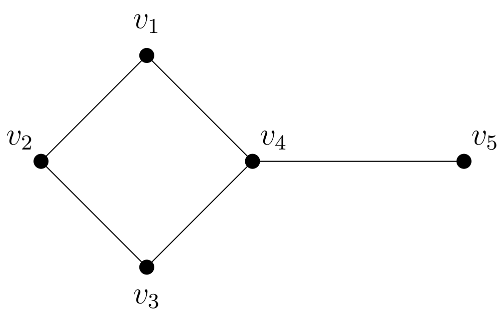

Figure 2: Pan graph

2. MAIN RESULTS

AVD Total Coloring of Pan graph

Theorem 2.1. *AVD Total Chromatic number of the Pan graph is* $\triangle\ (G) + 1 = \chi''(G)$

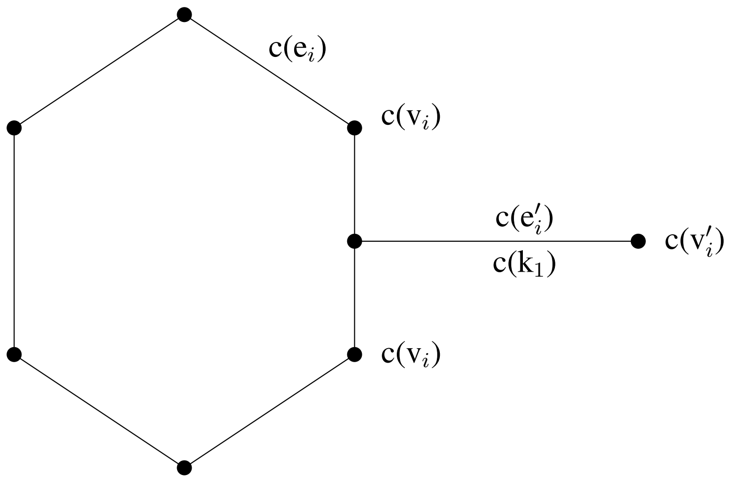

Figure 3: AVD Total Coloring of Pan graph

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From the definition of Pan graph 1.6 we observe that Pan graph is obtained by joining a cycle graph C_n to a singleton graph k_1 with a bridge. Let $\{v_1, v_2, v_3, ...v_n\}$ be the vertices of the cycle graph C_n and $\{e_1, e_2, e_3, ...e_n\}$ where $\{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the edges of the cycle graph C_n.
Let v be the vertices of the path of singleton bridge k_1 and let be the edges of singleton bridgek_1. In $L_{m,n}$ the vertex set and the edge set is given by

$$V(L_{m,n}) = \{v_i : 1 \leq i \leq n\} \cup V_i \text{ or } V_1',$$

we can define the AVD total coloring **f** such that

$$f : L \to C$$

where $L = V(L_{m,n} \cup E(L_{m,n})\ and\ C = \{1, 2, 3, 4\}$ be colors. Now assigning the color 1,2,3 consecutively to the edges of the cycle graph C_n such that $L_{m,n}$(no two adjacent vertices get the same color). Thus by combining the above steps we get no two adjacent or incident vertices and edges of L m,n are assigned the same color.This is total AVD coloring. Here C(u): set of colors that occurs in a vertex u Two vertices $u, v \in V(G)$ distinguishable when $c(u) \neq c(v)$

Here let us color the bridge C_k first as it is singleton followed by the vertex and edges of the cycle graph C_n.

Let us first color pendent vertex (bridge)with a color c_1 such that the color c_1 of the vertex is distinguish from any of the colors on other vertices of cycle C_n.

Next we will color the edge of the bridge with(another color)C_2 in such a way that the alternative edges e_i of the cycle C_n will take the same color.

Let us consider another vertex of the singleton bridge which is adjacent to cycle C_n and alternative vertices of the cycle C_n takes the color C_3.

Then remaining vertices of the cycle will take color 4. In general,with this pattern if we color the Pan graph we get the following cases.
Case 1: When the cycle C_n is of length Even C_n when n is Even. The color of the graph with singleton bridge k_1 takes same color for alternative edges of cycle where as the cycle C_n which is vertex adjacent to the singleton graph with bridge k_1 takes the same color
Case 2: C_n When n is odd But this not in the case of odd. Thus AVD total coloring of pan graph. Hence AVD total coloring can be generalized as

$$\big(C_n, C_(K_1)\big) = \begin{cases} c(k_1) = c(v_i) & \text{when } n \text{ is even} \\ c(k_1) \neq c(v_i) & \text{when } n \text{ is odd} \end{cases}$$

Therefore AVD total coloring of Pan graph is $= \Delta(G) + 1,\ n \geq 3$.

Definition 2.1. *([15]) n- sunlet graph is a graph on 2n vertices which is obtained by attaching n - pendant edges to the cycle C_n and it is denoted by S_n.*

Theorem 2.2. *AVD total coloring of Sunlet graph C_n where $0 \leq k \leq i$ and $i \in Z^+$ and n is number of vertices of a cycle.Then*

$$AVD\ Total\ coloring = \begin{cases} c_n - 2(k) & \textit{if } n \textit{ is even} \\ c_{n+1} - 2(k) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. The n-sunlet graph 2.1 on 2n-vertices is obtained by attaching the n-pendent edges to the cycle C_n and it is denoted by S_n i.e the coronas The Sunlet graph is colorable,
For $0 \leq k \leq n$ and $n \geq 3\forall$, $n \in \mathbb{Z}^+$
Let us consider few cases below:
Special case: For k=0, where n represents cycle

When n is **odd** i.e $n = 3$

$$c_n - 2(k)$$
$$c_4 - 2(0)$$
$$4 - 0$$
$$4 \text{ colorable}$$

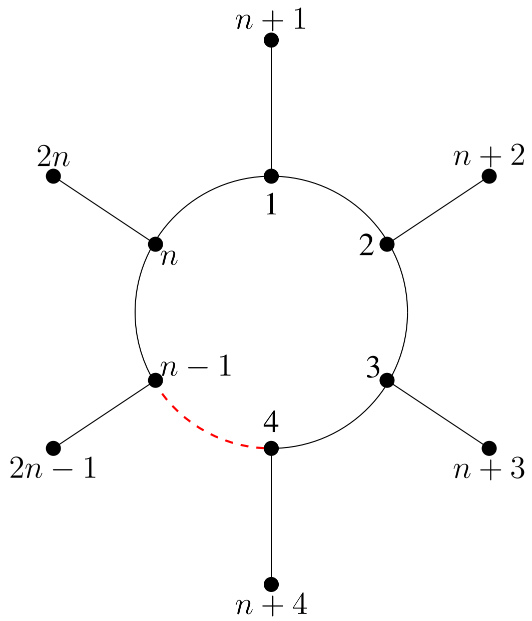


Figure 4: $n-$sunlet graph

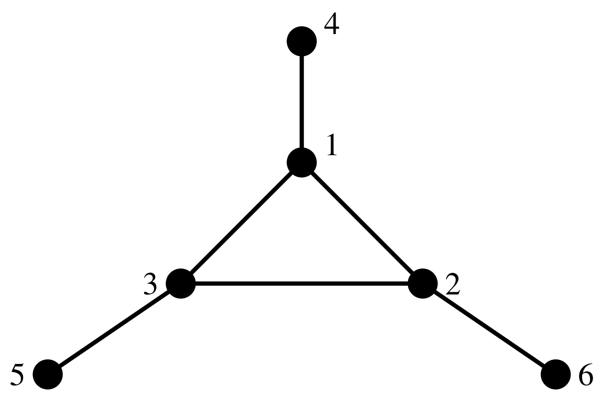


Figure 5: $3-$sunlet graph

As the distance between pair of vertices is 1 and AVD total coloring of a cycle C_3 will take colors and pendent vertex takes the 4^{th} color lying on cycle. Hence the Sunlet graph on 3 vertices is C_3 is 4 colorable.

Case1: For k=0,

(i.) When n is **even** i.e $n = 4$

$$c_(n) - 2(k)$$
$$c_(4) - 2(0)$$
$$c_4 - 0$$
$$4 \text{ colorable}$$

Hence the Sunlet graph on 4 vertices C_4 is 4 colorable.

(ii.) When n+1 is **odd** i.e $n + 1 = 5$

$$c_(n + 1) - 2(k)$$
$$c_(4 + 1) - 2(0)$$
$$c_5 - 0$$
$$5 \text{ colorable}$$

Hence the Sunlet graph on 5 vertices C_4 is 5 colorable.

Case2: For k=1,

(i.) When n is **even** i.e $n = 6$

$$c_(n) - 2(k)$$
$$c_(6) - 2(1)$$
$$c_6 - 2$$
$$4 \text{ colorable}$$

Hence the Sunlet graph on 6 vertices C_6 is 4 colorable.

(ii.) When n+1 is **odd** i.e $n + 1 = 7$

$$c_(n + 1) - 2(k)$$
$$c_(6 + 1) - 2(1)$$
$$c_7 - 2$$
$$5 \text{ colorable}$$

Hence the Sunlet graph on 7 vertices C_7 is 5 colorable.

Case3: For k=2,

(i.) When n is **even** i.e $n = 8$

$$c_(n) - 2(k)$$
$$c_(8) - 2(2)$$
$$c_8 - 4$$
$$4 \text{ colorable}$$

Hence the Sunlet graph on 8 vertices C_8 is 4 colorable.

(ii.) When n+1 is **odd** i.e $n + 1 = 9$

$$c_(n + 1) - 2(k)$$
$$c_(6 + 1) - 2(1)$$
$$c_7 - 2$$
$$5 \text{ colorable}$$

Hence the Sunlet graph on 9 vertices C_9 is 5 colorable.

Case4: For k=3,

(i.) When n is **even** i.e $n = 10$

$$c_(n) - 2(k)$$
$$c_(10) - 2(3)$$
$$c_1 0 - 6$$
$$4 \text{ colorable}$$

Hence the Sunlet graph on 10 vertices C_{10} is 4 colorable.

(ii.) When n+1 is **odd** i.e $n + 1 = 11$

$$c_(n + 1) - 2(k)$$
$$c_(10 + 1) - 2(3)$$
$$c_1 1 - 6$$
$$5 \text{ colorable}$$

Hence the Sunlet graph on 11 vertices C_{11} is 5 colorable.

Case5: For k=4,

(i.) When n is **even** i.e $n = 12$

$$c_(n) - 2(k)$$
$$c_(12) - 2(4)$$
$$c_12 - 8$$
$$4 \text{ colorable}$$

Hence the Sunlet graph on 12 vertices C_{12} is 4 colorable.

(ii.) When n+1 is **odd** i.e $n + 1 = 13$

$$c_(n + 1) - 2(k)$$
$$c_(13) - 2(4)$$
$$c_13 - 8$$
$$5 \text{ colorable}$$

Hence the Sunlet graph on 13 vertices C_{13} is 5 colorable.
Hence AVD Total coloring for the rest of the Sunlet graph can be found using the given table and following formulae.

Sl. No	**Cycle**	**Total no. of vertices**	**k**	C_n **no.of vertices on cycle**	**Pendent vertex**	**Colors used**
special case	odd	6	0	3	3	4
1	even	8	0	4	4	4
2	odd	10	0	5	5	5
3	even	12	1	6	6	4
4	odd	14	1	7	7	5
5	even	16	2	8	8	4
6	odd	18	2	9	9	5
7	even	20	3	10	10	4
8	odd	22	3	11	11	5
9	even	24	4	12	12	4
10	odd	26	4	13	13	5
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Hence AVD Total coloring holds good for given formula. □

Definition 2.2. *By linking n copies of the cycle graph C_3 with a common vertex, a Friendship graph F_n may be created.*

Example 2.1.

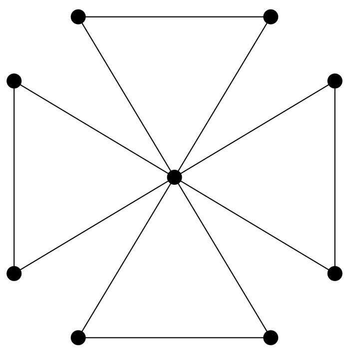


Figure 6: Friendship graph F_n

Proposition 2.1. *AVD total chromatic number of Friendship graph is always*

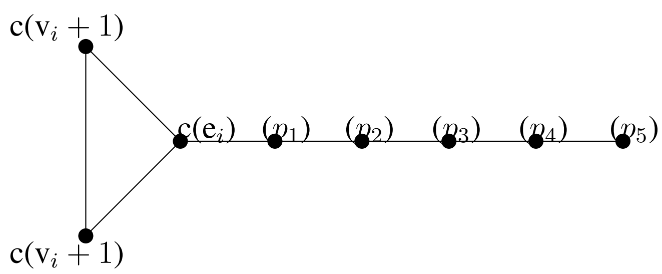
$$F_n = n \times 2 + 1, n \geq 3$$

.

A Friendship graph[2.2] G adheres the edge(which is not incident on a common vertex) $c(e_i)$ will take the same color as that of $c(v_i)$. Since in Friendship graph (F_n) where n determines the number of cyclic graph C_3 attached to the common node or common vertex. AVD total coloring of Friendship graph can be done in such a way that we fix a color of the common node or vertex and then color such that no two colors repeats. Further the edge which is not incident on a common node or vertex will take the same color as that of the common node or vertices.

Theorem 2.3. *AVD Total coloring of Tadpole graph is $\triangle G + 1$.*

Proof. .



Consider Tadpole graph consisting of a cycle graph on P (with atleast 3) the path of the graph on n vertices is connected with a bridge.

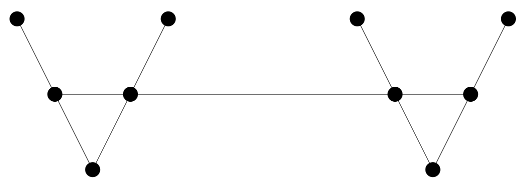
Let v_i be the vertex of the bridge connecting the cycle of a tadpole graph with path $p_n\{p_1, p_2, p_3, \ldots, p_n\}$ the vertices of the path and the cycle $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+n}\}$.

Let us now consider the vertex v_i connecting the cycle and bridge be colored with $c = \{1, 2, 3, 4\}$ any of one color in the beginning followed by alternatively coloring vertices and edges of different color. According to AVD coloring such that no two incident vertices nor edges are colored with the same color. Since the degree of V_i is always 3 it is obvious that it is colored with 3 different colors while bridge is been connected to the vertex v_i of cycle such that other than 3 colors,additional color has to be used so totally 4 colors are used in Tadpole graph irrespective of length of the path m_n where path is colored $c = \{1, 2, 3, 4\}$ alternatively. Where the total chromatic number $\chi"(G)$ of a graph G is minimum cardinality k such that G has

$$\Delta(G) + 1 \leq \chi_a^{''}(G) \leq \Delta(G) + 2$$

where $\triangle(G)$ is maximum degree of the Thus, AVD Total coloring of Tadpole graph is ΔG + 1. □

Definition 2.3. *Barbell bull graph: $n-$ barbell bull graph is the simple graph which can be obtained by connecting two copies of a bull graph (is a planar) can also be undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendent edges by a bridge.*

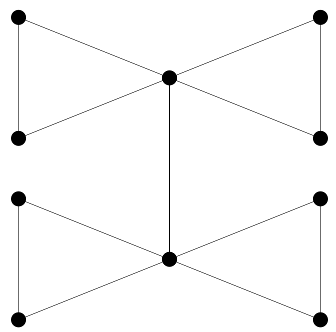


$\boldsymbol{B_L[(C_3, 2), K}$

Barbell bull graph

where B_L -Barbell bull graph
C_3-Cycle on 3 vertices K-Number of bridges$(1 \leq k \leq n)$

Definition 2.4. *Barbell butterfly graph: Barbell butterfly graph is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph C_3 with a common vertex and is therefore isomorphic to the friendship graph F_2.*

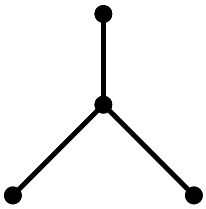
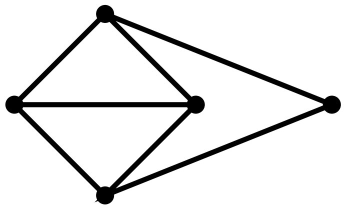
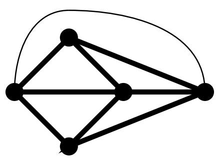
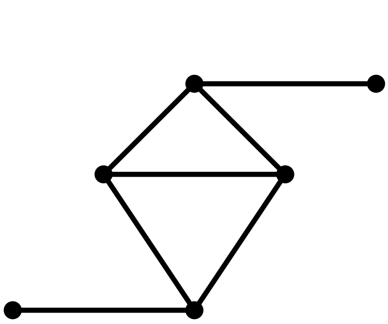
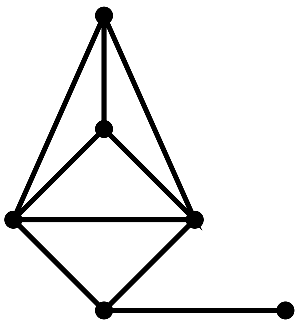
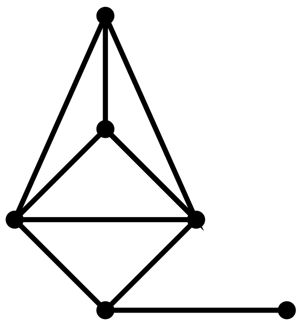
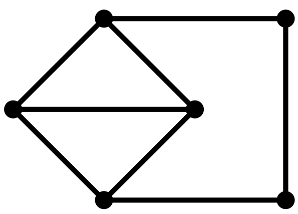
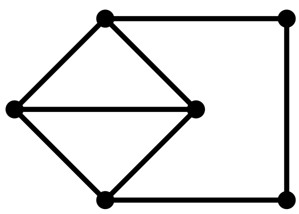
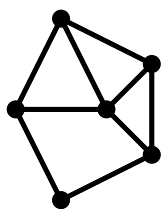


$\boldsymbol{B_F[(C_3, C3), K}$

Barbell Butterfly graph

where B_F - Barbell Butterfly graph, C_3-Cycle on 3 vertices, K-Number of bridges$(1 \leq k \leq n)$.

Remark 2.1. AVD-Total-coloring of forbidden subgraph for line graphs:

Forbidden graph G_n		
G_1 	G_2 	G_3 
G_4 	G_5 	G_6 
G_7 	G_8 	G_9 

Forbidden graph	Total number of vertices	$\Delta(G)$	$\chi(G)$	Total no.of complete graphs T_n
G_1	4	3	5	0
G_2	5	3	5	2
G_3	5	4	6	4
G_4	6	3	5	2
G_5	6	4	6	5
G_6	6	5	6	8
G_7	6	3	4	2
G_8	6	4	6	4
G_9	6	5	6	5

CONCLUSION:

In this paper we determine Adjacent-vertex-distinguishing(AVD)-total coloring of Pan graph, Sunlet graph,Tadpole and we define Barbell bull graph and barbell butterfly graph further this work can be extended for several sets of graphs in future.

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