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On the Study of Uniqueness of Meromorphic Functions that Shares Small Functions Partially with the Difference Operator

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Abstract

There are some uniqueness problems with meromorphic functions with difference operators that we looked into in this paper. We looked at them in the light of partial sharing. Specifically, we have obtained two uniqueness results by considering sharing and partial sharing of small functions. In the first theorem $\Delta\mathfrak{g}(\mathfrak{z})$ and $\mathfrak{g}(\mathfrak{z})$ shares $\mathfrak{a}_1(\mathfrak{z}), \mathfrak{a}_2(\mathfrak{z}), \infty$ CM, whereas in the second theorem $\mathfrak{g}(\mathfrak{z})$ and $\Delta\mathfrak{g}(\mathfrak{z})$ partially share $\mathfrak{a}_1(\mathfrak{z}), \mathfrak{a}_2(\mathfrak{z})$ CM.

1. INTRODUCTION

We presume that the reader is familiar with the notations of the Nevanlinna theory and for the basic ([1, 2, 3]). We mean $\mathcal{S}(\mathfrak{r}, \mathfrak{g}) = o(\mathcal{T}(\mathfrak{r}, \mathfrak{g}))\ \forall \mathfrak{r} \in (1, \infty)$. We denote the set meromorphic functions a_i for $i = 1, 2$ by $\mathcal{S}(\mathfrak{f})$.

The set of all $\mathfrak{a}$-points (counting multiplicities or CM) of $\mathfrak{f}$ is denoted by $E(\mathfrak{a}, \mathfrak{f})$. and all different $\mathfrak{a}$-points of $\mathfrak{f}$ by $\overline{E}(\mathfrak{a}, \mathfrak{f})$.

We use the following fundamental definitions to prove our results

Definition 1.1. *([4] It is claimed that a meromorphic function* $\mathfrak{f}$ *shares* $\mathfrak{a} \in \mathcal{S}(\mathfrak{f})$ *partially CM with a meromorphic function* $\mathfrak{g}$ *if* $E(\mathfrak{a}, \mathfrak{f}) \subseteq E(\mathfrak{a}, \mathfrak{g})$.

Definition 1.2. *([5]) It is claimed that a meromorphic function* $\mathfrak{f}$ *shares* $\mathfrak{a} \in \mathcal{S}(\mathfrak{f})$ *partially IM with a meromorphic function* $\mathfrak{g}$ *if* $\overline{E}(\mathfrak{a}, \mathfrak{g}) \subseteq \overline{E}(\mathfrak{a}, \mathfrak{f})$.

Definition 1.3. *([1]) It is claimed that if* $\mathfrak{f}$ *and* $\mathfrak{g}$ *share the value* $\mathfrak{a}$ *CM if* $E(\mathfrak{a}, \mathfrak{f}) = E(\mathfrak{a}, \mathfrak{g})$. $\mathfrak{f}$ *and* $\mathfrak{g}$ *share the value* $\mathfrak{a}$ *IM if* $\overline{E}(\mathfrak{a}, \mathfrak{f}) = \overline{E}(\mathfrak{a}, \mathfrak{g})$.

Halburd-Korhonen [17] and Chiang-Feng [18] started the counterpart of renowned Nevanlinna's theory for difference operator. Several noteworthy results [5, 12, 11] followed, of which we would like to highlight a few.

Heittokangas et al. [8] looked into the relationship between a meromorphic function's shift operator and meromorphic function when they share a, ∞ CM in 2009. The outcome is as follows.

Theorem 1.1. *([8]) Let* $\mathfrak{f}(\mathfrak{z})$ *be a meromorphic function and* $c \in \mathbb{C}$. *If* $\mathfrak{f}(\mathfrak{z}+\mathfrak{c})$ *and* $\mathfrak{f}(\mathfrak{z})$ *share* $\mathfrak{a}, \infty$ *CM, where* $\mathfrak{a} \in \mathbb{C}$, *then for some constant* τ,

$$\frac{\mathfrak{f}(\mathfrak{z}+\mathfrak{c}) - \mathfrak{a}}{\mathfrak{f}(\mathfrak{z}) - \mathfrak{a}} = \tau.$$

In [7], by considering three small functions CM, two small functions CM and one small function IM, Heittokangas et al., looked into the relation between $\mathfrak{f}(\mathfrak{z})$ and $\mathfrak{f}(\mathfrak{z}+\mathfrak{c})$. And by considering the entire function, Huang-Zhang in [11] got a result as in Theorem 1.1.

Theorem 1.2. *([11]) Let* $\mathfrak{f}(\mathfrak{z})$ *be a transcendental entire function of order* $\rho(\mathfrak{f}) < 2$. *If* $\Delta_c^k \mathfrak{f}(\mathfrak{z})$ *and* $\mathfrak{f}(\mathfrak{z})$ *share 0 CM, where* $k \in \mathbb{N}$ *and* $c \in \mathbb{C}\backslash\{0\}$ *are such that* $\Delta_c^k \mathfrak{f}(\mathfrak{z}) \neq 0$, *then*

$$\Delta_c^k \mathfrak{f}(\mathfrak{z}) \equiv \tau \mathfrak{f}(\mathfrak{z}),$$

for some constant T.

In order to obtain a similar result for a meromorphic function corresponding to Theorem 1.2, Chen-Yi [12] researched the uniqueness of $\Delta_c \mathfrak{f}(\mathfrak{z})$ and $\mathfrak{f}(\mathfrak{z})$ as follows.

Theorem 1.3. *([12]) Let* $\mathfrak{f}(\mathfrak{z})$ *be a transcendental meromorphic function such that the order* $\rho(\mathfrak{f})$ *is not an integer or infinite and* $c \in \mathbb{C}$ *be a constant such that* $f(z+c) \neq \mathfrak{f}(\mathfrak{z})$. *If* $\Delta_c \mathfrak{f}(\mathfrak{z})$ *and* $\mathfrak{f}(\mathfrak{z})$ *share three distinct values* a, b, ∞ *CM, then* $f(z+c) \equiv 2\mathfrak{f}(\mathfrak{z})$.

When Zhang-Liao [9] worked on the entire function in 2014, he removed the restriction that "$\rho(\mathfrak{f})$ is not an integer", Zhang-Liao did this in the following way:

Theorem 1.4. *[16] Let* $\mathfrak{f}(\mathfrak{z})$ *be a transcendental entire function of finite order* $\mathfrak{c}$ *be a non-zero constant;* a, b *be two distinct finite constants. If* $\Delta_c \mathfrak{f}(\mathfrak{z})(\neq 0)$ *and* $\mathfrak{f}(\mathfrak{z})$ *shore a, b CM, then* $\Delta_c \mathfrak{f}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z})$.

Theorem 1.5. *[15] Let* $\mathfrak{f}(\mathfrak{z})$ *be a non-constant meromorphic function of finite order such that* $\mathcal{N}(\mathfrak{r}, \mathfrak{f}) = \mathcal{S}(\mathfrak{r}, \mathfrak{f})$, *let* $c \in \mathbb{C}\backslash\{0\}$ *be a constant such that* $\Delta_c \mathfrak{f}(\mathfrak{z}) \neq 0$ *and let* a, b *be two non-zero distinct finite complex constants. If* $\Delta_c \mathfrak{f}(\mathfrak{z})$ *and* $\mathfrak{f}(\mathfrak{z})$ *share* a, b *CM, then* $f(z+c) = 2\mathfrak{f}(\mathfrak{z})$.

In the year 2017, L*ü*-L*ü* [14] removed the order restriction from the Theorem 1.5. For meromorphic functions, without any extra conditions, he proved uniqueness.

Theorem 1.6. *[14] Let $\mathfrak{f}(\mathfrak{z})$ be a transcendental meromorphic function of finite order and let $c \in \mathbb{C}$ be a constant such that $f(z+c) \neq \mathfrak{f}(\mathfrak{z})$. If $\Delta_c\mathfrak{f}(\mathfrak{z})$ and $\mathfrak{f}(\mathfrak{z})$ share three distinct values a, b, ∞ CM, then $f(z+c) \equiv 2\mathfrak{f}(\mathfrak{z})$.*

In the year 2019, Zhen [13] almost followed the same steps as the proof of Theorem 1.6, but instead of looking at value sharing, he looked at polynomial sharing. This made Theorem 1.6 better.

Theorem 1.7. *[13] Let $\mathfrak{f}(\mathfrak{z})$ be a transcendental meromorphic function of finite order and let $c(\neq 0)$ be a finite number. If $\Delta_c\mathfrak{f}(\mathfrak{z})$ and $\mathfrak{f}(\mathfrak{z})$ share three distinct polynomials P_1, P_2, ∞ CM, then $\Delta_c\mathfrak{f}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z})$.*

2. LEMMAS

Lemma 2.1. *([19]) Let $\mathfrak{f}$ be non-constant meromorphic function in $\mathbb{C}$. Let a_1, a_2, a_3 be pairwise distinct small meromorphic functions in $\mathbb{C}$ such that $a_1, a_2 \in S(\mathfrak{f})$ and*

$$\mathcal{T}(\mathfrak{r}, \mathfrak{a}_v) \leq v\mathcal{T}(\mathfrak{r}, \mathfrak{f}) + \mathcal{S}(\mathfrak{r}, \mathfrak{f})$$

for some $v \in [0, 1/3)$. Then

$$(1 - 3v - \epsilon)\mathcal{T}(\mathfrak{r}, \mathfrak{f}) \leq \sum_{\mathfrak{i}=g}^{\mathfrak{q}} \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{f} - \mathfrak{a}_i}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{f}).$$

Lemma 2.2. *([20]) Let $\mathfrak{f}$ be a meromorphic function of finite order, and let $\mathfrak{c} \in \mathbb{C}, \mathfrak{n} \in \mathbb{N}$. Then for any small periodic function $\mathfrak{a}(\mathfrak{z}) \in S(\mathfrak{f})$ with period $\mathfrak{c}$,*

$$m\left(r, \frac{\Delta_{\mathfrak{c}}^{\mathfrak{n}}\mathfrak{f}}{\mathfrak{f}(\mathfrak{z}) - \mathfrak{a}(z)}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}).$$

Lemma 2.3. *([15]) Let $\mathfrak{f}(\mathfrak{z})$ be a meromorphic function, and let η be a fixed non-zero complex number, then for each $\epsilon > 0$, we have $\mathcal{T}(\mathfrak{r}, \mathfrak{f}(\mathfrak{z} + \eta)) = \mathcal{T}(\mathfrak{r}, \mathfrak{f}) + \mathcal{S}(\mathfrak{r}, \mathfrak{f})$.*

Lemma 2.4. *([22]) Let $\mathfrak{f}$ be a meromorphic function of hyper-order $\gamma(\mathfrak{f}) < 1$ and let $\mathfrak{c} \in \mathbb{C}\backslash\{0\}$. Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in S(\mathfrak{f})$ be three distinct periodic functions with period $\mathfrak{c}$. Assume that $\mathfrak{f}(\mathfrak{z})$ and $\mathfrak{f}(\mathfrak{z} + \mathfrak{c})$ share partially $\mathfrak{a}_1, \mathfrak{a}_2$ CM and share partially $\mathfrak{a}_3$ IM, i.e.,*

$$E(\mathfrak{a}_1, \mathfrak{f}(\mathfrak{z})) \subseteq E(\mathfrak{a}_1, \mathfrak{f}(\mathfrak{z} + \mathfrak{c})), \quad E(\mathfrak{a}_2, \mathfrak{f}(\mathfrak{z})) \subseteq E(\mathfrak{a}_2, \mathfrak{f}(\mathfrak{z} + \mathfrak{c})),$$

and

$$\overline{E}(\mathfrak{a}_3, \mathfrak{f}(\mathfrak{z})) \subseteq \overline{E}(\mathfrak{a}_3, \mathfrak{f}(\mathfrak{z} + \mathfrak{c})).$$

If $\mathfrak{p}(\mathfrak{a},\mathfrak{f}) > 0$ *for some* $\mathfrak{a} \in S(\mathfrak{f})\backslash\{\mathfrak{a}_3\}$, *then* $\mathfrak{f}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z}+\mathfrak{c})$ *for all* $\mathfrak{z} \in \mathbb{C}$.

Lemma 2.5. *([22])() Let* $\mathcal{T} : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function and let* $\mathfrak{s} \in (0, +\infty)$ *such that hyper-order of* $\mathcal{T}$ *is strictly less than one, i.e.,*

$$\gamma = \limsup_{r\to\infty} \frac{log^+log^+\mathcal{T}(\mathfrak{r})}{logr} < 1,$$

then

$$\mathcal{T}(\mathfrak{r}+\mathfrak{s}) = \mathcal{T}(\mathfrak{r}) + o\left(\frac{\mathcal{T}(\mathfrak{r})}{\mathfrak{r}^{1-\gamma-\epsilon}}\right),$$

where $\epsilon > 0$ *and* $\mathfrak{r} \to \infty$ *outside a subset of finite logarithmic measure.*

Lemma 2.6. *([2]) Suppose that* $f(z)$ *is a non-constant meromorphic function and* $P(f) = a_0f^p + a_1f^{p-1} + \cdots + a_p$ $(a_0 \neq 0)$ *is a polynomial in* f *with degree* p *and coefficients* $a_j(j = 0, 1, \ldots, p)$ *are constants, suppose furthermore that* $b_j(j = 1, 2, \ldots, q)(q > p)$ *are distinct finite values. Then*

$$m\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)\cdots\left(f-b_q\right)}\right) = S(r, f).$$

3. MAIN RESULTS

Theorem 3.1. *Considering* $\mathfrak{g}(\mathfrak{z})$ *as a non-constant meromorphic function. Suppose that* $\mathfrak{c} \in \mathbb{C}\backslash\{0\}$, $\mathfrak{b}_0 \neq 0$ *and* $\mathfrak{a}_1(\mathfrak{z}), \mathfrak{a}_2(\mathfrak{z}) \in \mathcal{S}(\mathfrak{f})$ *are two small functions. If* $\Delta\mathfrak{g}(\mathfrak{z}) \not\equiv 0$ *and* $\Delta\mathfrak{g}(\mathfrak{z})$, $\mathfrak{g}$ *share* $\mathfrak{a}_1, \mathfrak{a}_2, \infty$ *CM, then* $\Delta\mathfrak{g}(\mathfrak{z}) \equiv \mathfrak{g}(\mathfrak{z})$.

Proof. Due to the fact that $\Delta\mathfrak{g}(\mathfrak{z})$ and $\mathfrak{g}$ share ∞ CM, we have

$$\begin{aligned}\mathcal{T}(\mathfrak{r}, \Delta\mathfrak{g}(\mathfrak{z})) &\leq m\left(\mathfrak{r}, \frac{\Delta\mathfrak{g}(\mathfrak{z})}{\mathfrak{f}}\right) + m(\mathfrak{r},\mathfrak{g}) + \mathcal{N}(\mathfrak{r}, L_\mathfrak{c}\mathfrak{g}) + O(1)\\ &= m(\mathfrak{r},\mathfrak{g}) + \mathcal{N}(\mathfrak{r},\mathfrak{g}) + \mathcal{S}(\mathfrak{r},\mathfrak{g})\\ &= \mathcal{T}(\mathfrak{r},\mathfrak{g}) + \mathcal{S}(\mathfrak{r},\mathfrak{g}).\end{aligned}$$

Thus

$$S(\mathfrak{r}, \Delta\mathfrak{g}(\mathfrak{z})) = \mathcal{S}(\mathfrak{r},\mathfrak{g}). \tag{3.1}$$

In the same way that $\Delta\mathfrak{g}(\mathfrak{z})$ and $\mathfrak{g}$ share $\mathfrak{a}_1, \mathfrak{a}_2, \infty$ CM. As such two polynomials $\mathfrak{p}_1(\mathfrak{z}), \mathfrak{p}_2(\mathfrak{z})$ exists, such that

$$\frac{\Delta\mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1}{\mathfrak{g} - \mathfrak{a}_1} = \mathfrak{e}^{\mathfrak{p}_1(\mathfrak{z})} \tag{3.2}$$

and

$$\frac{\Delta\mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_2}{\mathfrak{g} - \mathfrak{a}_2} = \mathfrak{e}^{\mathfrak{p}_2(\mathfrak{z})} \tag{3.3}$$

Case 1: Presuming $e^{p_1(z)} \equiv 1$ or $e^{p_2(z)} \equiv 1$, then $\Delta g(z) \equiv g(z)$.
Case 2: Presuming $e^{p_1(z)} \neq 1$ and $e^{p_2(z)} \neq 1$, however if suppose $e^{p_1(z)} \equiv e^{p_2(z)}$, then

$$\frac{\Delta g(z) - a_1}{g(z) - a_1} = \frac{\Delta g(z) - a_2}{g(z) - a_2},$$

we obtain via easy computation that $\Delta g(z) \equiv g(z)$.
Case 3: In this case we presume that $e^{p_1(z)} \not\equiv 1$ and $e^{p_2(z)} \not\equiv 1$ with $e^{p_1(z)} \not\equiv e^{p_2(z)}$. By (3.2) and (3.3)

$$g(z)e^{p_1(z)} = \Delta g(z) - a_1 + a_1 e^{p_1(z)}. \tag{3.4}$$

Similarly

$$g(z)e^{p_2(z)} = \Delta g(z) - a_2 + a_2 e^{p_2(z)}. \tag{3.5}$$

Now by (3.4) and (3.5), we get

$$g(z) = \frac{a_2 - a_1 + a_1 e^{p_1(z)} - a_2 e^{p_2(z)}}{e^{p_1(z)} - e^{p_2(z)}} \tag{3.6}$$

Sub-case 3.1: Presuming that $p_1(z)$ and $p_2(z)$ both polynomials are constants. Now from (3.6) we see that $g(z)$ is also a constant, so is not true.
Sub-case 3.2: Now, for this case, without loss of generality we'll assume that $p_2(z)$ is constant between $p_1(z)$ and $p_2(z)$. Now, using (3.6) we get

$$\mathcal{T}(r, g) = \mathcal{T}\left(r, e^{p_1(z)}\right) + S\left(r, e^{p_1(z)}\right) \tag{3.7}$$

and

$$T\left(r, e^{p_2}\right) = S\left(r, e^{p_1(z)}\right). \tag{3.8}$$

Now from (3.3) let $\mathcal{P}(z, g) = (\Delta g(z) - a_2) - e^{p_2}(g - a_2)$. Because e^{p_2} is constant, $\mathcal{P}(z, g)$ is a polynomial in $g(z)$ and its shifts whose coefficients are small functions of $g(z)$. From (3.3), we have $\mathcal{P}(z, g) = 0$. So $\mathcal{P}(z, a_1) = \Delta a_1(z) - a_2 - e^{p_2}(a_1 - a_2)$. We assert that $\mathcal{P}(z, a_1) \neq 0$, on the other hand, presuming $\mathcal{P}(z, a_1) = 0$ then $e^{p_2} = \frac{\Delta a_1(z) - a_2}{a_1 - a_2}$. From (3.6) we obtain

$$g(z) - a_1 = \frac{\Delta a_1(z) - a_1}{e^{p_1(z)} - e^{p_2}}. \tag{3.9}$$

Combining (3.9) with (3.2) we get

$$\Delta g(z) = \frac{\Delta a_1(z) e^{p_1} - a_1 e^{p_2}}{e^{p_1(z)} - e^{p_1}}. \tag{3.10}$$

From (3.9), we get

$$\begin{aligned}\Delta \mathfrak{g}(\mathfrak{z}) &= \Delta\left(\mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1\right) \\ &= \frac{\Delta \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c}) - \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c})}{e^{\mathfrak{p}_1(\mathfrak{z}+\mathfrak{c})} - e^{\mathfrak{p}_2}} - \frac{\Delta \mathfrak{a}_1(\mathfrak{z}) - \mathfrak{a}_1}{e^{\mathfrak{p}_1(\mathfrak{z})} - e^{\mathfrak{p}_1}}\end{aligned} \tag{3.11}$$

From (3.10) and (3.11)

$$\frac{\Delta \mathfrak{a}_1(\mathfrak{z})e^{\mathfrak{p}_1} - \mathfrak{a}_1 e^{\mathfrak{p}_2}}{e^{\mathfrak{p}_1(\mathfrak{z})} - e^{\mathfrak{p}_1}} = \frac{\Delta \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c}) - \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c})}{e^{\mathfrak{p}_1(\mathfrak{z}+\mathfrak{c})} - e^{\mathfrak{p}_2}} - \frac{\Delta \mathfrak{a}_1(\mathfrak{z}) - \mathfrak{a}_1}{e^{\mathfrak{p}_1(\mathfrak{z})} - e^{\mathfrak{p}_1}}. \tag{3.12}$$

$$\begin{gathered}\mathcal{T}\left(\mathfrak{r}, \frac{\Delta \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c}) - \mathfrak{a}_1(\mathfrak{z}+\mathfrak{c})}{e^{\mathfrak{p}_1(\mathfrak{z}+\mathfrak{c})} - e^{\mathfrak{p}_2}} - \frac{\Delta \mathfrak{a}_1(\mathfrak{z}) - \mathfrak{a}_1}{e^{\mathfrak{p}_1(\mathfrak{z})} - e^{\mathfrak{p}_1}}\right) = \mathcal{T}\left(\mathfrak{r}, \frac{\Delta \mathfrak{a}_1(\mathfrak{z})e^{\mathfrak{p}_1} - \mathfrak{a}_1 e^{\mathfrak{p}_2}}{e^{\mathfrak{p}_1(\mathfrak{z})} - e^{\mathfrak{p}_1}}\right). \\ \mathcal{T}(\mathfrak{r}, \mathfrak{g}) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\end{gathered} \tag{3.13}$$

we arrive at a contradiction.

Sub-case 3.3: If both $\mathfrak{p}_1(z)$ and $\mathfrak{p}_2(\mathfrak{z})$ are non-constant, then $\mathfrak{p}_1'(\mathfrak{z}) \neq 0$ and $\mathfrak{p}_2'(\mathfrak{z}) \neq 0$. By (3.2) we write

$$(\Delta \mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1) = e^{\mathfrak{p}_1(\mathfrak{z})}\left(\mathfrak{g} - \mathfrak{a}_1\right). \tag{3.14}$$

Differentiating (3.12) we get

$$\begin{aligned}(\Delta \mathfrak{g}(\mathfrak{z}))' &= e^{\mathfrak{p}_1(\mathfrak{z})}\mathfrak{p}_1'(\mathfrak{z})\left(\mathfrak{g} - \mathfrak{a}_1\right) + e^{\mathfrak{p}_1(\mathfrak{z})}\mathfrak{g}' \\ \frac{(\Delta \mathfrak{g}(\mathfrak{z}))'}{\Delta \mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1} &= \frac{e^{\mathfrak{p}_1(\mathfrak{z})}\mathfrak{p}_1'(\mathfrak{z})\left(\mathfrak{g} - \mathfrak{a}_1\right) + e^{\mathfrak{p}_1(\mathfrak{z})}\mathfrak{g}'}{\Delta \mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1}\end{aligned}$$

Now we get,

$$\frac{(\Delta \mathfrak{g}(\mathfrak{z}))'}{\Delta \mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1} = \mathfrak{p}_1'(\mathfrak{z}) + \frac{\mathfrak{g}'}{\mathfrak{g} - \mathfrak{a}_1}$$

This implies

$$\mathfrak{p}_1'(\mathfrak{z}) = \frac{(\Delta \mathfrak{g}(\mathfrak{z}))'}{\Delta \mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1} - \frac{\mathfrak{g}'}{\mathfrak{g} - \mathfrak{a}_1}. \tag{3.15}$$

$\mathfrak{p}_1'(\mathfrak{z})$ is an entire function, since $\mathfrak{p}_1(\mathfrak{z})$ is a polynomial. By (3.15) and (3.1) we obtain

$$\mathcal{T}\left(\mathfrak{r}, \mathfrak{p}_1'(\mathfrak{z})\right) = m\left(r, \mathfrak{p}_1'(\mathfrak{z})\right) \leq S\left(r, \Delta \mathfrak{g}(\mathfrak{z})\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.16}$$

From (3.15) we obtain

$$\frac{\mathfrak{p}_1'(\mathfrak{z})}{\mathfrak{g}-\mathfrak{a}_2} = \frac{(\Delta\mathfrak{g}(\mathfrak{z}))'}{(\mathfrak{g}-\mathfrak{a}_2)(\Delta\mathfrak{g}(\mathfrak{z})-\mathfrak{a}_1)} - \frac{\mathfrak{g}'}{(\mathfrak{g}-\mathfrak{a}_2)(\mathfrak{g}-\mathfrak{a}_1)}$$
$$= \frac{\Delta\mathfrak{g}(\mathfrak{z})}{(\mathfrak{g}-\mathfrak{a}_2)} \frac{(\Delta\mathfrak{g}(\mathfrak{z}))'}{\Delta\mathfrak{g}(\mathfrak{z})(\Delta\mathfrak{g}(\mathfrak{z})-\mathfrak{a}_1)} - \frac{\mathfrak{g}'}{(\mathfrak{g}-\mathfrak{a}_2)(\mathfrak{g}-\mathfrak{a}_1)}.$$

From the equation (3.1), Lemma (2.2) and Lemma (2.6), we obtain

$$m\left(\mathfrak{r}, \frac{\mathfrak{p}_1'(\mathfrak{z})}{\mathfrak{g}-\mathfrak{a}_2}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.17}$$

From (3.16) and (3.17) we get

$$m\left(r, \frac{1}{\mathfrak{g}-\mathfrak{a}_2}\right) \le m\left(\mathfrak{r}, \frac{\mathfrak{p}_1'(\mathfrak{z})}{\mathfrak{g}-\mathfrak{a}_2}\right) + m\left(\mathfrak{r}, \frac{1}{\mathfrak{p}_1'(\mathfrak{z})}\right)$$
$$\le \mathcal{S}(\mathfrak{r}, \mathfrak{g}) + \mathcal{T}(\mathfrak{r}, \mathfrak{p}_1'(\mathfrak{z}))$$

Therefore

$$m\left(\mathfrak{r}, \frac{1}{\mathfrak{g}-\mathfrak{a}_2}\right) \le \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.18}$$

Also, in the similar way, we obtain

$$m\left(\mathfrak{r}, \frac{1}{\mathfrak{g}-\mathfrak{a}_1}\right) \le \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.19}$$

By (3.2), (3.19) and the Lemma (2.2), we obtain

$$\begin{aligned} &\mathcal{T}\left(\mathfrak{r}, e^{\mathfrak{p}_1(\mathfrak{z})}\right) \\ =&m\left(\mathfrak{r}, e^{\mathfrak{a}_1(\mathfrak{z})}\right) \\ \le&m\left(\mathfrak{r}, \frac{\Delta\mathfrak{g}(\mathfrak{z})}{\mathfrak{g}-\mathfrak{a}_1}\right) + m\left(\mathfrak{r}, \frac{1}{\mathfrak{g}-\mathfrak{a}_1}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \end{aligned} \tag{3.20}$$

Also for $\mathcal{T}\left(\mathfrak{r}, e^{\mathfrak{p}_2(\mathfrak{z})}\right)$, we obtain

$$\mathcal{T}\left(\mathfrak{r}, e^{\mathfrak{p}_2(\mathfrak{z})}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.21}$$

By (3.6), (3.20) and (3.21), we obtain $\mathcal{T}(\mathfrak{r}, \mathfrak{g}) = \mathcal{S}(\mathfrak{r}, \mathfrak{g})$, we arrive at a contradiction. Therefore by the case 1, 2 and 3, concluding that $\Delta\mathfrak{g}(\mathfrak{z}) \equiv \mathfrak{g}(\mathfrak{z})$. □

Theorem 3.2. *Let* $\mathfrak{g}(\mathfrak{z})$ *be a non-constant meromorphic function. Let* $\mathfrak{a}_1, \mathfrak{a}_2 \in S(\mathfrak{g})$ *such* *that* $\mathfrak{g}(\mathfrak{z})$ *and* $\Delta\mathfrak{g}(\mathfrak{z})$ *partially share* $\mathfrak{a}_1, \mathfrak{a}_2 \in S(\mathfrak{g})$ *CM,* $\mathfrak{a}_3 = \tau$, *If*

$$\overline{E}(\mathfrak{a}_i, \mathfrak{g}(\mathfrak{z})) \subseteq \overline{E}(\mathfrak{a}_i, \Delta\mathfrak{g}(\mathfrak{z}), \quad \textit{for j=1,2},$$

and

$$\textit{for all } \nu \in \left[0, \frac{1}{3}\right) \textit{ and } 0 < \epsilon < \frac{1}{4}, \ (1 - 3\nu - \epsilon) > 4,$$

then $\Delta\mathfrak{g}(\mathfrak{z}) \equiv \mathfrak{g}(\mathfrak{z})$.

Proof. We know that, $S_1(r, \mathfrak{g}(\mathfrak{z})) = S_1(r, \Delta\mathfrak{g}(\mathfrak{z}))$. Suppose $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in S(\mathfrak{g})$. Denote

$$\mathfrak{g}(\mathfrak{z}) = \frac{\mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_1(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z}) - \mathfrak{a}_2(\mathfrak{z})} . \frac{a_3(\mathfrak{z}) - a_2(\mathfrak{z})}{a_3(\mathfrak{z}) - a_1(\mathfrak{z})}.$$

Then

$$\Delta\mathfrak{g}(\mathfrak{z}) = \frac{\mathfrak{g}(\mathfrak{z} + \mathfrak{c}) - \mathfrak{g}(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z} + \mathfrak{c}) - \mathfrak{a}_2(\mathfrak{z})} . \frac{\mathfrak{a}_3(\mathfrak{z}) - a_2(\mathfrak{z})}{a_3(\mathfrak{z}) - a_1(\mathfrak{z})}.$$

By the assumption that $\Delta\mathfrak{g}(\mathfrak{z})$ and $\mathfrak{g}(\mathfrak{z})$ share $\mathfrak{a}$ CM, we have that

$$\mathcal{N}\left(\mathfrak{r}, \frac{1}{\Delta\mathfrak{g} - \mathfrak{a}_1(\mathfrak{z})}\right) = \mathcal{N}\left(\mathfrak{r}, \frac{1}{\mathfrak{g} - \mathfrak{a}_1(\mathfrak{z})}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.22}$$

Similarly we can write for $\Delta\mathfrak{g}(\mathfrak{z})$ and $\mathfrak{g}(\mathfrak{z})$ share $\mathfrak{a}_2$ CM as

$$\mathcal{N}\left(\mathfrak{r}, \frac{1}{\Delta\mathfrak{g} - \mathfrak{a}_2(\mathfrak{z})}\right) = \mathcal{N}\left(\mathfrak{r}, \frac{1}{\mathfrak{g} - \mathfrak{a}_2(\mathfrak{z})}\right) = \mathcal{S}(\mathfrak{r}, \mathfrak{g}). \tag{3.23}$$

It suffices to show that $\Delta\mathfrak{g}(\mathfrak{z}) = \mathfrak{g}(\mathfrak{z})$. Since $\mathfrak{g}(\mathfrak{z})$ and $\Delta\mathfrak{g}(\mathfrak{z})$ share $0, \infty$ CM

$$\frac{\Delta\mathfrak{g}(\mathfrak{z})}{\mathfrak{g}(\mathfrak{z})} = \tau.$$

By Lemma 2.4, we have

$$\overline{E}(\mathfrak{a}_1, \mathfrak{g}(\mathfrak{z})) \subseteq \overline{E}(\mathfrak{a}_1, \Delta\mathfrak{g}(\mathfrak{z})),$$

and

$$\overline{E}(\mathfrak{a}_2, \mathfrak{g}(\mathfrak{z})) \subseteq \overline{E}(\mathfrak{a}_2, \Delta\mathfrak{g}(\mathfrak{z})).$$

Also by Lemma 2.5, for any $\mathfrak{a} \in \mathcal{S}(\mathfrak{f})$ we have

$$\begin{aligned}\overline{N}\left(\mathfrak{r}, \frac{1}{\Delta\mathfrak{g} - \mathfrak{a}}\right) &\leq \overline{\mathcal{N}}\left(\mathfrak{r} + |\mathfrak{c}|, \frac{1}{\mathfrak{g} - \mathfrak{a}}\right), \\ &= \overline{N}\left(\mathfrak{r}, \frac{1}{\mathfrak{g} - \mathfrak{a}}\right) + o\left(\overline{\mathcal{N}}\left(\mathfrak{r}, \frac{1}{\mathfrak{g} - \mathfrak{a}}\right)\right), = \overline{\mathcal{N}}\left(\mathfrak{r}, \frac{1}{\mathfrak{g} - \mathfrak{a}}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}).\end{aligned}$$

Case 1: If $\tau = 1$. Then it is clear that

$$\Delta \mathfrak{g}(\mathfrak{z}) \equiv \mathfrak{g}(\mathfrak{z}).$$

Case 2: $\tau \neq 1$. By Lemma 2.1 and Lemma 2.4, we get

$$\begin{aligned}
(1 - 3\nu - \epsilon)\mathcal{T}(\mathfrak{r}, \mathfrak{g}) &\leq \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_j}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_3}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \tau}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\frac{\Delta \mathfrak{g}(\mathfrak{z}) - \tau^2}{\tau}}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}\left(r, \frac{\tau}{\Delta \mathfrak{g}(\mathfrak{z}) - \tau^2}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}(r, \tau) + \overline{\mathcal{N}}\left(r, \frac{1}{\Delta \mathfrak{g}(\mathfrak{z}) - \tau^2}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}(r, \tau) + \overline{\mathcal{N}}\left(r, \frac{1}{g(z + c) - \mathfrak{g}(\mathfrak{z}) - \tau^2}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}),\\
&\leq \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_1}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g} - \mathfrak{a}_2}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g}(\mathfrak{z} + \mathfrak{c})}\right) + \overline{\mathcal{N}}\left(r, \frac{1}{\mathfrak{g}(\mathfrak{z})}\right) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}).
\end{aligned}$$

Therefore, using Lemma 2.3 in the above inequality, we get

$$(1 - 3\nu - \epsilon)\mathcal{T}(\mathfrak{r}, \mathfrak{g}) \leq 4\mathcal{T}(\mathfrak{r}, \mathfrak{g}) + \mathcal{S}(\mathfrak{r}, \mathfrak{g}), \tag{3.24}$$

now, $\forall\ \nu \in \left[0, \frac{1}{3}\right)$, ϵ is negative which is not possible and contradicts to our assumption. Hence $\Delta \mathfrak{g}(\mathfrak{z}) \equiv \mathfrak{g}(\mathfrak{z})$. □

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