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Some Aspects of Γ-Idempotent Graphs

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Abstract

Let us consider two non empty sets $\mathbb{Z}_n$ and $\Gamma \subseteq \mathbb{Z}_n$. It is obvious that $\mathbb{Z}_n$ forms a Γ-semigroup. In this paper we have defined two graph structures, graph $BG_1(\mathbb{Z}_n(\Gamma))$ and $BG_2(\mathbb{Z}_n(\Gamma))$. The adjacency is defined by considering any two distinct vertices x and y of $\mathbb{Z}_n$, which are adjacent if and only if $x + \alpha + y \in U(\mathbb{Z}_n)$ in graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ and the same two elements are adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$ if and only if $x + \alpha + y \in Z(\mathbb{Z}_n)$. We have compared this two graph structures through the degrees of their vertices. We have also studied the various other important properties such as planarity and traversibility of the graph structures.

Keywords: Γ-semigroup, Idempotent, Γ-Idempotent

1. INTRODUCTION

In the year 1988 I. Beck[6] constructed zero divisor graph where he established a connection between ring theory and graph theory. In his graph structure he considerd any two distinct elements of a commutative ring R as vertices and made them adjacent if and only if $xy = 0$. He was interested in coloring of the vertices of the above graph structure.

Above graph structure motivated many mathematicians to construct more bridges between algebraic structures and graph structures. In 1993 D.D Anderson and M. Naser[3] and D.F.Anderson and P.S. Livingston[4] redefined the graph structure given by I. Beck[6] by considering any two non zero zero divisors of $Z(R)$ as the set of vertices and any two distinct vertices x and y are adjacent if and only if $xy = 0$.

Another graph structure connecting ring theory with graph theory is total graph of a commutative ring discussed by D.F. Anderson and A. Badwi[2] in 2008 where they considered all the elements of R as vertices and x and y of R are adjacent if

and only if $x + y \in Z(R)$. Motivated by these concepts many more research articles were published discussing various parameters of graph structure such as planarity, hamiltonicity etc of the total graph were discussed in their respective articles .
A well known structure of Γ-algebraic structure was introduced by N. Nobusawa[8]in 1964 which was also considered as Γ-ring. Following the above definition mathematicians like M.k Sen and in [9] introduced the concept of Γ-semigroup in the year 1981. M.K Sen and N.K. Saha[10] in 1986 gave a stronger definition of Γ-semigroup by considering two non empty sets S and Γ and the set S is said to be a Γ semigroup if for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$ we have $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$.
In our graph structure we have considered the non empty sets $\mathbb{Z}_n$ and $\Gamma = U(\mathbb{Z}_n)$(unit elements of $\mathbb{Z}_n$). Any two distinct elements x and y of $\mathbb{Z}_n$ are adjacent in $BG_1\mathbb{Z}_n(\Gamma)$ if and only if $x + \alpha + y \in U(\mathbb{Z}_n)$ and same two vertices are adjacent in $BG_2\mathbb{Z}_n(\Gamma)$ if and only if $x + \alpha + y \in Z(\mathbb{Z}_n)$. we studied the various parameters of these graph structure.

2. PRELIMINARIES

For basic preliminary terms and definitions we have referred to F. Harary[5], N. Biggs[7] and Godsil and Royle[1]. Some of the preliminary definitions are given below.
DEFINITION 2.1: A graph G is said to be connected if any two distinct points of the graph is connected by a path or else it is disconnected.
DEFINITION 2.2: If a graph can be partitioned into two or more vertex sets where the vertices of the partitioned sets are adjacent among themselves and not adjacent to the other vertex sets then the graph is said to form components of the graph G.
DEFINITION 2.3: A graph G is said to be complete any two distinct vertices are adjacent. If a graph G is complete of n vertices then degree of any vertex of G is $n - 1$.
DEFINITION 2.4: The degree of a vertex x of a graph G is the number of edges adjacent to x.
DEFINITION 2.5: A graph G is said to be regular if every vertex of G have the same degree and semi regular if the vertices has only two degrees for the graph G.
DEFINITION 2.6: If the degree of each of the vertices of a graph G is 2 then the graph G is said to be a cycle and the length of the smallest cycle of the graph G is known as the girth of the graph G.
DEFINITION 2.7: A graph G is said to be Eulerian if it can be traversed by crossing each edge exactly once and if it can be traversed by each vertices by exactly once then the graph G is termed as Hamiltonian.
DEFINITION 2.8: If a graph G can be drawn in a plane in such a way that no two edges of the graph can intersect except at the incident points then the graph G is said to

be a planar graph.

DEFINITION 2.9: An element a of S is a regular element of Γ-semigroup if $a \in a\Gamma S\Gamma a$. If every element of the set S is regular then S is considered as regular set.

DEFINITION 2.10: An element $e \in S$ is termed as an α-idempotent of a Γ-semigroup S if we have the relation $e\alpha e = e$ and the set of all α- idempotents elements are denoted by $E(S)$.

Some of the examples of the two graph structures are shown below:

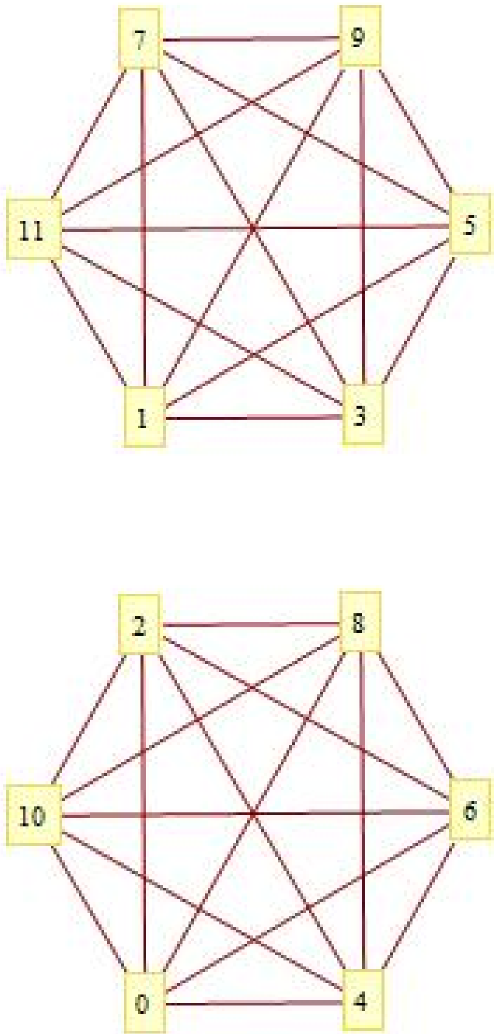

Figure 1: Graph of $BG_1(\mathbb{Z}_8(\Gamma))$

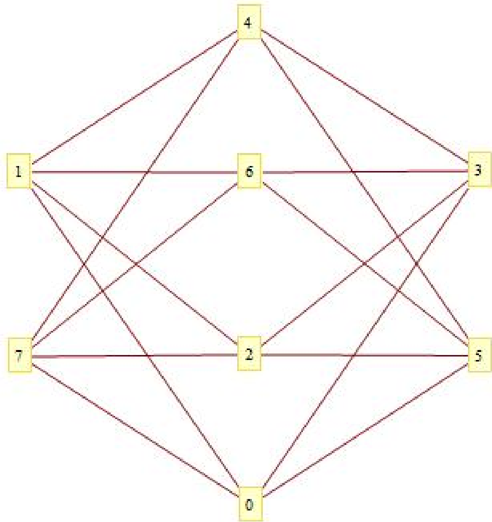

Figure 2: Graph of $BG_2(\mathbb{Z}_8(\Gamma))$

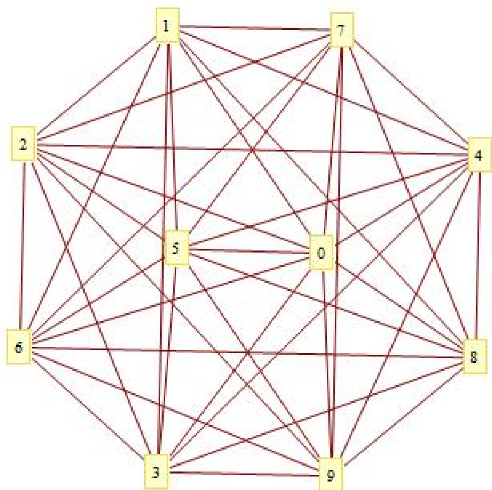

Figure 3: Graph of $BG_2(\mathbb{Z}_{1_0}(\Gamma))$

3. MAIN RESULTS

Throughout this paper we have cosidered two non empty sets $\mathbb{Z}_n$(set of integers modulo n) and $\Gamma = U_n$(set of all unit elements of $\mathbb{Z}_n$).

Proposition 1: The set $\mathbb{Z}_n$ forms a Γ-semigroup.

Proposition 2: The set $\mathbb{Z}_n$ is a regular set with respect to Γ-semigroup.

Proposition 3: The set of all α-idempotent elements of $\mathbb{Z}_n$ which is a Γ- semigroup is Γ itself.

Proof: Let us consider an element $e \in \mathbb{Z}_n$ be an arbitrary element. Now for e to be an α idempotent element we must have $e + \alpha + e = e$ which imply $e + \alpha = 0$ this means $e = n - \alpha$. Therefore we can conclude that e is nothing but an inverse element of α in $\mathbb{Z}_n$. Again we have that inverse of every unit element in $\mathbb{Z}_n$ are again an unit element of $\mathbb{Z}_n$. Hence the result.

Considering two distinct vertices x and y of $\mathbb{Z}_n$ and this two vertices will be adjacent if and only if they satisfy $x + \alpha + y \in U_n$ for some $\alpha \in U_n$. We denote this graph structure by $BG_1(\mathbb{Z}_n(\Gamma))$ and various parameters of this graph structures are studied in the following results.

Theorem 3.1: $BG_1(\mathbb{Z}_n(\Gamma))$ is a complete graph for n prime.

Proof: Let us consider two distinct elements x and y of $\mathbb{Z}_n$ as vertices of $BG_1(\mathbb{Z}_n(\Gamma))$ now since n is prime each of the element of $\mathbb{Z}_n$ is an unit element hence we can conclude that $x + y$ is also an unit element.

Now for $\alpha \in \Gamma = U_n$ we have $x + \alpha + y \in U_n$

Thus we can conclude that any two distinct vertices x and y of $BG_1(\mathbb{Z}_n(\Gamma))$ are adjacent. Hence the result.

Theorem 3.2: $BG_1(\mathbb{Z}_n(\Gamma))$ partitions itself into two components both of which are complete when n is an even number.

Proof: Here n is an even integer therefore $\Gamma = U_n$ will contain only the odd elements of $\mathbb{Z}_n$.

Now, let us consider any two distinct vertices x and y of $\mathbb{Z}_n$

if x is even element of $\mathbb{Z}_n$ and y be an odd element of $\mathbb{Z}_n$ then $x + y$ is an odd element and hence $x + \alpha + y$ is an even element for all $\alpha \in \Gamma = U_n$

therefore x and y cannot be adjacent in $BG_1(\mathbb{Z}_n(\Gamma))$

since $x + \alpha + y$ does not contain in U_n for x even and y odd.

Hence we can conclude that all odd elements of $\mathbb{Z}_n$ are adjacent to odd elements of $\mathbb{Z}_n$ and all even elements are adjacent to even elements of $\mathbb{Z}_n$.

Thus the graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ partitions into two vertex set V_1 and V_2 one of odd integers of $\mathbb{Z}_n$ and other is of even integers of $\mathbb{Z}_n$ and both the component are complete.

Hence $BG_1(\mathbb{Z}_n(\Gamma))$ is component wise complete graph for n even.

Theorem 3.3: The graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ is a complete graph for all odd values of n.

Proof: Let x and y be any two distinct values of $\mathbb{Z}_n$

let us suppose that x and y be the zero divisor element and unit element of $\mathbb{Z}_n$ respectively.

Now for adjacency of x and y we must have $x + \alpha + y \in U_n$ for some $\alpha \in U_n$.
Two cases arises :
Case 1: If $x + y$ is an unit element.
then we can always find an unit element $\alpha \in U_n$ such that $x + \alpha + y$ is an unit element. Hence x and y are adjacent in $BG_1(\mathbb{Z}_n(\Gamma))$.
Case 2: if $x + y$ is a zero divisor element of $\mathbb{Z}_n$
now for each zero divisor element of $\mathbb{Z}_n$ we can always find an unit element α in U_n such that their sum is an unit element.
Hence $x + \alpha + y \in U_n$. Thus x and y are adjacent.
Hence $BG_1(\mathbb{Z}_n(\Gamma))$ is complete graph for all n except n is even.
For the second graph structure of Γ- idempotent graph can be obtained by taking two distinct elements x and y of $\mathbb{Z}_n$ as vertices of $BG_2(\mathbb{Z}_n(\Gamma))$ and these two vertices will be adjacent if and only if $x+\alpha+y \in Z(\mathbb{Z}_n)$. Throughout the remaining of the paper we will be interested in study of the various parameters of the graph structure $BG_2(\mathbb{Z}_n(\Gamma))$ and compare it with the graph structure $BG_1(\mathbb{Z}_n(\Gamma))$.
Theorem 3.4: A vertex $BG_2(\mathbb{Z}_n(\Gamma))$ can never be adjacent to its inverse element.
Proof: let us consider an element of $\mathbb{Z}_n$ which is a vertice of the graph structure $BG_2(\mathbb{Z}_n(\Gamma))$.
now we know that inverse of x in $\mathbb{Z}_n$ is $n - x$
therefore we can conclude $x + n - x = n \equiv 0(mod n)$ and $0 + \alpha = \alpha$ is again an unit element.
Hence x and $n - x$ are not adjacent.
Theorem 3.5: The degree of each of the vertices of Γ- idempotent graph $BG_2(\mathbb{Z}_n(\Gamma))$ is $\phi(n)$ for all $n = 2^m$.
Proof: Let us consider the vertex set as $\mathbb{Z}_n$, where $n = 2^m$
Here $\Gamma = \{1, 3, 5, 7, 9, ..., 2^m - 1\} = U_n$
Let us consider arbitrary elements x and y of $\mathbb{Z}_n$
now, if x and y are both even then their sum $x + y$ is also even and therefore $x + y \in Z(\mathbb{Z}_n)$ a zero divisor element.
Then for all $\alpha \in \Gamma$ we have $x + \alpha + y$ is odd. Hence an unit element.
Therefore x and y are not adjacent.
Hence for adjacency of x and y we must have one odd and other even respectively.
Since we have $\phi(n)$ number of even and $\phi(n)$ number of odd numbers. Thus degree of any arbitrary vertex x of $\mathbb{Z}_n$ in $BG_2(\mathbb{Z}_n(\Gamma))$ is $\phi(n)$.
Theorem: 3.6 The $\Gamma-$ idempotent graph $BG_2(\mathbb{Z}_n(\Gamma))$ has degree $n-1$ for the element 0 and self inverse elements of $\mathbb{Z}_n$ and has degree $n - 2$ for the remaining vertices of $\mathbb{Z}_n$ for all $n = p_1p_2p_3p_4......p_m$ where p_i are prime integers.
Proof: Let us consider $n = p_1p_2p_3p_4......p_m$

therefore $\Gamma = \{U(\mathbb{Z}_n)\}$ (set of unit elements of $\mathbb{Z}_n$.
i.e no elements of Γ can be a factor of $p_1, p_2, p_3, p_4, ...p_n$
Let us consider any two vertices x and y of $BG_2(\mathbb{Z}_n(\Gamma))$ and they are not the inverses of each other in $\mathbb{Z}_n$
Now, if $x + y$ is a factor of some p_i i.e if $x + y \in \mathbb{Z}_n$ then we can always find an α in Γ which will make $x + \alpha + y$ a factor of p_j and $p_i \neq p_j$ where $i, j = 1, 2, 3, ..., m$
therefore $x + \alpha + y \in Z(\mathbb{Z}_n)$
hence x and y are adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$.
If $x + y$ is not a factor of p_i for all $i = 1, 2, 3, ...m$ i.e if $x + y$ is an unit element of $\mathbb{Z}_n$
Then for every such x and y we can choose $\alpha \in \Gamma$ making $x + \alpha + y \in Z(\mathbb{Z}_n)$
Therefore x and y are adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$
Again by Theorem 3.4 we have that a vertex can not be adjacent to its inverse.
Therefore $deg(0) = n - 1$ and $deg(x) = n - 2$.

Theorem 3.7: The degree of each of the vertices of an Γ- idempotent graph $BG_2(\mathbb{Z}_n(\Gamma))$ where $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} ......p_m^{a_m}$ for each prime p_i where $i = 1, 2, ..., m$ is as follows:
deg(0 and self inverse element)= $n - [\frac{n}{p_1 \cdot p_2 \cdot p_3 .... p_m}]$ and
deg(x)= $n - [\frac{n}{p_1 \cdot p_2 \cdot p_3 .... p_m} + 1]$.

Proof: Let x and y be any two distinct elements of $\mathbb{Z}_n$ and $\Gamma =$ *Unit elements of* $\mathbb{Z}_n$.
this imply $\Gamma = \{b \in \mathbb{Z}_n : b$ is not a factor of $p_1, p_2, p_3, ...p_m\}$.// Now if $x + y$ is a factor of $p_1, p_2, p_3, ...p_m$ then $\forall \alpha \in \Gamma$ we will have $x + \alpha + y \in \Gamma$(*unit elements of* $\mathbb{Z}_n$)
Thus we can conclude that two distinct vertices x and y of $\mathbb{Z}_n$ are not adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$ if $x + y$ is a factor of $p_1, p_2, p_3, ...p_m$, consequently any two distinct vertices x and y of $\mathbb{Z}_n$ are adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$ if and only if $x + y$ is not a factor of $p_1, p_2, p_3, ...p_m$
Now there are $[\frac{n}{p_1 \cdot ??_2 \cdot p_3 .... p_m}]$ elements in $\mathbb{Z}_n$ which are factors of $p_1, p_2, p_3, ...p_m$ including 0.

Case1: If we consider $x = 0$ or self inverse element and y be any other vertex other then 0 of $\mathbb{Z}_n$
Now $0 + y \not\equiv 0 (mod n)$ for any $y \in \mathbb{Z}_n$
Thus 0 is adjacent to all the vertices of y where y is not a factor of $p_1, p_2, p_3, ...p_m$.
Thus 0 is adjacent to $n - [\frac{n}{p_1 \cdot p_2 \cdot p_3 .... p_m}$ vertices of $\mathbb{Z}_n$ again since 0 is not adjacent to 0 by our definition
Thus *deg(0)*= $n - [\frac{n}{p_1 \cdot p_2 \cdot p_3 .... p_m}]$

case2: If we consider two non zero vertices x and y of $\mathbb{Z}_n$
we have already shown that any two distinct vertices x and y of $\mathbb{Z}_n$ are adjacent in $BG_2(\mathbb{Z}_n(\Gamma))$ if and only if $x + y$ is not a factor of $p_1, p_2, p_3, ...p_m$
and clearly x is not adjacent to itsself and its inverse element $n - x$ as $x + n - x + \alpha =$

$n + \alpha \equiv \alpha \in \Gamma$
thus we can conclude that *deg(x)*=$n - [\frac{n}{p_1.p_2.p_3........p_m} + 1]$.
in next theorem we are going to show for what values of n the two graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ and $BG_2(\mathbb{Z}_n(\Gamma))$ are Eulerian for which we will require the following lemma: **Lemma:** : A Graph $G(V, E)$ is Eulerian if and only If the degree of every vertex of a graph $G(V, E)$ is even.

Theorem 3.8: The $\Gamma-$ idempotent graph structures $BG_1(\mathbb{Z}_n(\Gamma))$ is eulerian for all odd n and $BG_2(\mathbb{Z}_n(\Gamma))$ is eulerian for $n = 2^m$ eulerian.

Proof: By **theorem 3.1** and **theorem 3.3** we have that the graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ is complete. Thus for all odd values of n the degree of the vertices is $n - 1$ which is even and therefore eulerian.

Again by **theorem 3.5** we have that the degree of all the vertices of $n = 2^m$ is $\phi(n)$ which is even thus the graph structure $BG_2(\mathbb{Z}_n(\Gamma))$ eulerian for such n.

Again from **theorem 3.6** and **theorem 3.7** we can clearly observe that there exist an element of $\mathbb{Z}_n$ whose degree in $BG_2(\mathbb{Z}_n(\Gamma))$ is odd hence non eulerian.

For the study of planarity of the graph structures $BG_1(\mathbb{Z}_n(\Gamma))$ and $BG_2(\mathbb{Z}_n(\Gamma))$ we are going to use the following lemmas:

Lemma: A graph is non planar if and only if it contains a subgraph homeomorphic to complete graph K_5 or $K_3, 3$.

Lemma: If a graph $G(V, E)$is planar then $q \leq 3p - 6$ containing p vertices and q edges.

Lemma: A graph $G(V, E)$ is maximal planar if the graph satisfies
$q = 3p - 6$.

Theorem 3.9: The graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ is planar for only $n = 3$ and the graph structure $BG_2(\mathbb{Z}_n(\Gamma))$ is planar for n=2, 3, 4 and 8.

Proof: The graph structure $BG_1(\mathbb{Z}_n(\Gamma))$ is complete for all odd n [by theorem 3.1 and theorem 3.3]. Thus we can clearly observe that $BG_1(\mathbb{Z}_n(\Gamma))$ is not planar for odd n other then 3 as $BG_1(\mathbb{Z}_n(\Gamma))$ is isomorphic to K_5. Hence the result.

Again, the graph structure $BG_2(\mathbb{Z}_n(\Gamma))$ is not planar for $n \geq 6$ other than 8 as they are isomorphic to K_5 by theorem 3.5, theorem 3.6 and theorem 3.7. Hence we can conclude that $BG_2(\mathbb{Z}_n(\Gamma))$ is planar for n=2, 3, 4 and 8.

Conclusion: In this paper we have defined two graph structures $BG_1(\mathbb{Z}_n(\Gamma))$ and $BG_2(\mathbb{Z}_n(\Gamma))$ naming them as $\Gamma-$ Idempotent graph and have studied their degrees in **Theorem 3.1** , **Theorem 3.2**, **Theorem 3.3**, **Theorem 3.5**, **Theorem 3.6** and **Theorem 3.7**. We have also compared various aspects of these two graph structures like planarity and Eulerinity by **Theorem 3.8** and **Theorem 3.9**.

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