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On the Equivalent Relations with the J. L. Lions Lemma and an Application to the Korn Inequality

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Abstract

In this paper, we consider the equivalent conditions with $W^{-m,p}$-version ($m \geq 0$ integer and $1 < p < \infty$) of the J. L. Lions lemma. As an applications, we consider the Korn inequality. Furthermore, we consider the other fundamental results.

Keywords and phrase: J. L. Lions lemma, de Rham theorem, Korn's inequality.
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1. INTRODUCTION

Assume that Ω is a bounded domain of $\mathbb{R}^d$ with a Lipcshitz-continuous boundary. In this paper, this means that Ω is a bounded and connected open subset of $\mathbb{R}^d$ whose boundary $\Gamma = \partial\Omega$ is Lipschitz-continuous and Ω is locally on the same side of Γ. The classical J. L. Lions lemma asserts that any distribution in the space of $H^{-1}(\Omega)$ with the gradient (in the distribution sense) belonging to $\boldsymbol{H}^{-1}(\Omega)$ is a function in $L^2(\Omega)$.

Amrouche et al. [1] derived the equivalent conditions with the J. L. Lions lemma. The conditions are the classical and the general J. L. Lions lemma, the Nečas inequality, the coarse version of the de Rham theorem, the surjectivity of the operator $\mathrm{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ and an approximation lemma. Some of these equivalent properties can be given by a "direct" proof.

However, these equivalent conditions of L^2-version of the J. L. Lions lemma are insufficient for considering the Maxwell-Stokes type system containing p-curlcurl operator. Thus it is important for us to improve the result to the L^p-version of the equivalent relations with the J. L. Lions lemma. For an application to the Maxwell-Stokes problem, see Aramaki [5, 3, 4] and Pan [12]. As an another application, we derived the Korn inequality of L^p-version. In order to show an extension of the Korn inequality in L^p to $W^{-m,p}$-version, we have to extend the equivalent relation to $W^{-m,p}$-version ($m \geq 0$ integer and $1 < p < \infty$) of the J. L. Lions lemma.

One of the purpose of this paper is an improvement of the result in the previous paper [3] in which we derived the equivalent relation with the classical J. L. Lions lemma, that is, if $f \in W^{-1,p}(\Omega)$ satisfies $\nabla f \in \boldsymbol{W}^{-1,p}(\Omega)$ $(1 < p < \infty)$, then $f \in L^p(\Omega)$. In this paper, we derive the classical J. L. Lions lemma: when $m \geq 0$ integer and $1 < p < \infty$, if $f \in W^{-m-1,p}(\Omega)$ satisfies $\nabla f \in \boldsymbol{W}^{-m-1,p}(\Omega)$, then $f \in W^{-m,p}(\Omega)$, and its equivalence relations. For example, $W^{-m,p}$- version of the Nečas inequality can be found in Nečas [11, Theorem 1], Geymonat and Suquet [10, Lemma 1] and Amrouche and Girault [2]. We show that using the Galdi [9, Theorem 3.2], we can derive $W^{-m,p}$-version of the J. L. Lions lemma directly.

The paper is organized as follows. In section 2, we give some preliminaries. In section 3, we derive $W^{-m,p}$ version of the J. L. Lions lemma and its equivalent relations. Section 4 is devoted to consider the Korn inequality. In section 5, we show the equivalence between the J. L. Lions lemma and a simplified version of the de Rham theorem. Finally, section 6 is devoted to the direct proof of the J. L. Lions lemma using a result of [9].

2. PRELIMINARIES

In this section, we shall state some preliminaries that are necessary in this paper. Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 2)$ (which means a bounded, connected open subset of $\mathbb{R}^d$) with a Lipschitz-continuous boundary Γ, let $1 < p < \infty$ and let p' be the conjugate exponent i.e., $(1/p) + (1/p') = 1$. From now on we use $\mathcal{D}(\Omega)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $W^{-m,p}(\Omega) = W_0^{m,p}(\Omega)' =$ the dual space of $W_0^{m,p}(\Omega)$, $(m \geq 0$, integer), for the standard real C^∞ functions with compact supports in Ω, L^p and Sobolev spaces of real valued functions. For any above space B, we denote B^d by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions. We denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. We denote the space of distributions in Ω by $\mathcal{D}'(\Omega)$. Moreover, for the dual space B' of B (resp. $\boldsymbol{B}'$ of $\boldsymbol{B}$), we denote the duality bracket between B' and B (resp. $\boldsymbol{B}'$ and $\boldsymbol{B}$) by $\langle \cdot, \cdot \rangle_{B',B}$ (resp. $\langle \cdot, \cdot \rangle_{\boldsymbol{B}',\boldsymbol{B}}$).

The gradient operator $\mathbf{grad} = \nabla : \mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}}'(\Omega)$ is defined by

$$\langle \nabla f, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)} = -\langle f, \operatorname{div} \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for } f \in \mathcal{D}'(\Omega) \text{ and } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega).$$

Since Ω is connected, we can see that if $f \in \mathcal{D}'(\Omega)$ satisfies $\nabla f = \mathbf{0}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$, then f is identified with a constant function (cf. Boyer and Fabrie [6, Chapter II. Lemma II.2.44]). For $f \in W^{-m,p}(\Omega)$, we can regard ∇f as an element of $\boldsymbol{W}^{-m-1,p}(\Omega)$ by the definition, and

$$\langle \nabla f, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} = -\langle f, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-m,p}(\Omega), W_0^{m,p'}(\Omega)} \tag{2.1}$$

for $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m+1,p'}(\Omega)$. Then it is clear that $\nabla : W^{-m,p}(\Omega) \to \boldsymbol{W}^{-m-1,p}(\Omega)$ is a linear and continuous operator. Since $\ker \nabla = \mathbb{R}$, we can also define an continuous operator $\nabla : W^{-m,p}(\Omega)/\mathbb{R} \to \boldsymbol{W}^{-m-1,p}(\Omega)$ by the definition $\nabla[f] = \nabla f$ for $[f] \in W^{-m,p}(\Omega)/\mathbb{R}$, where $[f]$ denotes the class in the quotient space $W^{-m,p}(\Omega)/\mathbb{R}$ with a representative f.

We introduce the closed subspace of a reflexive Banach space $\boldsymbol{W}_0^{m,p}(\Omega)$ which is a basic space in our argument.

$$\boldsymbol{W}_0^{m,p}(\Omega, \operatorname{div} 0) = \{\boldsymbol{u} \in \boldsymbol{W}_0^{m,p}(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}.$$

Moreover, we define a closed subspace of $W_0^{m,p}(\Omega)$ by

$$\begin{aligned} \dot{W}_0^{m,p}(\Omega) &= \left\{ f \in W_0^{m,p}(\Omega); \int_\Omega f dx = 0 \right\} \text{ if } m > 0, \\ L_0^p(\Omega) &:= \left\{ f \in L^p(\Omega); \int_\Omega f dx = 0 \right\} \text{ if } m = 0 \end{aligned}$$

endowed with the norm of $W^{m,p}(\Omega)$. We note that the dual space $(W^{-m,p}(\Omega)/\mathbb{R})'$ of $W^{-m,p}(\Omega)/\mathbb{R}$ is identified with $\dot{W}_0^{m,p'}(\Omega)$.

Now we state a theorem on the property of the domain Ω without its proof.

Theorem 2.1. *Let Ω be a bounded domain of $\mathbb{R}^d$. Then there exist domains Ω_j $(j = 1, 2, \dots)$ of $\mathbb{R}^d$ such that the boundary $\partial\Omega_j$ is of class C^∞, $\overline{\Omega_j} \subset \Omega_{j+1} \subset \Omega$ and $\Omega = \cup_{j=1}^\infty \Omega_j$.*

For the proof, see [6].

3. $W^{-M,P}$-VERSION OF THE J. L. LIONS LEMMA AND ITS EQUIVALENT RELATIONS

In this section, we assume that Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary.

We derive the following theorem.

Theorem 3.1. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary. Let $m \geq 0$ be an integer and $1 < p < \infty$. Then the following (a), (b), ..., (f) are equivalent.*

(a) Classical J. L. Lions lemma: *if $f \in W^{-m-1,p}(\Omega)$ satisfies*

$$\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega),$$

then $f \in W^{-m,p}(\Omega)$.

(b) The Nečas inequality: *there exists a constant $C = C(m, p, \Omega)$ such that*

$$\|f\|_{W^{-m,p}(\Omega)} \leq C(\|f\|_{W^{-m-1,p}(\Omega)} + \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}) \text{ for all } f \in W^{-m,p}(\Omega).$$

(c) The operator grad has a closed range: $\mathrm{grad}\ (W^{-m,p}(\Omega)/\mathbb{R})$ *is a closed subspace of $\boldsymbol{W}^{-m-1,p}(\Omega)$.*

(d) A coarse version of the de Rham theorem: *for any $\boldsymbol{h} \in \boldsymbol{W}^{-m-1,p}(\Omega)$, there exists a unique $[\pi] \in W^{-m,p}(\Omega)/\mathbb{R}$, where $[\pi]$ denotes the class in the quotient space $W^{-m,p}(\Omega)/\mathbb{R}$ with the representative π, such that $\boldsymbol{h} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-m-1,p}(\Omega)$ if and only if*

$$\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{m+1,p'}(\Omega, \mathrm{div}\, 0).$$

(e) The operator div **is surjective:** *the operator*

$$\mathrm{div}\ : \boldsymbol{W}_0^{m+1,p'}(\Omega) \to \dot{W}_0^{m,p'}(\Omega).$$

is linear, continuous and surjective.

Consequently, for any $f \in \dot{W}_0^{m,p'}(\Omega)$, there exists a unique

$$[\boldsymbol{u}_f] \in \boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\, \mathrm{div}\,,$$

where $\mathbf{Ker}\, \mathrm{div}\ = \boldsymbol{W}_0^{m+1,p'}(\Omega, \mathrm{div}\, 0)$ and $[\boldsymbol{u}_f]$ denotes the class in the quotient space $\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\, \mathrm{div}$ with the representative $\boldsymbol{u}_f$ such that $\mathrm{div}\, [\boldsymbol{u}_f] := \mathrm{div}\, \boldsymbol{u}_f = f$ in Ω. Therefore, the operator

$$\mathrm{div}\ : \boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\, \mathrm{div}\ \to \dot{W}_0^{m,p'}(\Omega)$$

is linear, continuous and bijective. Hence, by the Banach open mapping theorem, there exists a constant $c_1(m, p, \Omega) > 0$ such that

$$\|[\boldsymbol{u}_f]\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\, \mathrm{div}} \leq c_1(m, p, \Omega)\|f\|_{W^{m,p'}(\Omega)} \text{ for all } f \in \dot{W}_0^{m,p'}(\Omega).$$

In addition, for $\varphi \in \dot{\mathcal{D}}(\Omega)$, where

$$\dot{\mathcal{D}}(\Omega) = \left\{ \psi \in \mathcal{D}(\Omega); \int_\Omega \psi dx = 0 \right\},$$

we can choose $\boldsymbol{u}_\varphi \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{u}_\varphi = \varphi$.

(f) The J. L. Lions lemma: *if $f \in \mathcal{D}'(\Omega)$ satisfies $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega)$, then $f \in W^{-m,p}(\Omega)$.*

Remark 3.2. *Though the authors of [1] derived this theorem in L^2-framework in the classical J. L. Lions lemma in the sense that $f \in H^{-1}(\Omega)$ and $\boldsymbol{\nabla} f \in \boldsymbol{H}^{-1}(\Omega)$ implies $f \in L^2(\Omega)$, our Theorem 3.1 is an improvement of [1]. In the previous paper [3], we extended the results of [1] to L^p-version and applied the results to the Maxwell-Stokes problem containing p-curlcurl equation and the Korn inequality in L^p-version. However, according to Theorem 3.1, we will show the Korn inequality in $W^{-m,p}$-version in section 4.*

Proof of Theorem 3.1

(a) implies (b). Define a Banach space

$$V(\Omega) = \{f \in W^{-m-1,p}(\Omega); \boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega)\},$$

equipped with the norm

$$\|f\|_{V(\Omega)} = \|f\|_{W^{-m-1,p}(\Omega)} + \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}.$$

The canonical injection $i : W^{-m,p}(\Omega) \to V(\Omega)$ is linear, continuous and bijective according to (a). Hence, it follows from Banach open mapping theorem that i^{-1} is also linear and continuous, that is, there exists a constant $c_0(m, p, \Omega) > 0$ such that

$$\|f\|_{W^{-m,p}(\Omega)} \leq c_0(m, p, \Omega)(\|f\|_{W^{-m-1,p}(\Omega)} + \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)})$$

for all $f \in W^{-m,p}(\Omega)$.

(b) implies (c). It suffices to show that there exists a constant $C(m, p, \Omega) > 0$ such that

$$\|[f]\|_{W^{-m,p}(\Omega)/\mathbb{R}} \leq C(m, p, \Omega)\|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \tag{3.1}$$

for all $[f] \in W^{-m,p}(\Omega)/\mathbb{R}$. If the inequality (3.1) is false, then there exists a sequence $\{[f_k]\} \subset W^{-m,p}(\Omega)/\mathbb{R}$ such that

$$\|[f_k]\|_{W^{-m,p}(\Omega)/\mathbb{R}} = 1 \text{ for } k = 1, 2, \dots \text{ and } \|\boldsymbol{\nabla} f_k\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \to 0 \text{ as } k \to \infty.$$

We can easily see that the quotient norm $\|[f]\|_{W^{-m,p}(\Omega)/\mathbb{R}} = \inf\{\|f+c\|_{W^{-m,p}(\Omega)}; c \in \mathbb{R}\}$ is achieved, so we may assume that $f_k \in W^{-m,p}(\Omega)$, $\|f_k\|_{W^{-m,p}(\Omega)} = 1$ and $\boldsymbol{\nabla} f_k \to \mathbf{0}$ in $\boldsymbol{W}^{-m-1,p}(\Omega)$ as $k \to \infty$. Since $\{f_k\}$ is bounded in $W^{-m,p}(\Omega)$, there exist a subsequence $\{f_{k_l}\}$ of $\{f_k\}$ and $f \in W^{-m,p}(\Omega)$ such that $f_{k_l} \to f$ weakly in $W^{-m,p}(\Omega)$. Since the canonical injection operator

$$i_0 : W^{-m,p}(\Omega) \hookrightarrow W^{-m-1,p}(\Omega)$$

is compact, because i_0 is the dual operator of the compact canonical injection operator $W_0^{m+1,p'}(\Omega) \hookrightarrow W_0^{m,p'}(\Omega)$, thus we see that $f_{k_l} \to f$ strongly in $W^{-m-1,p}(\Omega)$. On the other hand, since $\boldsymbol{\nabla} f_{k_l} \to \mathbf{0}$ in $\boldsymbol{W}^{-m-1,p}(\Omega)$, it follows from the hypothesis (b) that we can see that $\{f_{k_l}\}$ is a Cauchy sequence in $W^{-m,p}(\Omega)$. Hence $f_{k_l} \to f$ strongly in $W^{-m,p}(\Omega)$ as $l \to \infty$. Since $\boldsymbol{\nabla} : W^{-m,p}(\Omega) \to \boldsymbol{W}^{-m-1,p}(\Omega)$ is continuous, we have $\boldsymbol{\nabla} f_{k_l} \to \boldsymbol{\nabla} f = \mathbf{0}$ in $\boldsymbol{W}^{-m-1,p}(\Omega)$. This implies that $f = \text{const.}$, so $[f] = 0$. Thus $\|[f_{k_l}]\|_{W^{-m,p}(\Omega)/\mathbb{R}} = \|[f_{k_l} - f]\|_{W^{-m,p}(\Omega)/\mathbb{R}} \le \|f_{k_l} - f\|_{W^{-m,p}(\Omega)} \to 0$ as $l \to \infty$. This is a contradiction.

(c) is equivalent to (d). We note that the operator $\mathbf{grad} = \boldsymbol{\nabla} : W^{-m,p}(\Omega)/\mathbb{R} \to \boldsymbol{W}^{-m-1.p}(\Omega)$ is the dual operator of

$$-\mathrm{div} : \boldsymbol{W}_0^{m+1,p'}(\Omega) \to (W^{-m,p}(\Omega)/\mathbb{R})' = \dot{W}_0^{m,p'}(\Omega)$$

and satisfies

$$\begin{aligned}\langle \boldsymbol{\nabla}[f], \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)} &= \langle \boldsymbol{\nabla} f, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)} \\ &= -\langle f, \mathrm{div}\,\boldsymbol{\varphi}\rangle_{W^{-m,p}(\Omega),\dot{W}_0^{m,p'}(\Omega)}.\end{aligned}$$

for all $[f] \in W^{-m,p}(\Omega)/\mathbb{R}$ and $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m+1,p'}(\Omega)$. Therefore, if we apply the Banach closed range theorem, $\mathrm{Im}\,\boldsymbol{\nabla}$ is a closed subspace of $\boldsymbol{W}^{-m-1,p}(\Omega)$ if and only if

$$\begin{aligned}\mathrm{Im}\,\boldsymbol{\nabla} = {}^{\perp}(\mathbf{Ker}\,\mathrm{div}\,) := \{\boldsymbol{h} \in \boldsymbol{W}^{-m-1,p}(\Omega); \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0 \\ \text{for all } \boldsymbol{\varphi} \in \mathbf{Ker}\mathrm{div}\, = \boldsymbol{W}_0^{m+1,p'}(\Omega, \mathrm{div}\, 0)\}.\end{aligned}$$

This means that (c) and (d) are equivalent.

(d) implies (e). Assume that (d) and (c) hold. Since $\mathbf{grad}\ (W^{-m,p}(\Omega)/\mathbb{R})$ is a closed subspace of $\boldsymbol{W}^{-m-1,p}(\Omega)$ from (c), it follows from the Banach closed range theorem that we have $\mathrm{Im}\,\mathrm{div}\, = (\mathrm{Ker}\,\boldsymbol{\nabla})^{\perp}$. Since $\mathrm{Ker}\,\boldsymbol{\nabla} = \mathbb{R}$, we have $\mathrm{Im}\,\mathrm{div}\, = \dot{W}_0^{m,p'}(\Omega)$, so we can see that

$$\mathrm{div}\, : \boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\,\mathrm{div}\, \to \dot{W}_0^{m,p'}(\Omega)$$

is a continuous and bijective linear operator. Therefore, from the Banach open mapping theorem, the inverse operator is continuous. Thus for any $f \in \dot{W}_0^{m,p'}(\Omega)$, there exists a unique $[\boldsymbol{u}_f] \in \boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\,\mathrm{div}$ such that $\mathrm{div}\,\boldsymbol{u}_f = f$ in $\dot{W}_0^{m,p'}(\Omega)$, and there exists a constant $C(m,p,\Omega)$ such that

$$\|[\boldsymbol{u}_f]\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\,\mathrm{div}} \leq C(m,p,\Omega)\|f\|_{\dot{W}_0^{m,p'}(\Omega)}.$$

In particular, if $\varphi \in \dot{\mathcal{D}}(\Omega)$, according to the Diening et al. [8, Thereom 14.3.15], we can choose $\boldsymbol{u}_\varphi \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathrm{div}\,\boldsymbol{u}_\varphi = \varphi$.

(e) implies (f). Let $f \in \mathcal{D}'(\Omega)$ satisfy $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega)$. It suffices to prove that there exists a constant $C_0 = C_0(m,p,f,\Omega)$ such that

$$|\langle f, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \leq C_0\|\varphi\|_{W_0^{m,p'}(\Omega)} \text{ for all } \varphi \in \mathcal{D}(\Omega). \tag{3.2}$$

Indeed, assume that (3.2) holds. Since $\mathcal{D}(\Omega)$ is contained in $W_0^{m,p'}(\Omega)$ densely, for any $\varphi \in W_0^{m,p'}(\Omega)$, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ such that $\varphi_n \to \varphi$ in $W_0^{m,p'}(\Omega)$. From (3.2), we have

$$|\langle f, \varphi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} - \langle f, \varphi_m\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \leq C_0\|\varphi_n - \varphi_m\|_{W_0^{m,p'}(\Omega)}.$$

Therefore, $\{\langle f, \varphi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\}$ is a Cauchy sequence in $\mathbb{R}$. Define a linear functional $\widehat{f}$ on $W_0^{m,p'}(\Omega)$ by

$$\langle \widehat{f}, \varphi\rangle = \lim_{n\to\infty}\langle f, \varphi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for } \varphi \in W_0^{m,p'}(\Omega). \tag{3.3}$$

We can clearly recognize that the definition is independent of the choice of $\{\varphi_n\}$ with $\varphi_n \to \varphi$ in $W_0^{m,p'}(\Omega)$. From (3.2),

$$\begin{aligned} |\langle \widehat{f}, \varphi\rangle| &= \lim_{n\to\infty}|\langle f, \varphi_n\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \\ &\leq C_0\lim_{n\to\infty}\|\varphi_n\|_{W_0^{m,p'}(\Omega)} \\ &= C_0\|\varphi\|_{W_0^{m,p'}(\Omega)} \text{ for all } \varphi \in W_0^{m,p'}(\Omega). \end{aligned}$$

Thus $\widehat{f} \in (W_0^{m,p'}(\Omega))' = W^{-m,p}(\Omega)$. In particular, if we choose $\varphi \in \mathcal{D}(\Omega)$ in (3.3), we have

$$\langle \widehat{f}, \varphi\rangle = \langle f, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for all } \varphi \in \mathcal{D}(\Omega).$$

That is to say, $f = \widehat{f} \in W^{-m,p}(\Omega)$.

We derive (3.2). Let $\varphi_1 \in \mathcal{D}(\Omega)$ such that $\int_\Omega \varphi_1 dx = 1$. For any $\varphi \in \mathcal{D}(\Omega)$, define

$$\varphi_0 = \varphi - \left(\int_\Omega \varphi dx\right)\varphi_1 \in \dot{\mathcal{D}}(\Omega).$$

Then it follows from the Hölder inequality that

$$\begin{aligned}\|\varphi_0\|_{W_0^{m,p'}(\Omega)} &\le \|\varphi\|_{W_0^{m,p'}(\Omega)} + \int_\Omega |\varphi| dx \|\varphi_1\|_{W_0^{m,p'}(\Omega)} \\ &\le \|\varphi\|_{W_0^{m,p'(\cdot)}(\Omega)} + C_1\|\varphi\|_{L^{p'(\cdot)}(\Omega)}\|\varphi_1\|_{W_0^{m,p'}(\Omega)} \\ &\le C_2\|\varphi\|_{W_0^{m,p'}(\Omega)}.\end{aligned}$$

Here we note that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$,

$$|\langle f, \operatorname{div} \boldsymbol{v}\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| = |\langle \boldsymbol{\nabla} f, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}| \le \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)}.$$

From (e), there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = \varphi_0$ and

$$\|[\boldsymbol{v}]\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\operatorname{div}} \le C_3\|\varphi_0\|_{W_0^{m,p'}(\Omega)} \le C_2C_3\|\varphi\|_{W_0^{m,p'}(\Omega)}.$$

Hence

$$\begin{aligned}\langle f, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= \langle f, \varphi_0\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_\Omega \varphi dx \langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= \langle f, \operatorname{div} \boldsymbol{v}\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_\Omega \varphi dx \langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.\end{aligned}$$

So, we have

$$\begin{aligned}|\langle f, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| &= \left|\langle f, \varphi_0\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_{\Omega)} \varphi dx \langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\right| \\ &\le |\langle \boldsymbol{\nabla} f, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}| + |\Omega|^{1/p}\|\varphi\|_{L^{p'}(\Omega)}|\langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \\ &\le \|\boldsymbol{\nabla} f\|_{(\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\operatorname{div})'}\|[\boldsymbol{v}]\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\operatorname{div}} \\ &\quad +C_4\|\varphi\|_{L^{p'}(\Omega)}|\langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \\ &\le C_5\|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}\|\varphi\|_{W_0^{m,p'}(\Omega)} \\ &\quad +C_6|\langle f, \varphi_1\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}|\|\varphi\|_{W_0^{m,p'}(\Omega)} \\ &\le C_7(\|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} + 1)\|\varphi\|_{W_0^{m,p'}(\Omega)}.\end{aligned}$$

Here we identified $(\boldsymbol{W}_0^{m+1,p'}(\Omega)/\mathbf{Ker}\operatorname{div})'$ with

$(\mathbf{Ker}\operatorname{div})^\perp = \{\boldsymbol{g} \in \boldsymbol{W}^{-m-1,p}(\Omega); \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0$ for all $\boldsymbol{v} \in \mathbf{Ker}\operatorname{div}\}$.

Thus (3.2) holds.

(f) implies (a). Clear.

This completes the proof of Theorem 3.1.

Remark 3.3. *Since we can prove that the classical J. L. Lions lemma (a) holds (cf. Ciarlet [7, p. 381 and the footnote]), or the Nečas inequality (b) (cf. [11, Theorem 1] or [6, Remark IV.1.1]) directly, consequently if Ω is a bounded domain with a Lipschitz-continuous boundary, then all of (a), ..., (f) are true. In section 6, we prove that the J. L. Loins lemma (Theorem 3.1 (f)) holds using [9, Theorem 3.2]. According to our best knowledge, it seems that the proof is new.*

4. AN APPLICATION OF THE J. L. LIONS LEMMA TO THE KORN INEQUALITY

In this section, we consider the Korn inequality which plays a crucial role in linearized elasticity.

We introduce the following Korn inequality in $\boldsymbol{W}^{1,p}(\Omega)$ which is proved in the previous paper [3].

Theorem 4.1. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary and $1 < p < \infty$. Then the J. L. Lions lemma (Theorem 3.1 (a) with $m = 0$) implies the following Korn inequality in $\boldsymbol{W}^{1,p}(\Omega)$: there exists a constant $C = C(p, \Omega) > 0$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{e}(\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}) \textit{ for all } \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \tag{4.1}$$

where $\boldsymbol{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v}))_{1 \leq i,j \leq d}$ with

$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j), \quad \boldsymbol{v} = (v_1, \ldots, v_d).$$

For the proof, see [3, Theorem 5.1].

In this paper, we consider the Korn inequality in $\boldsymbol{W}^{-m,p}(\Omega)$.

Theorem 4.2. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary, $m \geq -1$ is an integer and $1 < p < \infty$. Then the classical J. L. Lions lemma (Theorem 3.1 (a)) implies the following Korn inequality: there exists a constant $C = C(p, m, \Omega) > 0$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{-m,p}(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} + \|\boldsymbol{e}(\boldsymbol{v})\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}) \textit{ for all } \boldsymbol{v} \in \boldsymbol{W}^{-m,p}(\Omega), \tag{4.2}$$

Conversely, (4.2) *with $m \geq 0$ implies the classical J. L. Lions lemma in Theorem 3.1 (a).*

Proof. If $m = -1$, the theorem is just Theorem 4.1. Hence let $m \geq 0$.

Step 1. If we define

$$\boldsymbol{F}^{-m-1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{-m-1,p}(\Omega); \boldsymbol{e}(\boldsymbol{v}) \in \boldsymbol{W}^{-m-1,p}(\Omega)\}$$

equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{F}^{-m-1,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} + \|\boldsymbol{e}(\boldsymbol{v})\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}$, then we can see that $\boldsymbol{F}^{-m-1,p}(\Omega)$ is a Banach space. We claim that $\boldsymbol{F}^{-m-1,p}(\Omega) = \boldsymbol{W}^{-m,p}(\Omega)$. Indeed, clearly we can see that $\boldsymbol{W}^{-m,p}(\Omega) \subset \boldsymbol{F}^{-m-1,p}(\Omega)$. Let $\boldsymbol{v} = (v_1, \dots, v_d) \in \boldsymbol{F}^{-m-1,p}(\Omega)$. Since $v_i \in W^{-m-1,p}(\Omega)$, we have $\partial_k v_i \in W^{-m-2,p}(\Omega)$ and

$$\partial_j(\partial_k v_i) = \partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v}) \in W^{-m-2,p}(\Omega).$$

By Theorem 3.1 (a), we have $\partial_k v_i \in W^{-m-1,p}(\Omega)$ for every $k = 1, \dots, d$, so $\boldsymbol{\nabla} v_i \in \boldsymbol{W}^{-m-1,p}(\Omega)$. Using again Theorem 3.1 (a), we can see that $v_i \in W^{-m,p}(\Omega)$ for every $i = 1, \dots, d$, so $\boldsymbol{v} \in \boldsymbol{W}^{-m,p}(\Omega)$.

Step 2. The canonical injection $i : \boldsymbol{W}^{-m,p}(\Omega) \to \boldsymbol{F}^{-m-1,p}(\Omega)$ is linear, injective and continuous. From Step 1, we see that i is surjective. Using again the Banach open mapping theorem, i^{-1} is also continuous. This implies the estimate (4.2).

Step 3. Conversely, let $m \geq 0$ be an integer and assume that (4.2) holds. Since

$$\boldsymbol{e}(\boldsymbol{v}) = \frac{1}{2}((\boldsymbol{\nabla} \boldsymbol{v})^T + \boldsymbol{\nabla} \boldsymbol{v}),$$

where $(\boldsymbol{\nabla} \boldsymbol{v})^T$ denotes the transposed matrix of $\boldsymbol{\nabla} \boldsymbol{v}$. Thus

$$\|\boldsymbol{e}(\boldsymbol{v})\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \leq \|\boldsymbol{\nabla} \boldsymbol{v}\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}.$$

From (4.2), we have

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{-m,p}(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} + \|\boldsymbol{\nabla} \boldsymbol{v}\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}) \text{ for all } \boldsymbol{v} \in \boldsymbol{W}^{-m,p}(\Omega). \quad (4.3)$$

For $f \in W^{-m,p}(\Omega)$, if we put $\boldsymbol{v} = (f, 0, \dots, 0)$, then it follows from (4.3) that

$$\|f\|_{W^{-m,p}(\Omega)} \leq C(\|f\|_{W^{-m-1,p}(\Omega)} + \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)}) \text{ for all } f \in W^{-m,p}(\Omega).$$

This is the Nečas inequality in Theorem 3.1 (b). Thus it follows from Theorem 3.1 that the classical J. L. Lions lemma (Theorem 3.1 (a)) holds. □

5. RELATION BETWEEN THE J. L. LIONS LEMMA AND A SIMPLIFIED VERSION OF THE DE RHAM THEOREM

In this section, we discuss on a relation between the J. L. Lions lemma and a simplified version of the de Rham theorem, which is a the fundamental result.

We have the following theorem.

Theorem 5.1. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary, $m \geq 0$ is an integer and $1 < p < \infty$. Then the J. L. Lions lemma (Theorem 3.1 (a)) implies that the following simplified version of the de Rham theorem: for any $\boldsymbol{h} \in \boldsymbol{W}^{-m-1,p}(\Omega)$ satisfying*

$$\langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0 \textit{ for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) \textit{ with } \operatorname{div} \boldsymbol{\varphi} = 0 \textit{ in } \Omega,$$

there exists $\pi \in W^{-m,p}(\Omega)$ such that $\boldsymbol{h} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-m-1,p}(\Omega)$.

Conversely, the simplified version of the de Rham theorem implies the J. L. Lions lemma (Theorem 3.1 (a)).

Proof. Let $\boldsymbol{h} \in \boldsymbol{W}^{-m-1,p}(\Omega)$ satisfy

$$\langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0 \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) \text{ with } \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \Omega. \tag{5.1}$$

From Theorem 2.1, choose bounded domains $\Omega_j \subset \Omega$ $(j = 1, 2, \ldots)$ such that $\partial\Omega_j$ is of class C^∞, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $\Omega = \cup_{j=1}^\infty \Omega_j$. For any $\boldsymbol{v}_j \in \boldsymbol{W}_0^{m+1,p'}(\Omega_j)$ satisfying $\operatorname{div} \boldsymbol{v}_j = 0$ in Ω_j, define $\widetilde{\boldsymbol{v}}_j \in \boldsymbol{W}_0^{m+1,p'}(\mathbb{R}^d)$ as an extension of $\boldsymbol{v}_j$ by $\boldsymbol{0}$ on $\mathbb{R}^d \setminus \Omega_j$. Let $\{\rho_n\}_{n=1}^\infty$ be the standard mollifier, that is, choose $\rho \in \mathcal{D}(\mathbb{R}^d)$ such that $\rho \geq 0$, $\operatorname{supp} \rho \subset \{x \in \mathbb{R}^d; |x| \leq 1\}$ and $\int_{\mathbb{R}^d} \rho(x)dx = 1$, then ρ_n is defined by $\rho_n(x) = n^d \rho(nx)$. Then there exists $n_0(j)$ and a compact set $K_j \subset \Omega$ such that $\operatorname{supp}(\widetilde{\boldsymbol{v}}_j * \rho_n) \subset K_j$, so $\widetilde{\boldsymbol{v}}_j * \rho_n\big|_\Omega \in \boldsymbol{\mathcal{D}}(\Omega)$ for any $n \geq n_0(j)$, $\operatorname{div}(\widetilde{\boldsymbol{v}}_j * \rho_n) = (\operatorname{div} \widetilde{\boldsymbol{v}}_j) * \rho_n = 0$ in $\mathbb{R}^d$ for any $n \geq n_0(j)$ and $\lim_{n\to\infty} \|\widetilde{\boldsymbol{v}}_j * \rho_n - \widetilde{\boldsymbol{v}}_j\|_{\boldsymbol{W}^{m+1,p'}(\mathbb{R}^d)} = 0$. For $j \geq 1$, let $\boldsymbol{h}_j \in \boldsymbol{W}^{-m-1,p}(\Omega_j)$, where $\boldsymbol{h}_j$ is defined by

$$\langle \boldsymbol{h}_j, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j), \boldsymbol{W}_0^{m+1,p'}(\Omega_j)} = \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}^{m+1,p'}(\Omega)} \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{W}_0^{m+1,p'}(\Omega_j),$$

where we identified $\boldsymbol{W}_0^{m+1,p'}(\Omega_j)$ with a subspace of $\boldsymbol{W}_0^{m+1,p'}(\Omega)$. Then from (5.1), we have

$$\begin{aligned} \langle \boldsymbol{h}_j, \boldsymbol{v}_j \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j), \boldsymbol{W}_0^{m+1,p'}(\Omega_j)} &= \langle \boldsymbol{h}, \widetilde{\boldsymbol{v}}_j\big|_\Omega \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} \\ &= \lim_{n\to\infty} \langle \boldsymbol{h}, \widetilde{\boldsymbol{v}}_j * \rho_n\big|_\Omega \rangle_{\boldsymbol{W}^{-m-1,p}(\Omega), \boldsymbol{W}_0^{m+1,p'}(\Omega)} = 0 \end{aligned}$$

for all $\boldsymbol{v}_j \in \boldsymbol{W}_0^{m+1,p'}(\Omega_j)$ satisfying $\operatorname{div} \boldsymbol{v}_j = 0$ in Ω_j. By Theorem 3.1 (d), there exists $\pi_j \in W^{-m,p}(\Omega_j)$ such that $\boldsymbol{h}_j = \nabla\pi_j$ in $\boldsymbol{W}^{-m-1,p}(\Omega_j)$. Fix $\varphi_0 \in \mathcal{D}(\Omega_1)$ such that $\int_{\Omega_1} \varphi_0 dx = 1$. Since

$$\langle \pi_j - \langle \pi_j, \varphi_0 \rangle_{W^{-m,p}(\Omega_j), W_0^{m,p'}(\Omega_j)}, \varphi_0 \rangle_{W^{-m,p}(\Omega_j), W_0^{m,p'}(\Omega_j)} = 0,$$

if we replace π_j with $\pi_j - \langle \pi_j, \varphi_0 \rangle_{W^{-m,p}(\Omega_j), W_0^{m,p'}(\Omega_j)}$, then we can assume that

$$\langle \pi_j, \varphi_0 \rangle_{W^{-m,p}(\Omega_j), W_0^{m,p'}(\Omega_j)} = 0 \text{ for every } j.$$

For $j, k \geq 1$ and any $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m+1,p'}(\Omega_j)$, we have

$$\begin{aligned}\langle \boldsymbol{\nabla}\pi_{j+k}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),\boldsymbol{W}_0^{m+1,p'}(\Omega_j)} &= \langle \boldsymbol{h}_{j+k}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),\boldsymbol{W}_0^{m+1,p'}(\Omega_j)}\\ &= \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)}\\ &= \langle \boldsymbol{h}_j, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),\boldsymbol{W}_0^{m+1,p'}(\Omega_j)}\\ &= \langle \boldsymbol{\nabla}\pi_j, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),\boldsymbol{W}_0^{m+1,p'}(\Omega_j)}.\end{aligned}$$

Hence we have $\boldsymbol{\nabla}(\pi_{j+k} - \pi_j) = \boldsymbol{0}$ in $\boldsymbol{W}^{-m-1,p}(\Omega_j)$, so $\pi_{j+k} - \pi_j = c_{j,k} \in \mathbb{R}$. Since $\langle \pi_j, \varphi_0\rangle_{W^{-m,p}(\Omega),W_0^{m,p'}(\Omega)} = 0$ for every j, we see that $\pi_{j+k} = \pi_j$ in $\boldsymbol{W}^{-m,p}(\Omega_j)$, that is, $\pi_{j+k} \in W^{-m,p}(\Omega_{j+k})$ is an extension of $\pi_j \in W^{-m,p}(\Omega_j)$. Define a linear functional π on $\mathcal{D}(\Omega)$ as follows. For $\varphi \in \mathcal{D}(\Omega)$, there exists $j \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset \Omega_j$. Then we define

$$\langle \pi, \varphi\rangle = \langle \pi_j, \varphi\rangle_{W^{-m,p}(\Omega_j),W_0^{m,p'}(\Omega_j)}.$$

This definition is well defined and we can easily see that $\pi \in \mathcal{D}'(\Omega)$. For any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, choose $j(\boldsymbol{\varphi}) \geq 1$ such that $\operatorname{supp}\boldsymbol{\varphi} \subset \Omega_j$ for $j \geq j(\boldsymbol{\varphi})$. Hence we have

$$\begin{aligned}\langle \boldsymbol{\nabla}\pi, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)} &= -\langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\\ &= -\langle \pi_j, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-m,p}(\Omega_j),W_0^{m,p'}(\Omega_j)}\\ &= \langle \boldsymbol{\nabla}\pi_j, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),W_0^{m+1,p'}(\Omega_j)}\\ &= \langle \boldsymbol{h}_j, \boldsymbol{\varphi}\big|_{\Omega_j}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega_j),\boldsymbol{W}_0^{m+1,p'}(\Omega_j)}\\ &= \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m-1,p}(\Omega),\boldsymbol{W}_0^{m+1,p'}(\Omega)}\\ &= \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}.\end{aligned}$$

So we get $\boldsymbol{\nabla}\pi = \boldsymbol{h} \in \boldsymbol{W}^{-m-1,p}(\Omega)$. It follows from (f) in Theorem 3.1 that $\pi \in W^{-m.p}(\Omega)$.

Conversely, if the simplified version of the de Rham theorem holds, then (d) in Theorem 3.1 holds, so the J. L. Lions lemma in Theorem 3.1 (a) follows. □

6. THE DIRECT PROOF OF THEOREM 3.1 (F)

We can directly derive Theorem 3.1 (f) from the following result of [9].

Theorem 6.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, $m \geq 0$ integer and $1 < p < \infty$. For any $f \in \dot{\mathcal{D}}(\Omega)$, there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that*

$$\operatorname{div}\boldsymbol{v} = f \textit{ in } \Omega,$$

and moreover, there exists a constant $C = C(d, p, \Omega) > 0$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C\|f\|_{W^{m,p}(\Omega)}.$$

For the proof, see [9, Theorem 3.2].

Now we derive the J. L. Lions lemma (Theorem 3.1 (f)).

Theorem 6.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous boundary, $m \geq 0$ integer and $1 < p < \infty$. If $f \in \mathcal{D}'(\Omega)$ satisfies $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega)$, then $f \in W^{-m,p}(\Omega)$, that is, the J. L. Lions lemma (Theorem 3.1 (f)) holds.*

Proof. Let $f \in \mathcal{D}'(\Omega)$ satisfy $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p}(\Omega)$. It suffices to show that there exists a constant $C_0 = C_0(m, p, f, \Omega)$ such that

$$|\langle f, \varphi \rangle_{\mathcal{D}(\Omega), \mathcal{D}(\Omega)}| \leq C_0 \|\varphi\|_{W_0^{m,p'}(\Omega)} \text{ for all } \varphi \in \mathcal{D}(\Omega). \tag{6.1}$$

Indeed, let (6.1) be true. Since $\mathcal{D}(\Omega)$ is dense in $W_0^{m,p'}(\Omega)$, for any $\varphi \in W_0^{m,p'}(\Omega)$, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ such that $\varphi_n \to \varphi$ in $W_0^{m,p'}(\Omega)$. By (6.1),

$$|\langle f, \varphi_n - \varphi_m \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}| \leq C_0 \|\varphi_n - \varphi_m\|_{W^{m,p'}(\Omega)} \text{ as } n, m \to \infty.$$

Hence $\{\langle f, \varphi_n \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}\}$ is a Cauchy sequence in $\mathbb{R}$. Define a linear functional $\widehat{f}$ on $W_0^{m,p'}(\Omega)$ by

$$\langle \widehat{f}, \varphi \rangle = \lim_{n \to \infty} \langle f, \varphi_n \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \text{ for } \varphi \in W_0^{m,p'}(\Omega).$$

The definition is well defined, that is, it is independent of the choice of $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ with $\varphi_n \to \varphi$ in $W_0^{m,p'}(\Omega)$. Since

$$|\langle \widehat{f}, \varphi \rangle| = \lim_{n \to \infty} |\langle f, \varphi_n \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}| \leq C_0 \lim_{n \to \infty} \|\varphi_n\|_{W_0^{m,p'}(\Omega)} \leq C_0 \|\varphi\|_{W_0^{m,p'}(\Omega)}$$

for any $\varphi \in W_0^{m,p'}(\Omega)$. Hence $\widehat{f} \in W^{-m,p}(\Omega)$ and

$$\langle \widehat{f}, \varphi \rangle_{W^{-m,p}(\Omega), W^{m,p'}(\Omega)} = \langle \widehat{f}, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \text{ for any } \varphi \in \mathcal{D}(\Omega).$$

Thus if $\varphi \in \mathcal{D}(\Omega)$, then $\langle \widehat{f}, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$. This implies that $f = \widehat{f} \in W^{-m,p}(\Omega)$.

Now we derive the estimate (6.1). Choose $\varphi_1 \in \mathcal{D}(\Omega)$ such that $\int_\Omega \varphi_1 dx = 1$. For any $\varphi \in \mathcal{D}(\Omega)$, define

$$\varphi_0 = \varphi - \left(\int_\Omega \varphi dx \right) \varphi_1 \in \dot{\mathcal{D}}(\Omega). \tag{6.2}$$

Then using the Hölder inequality, we have

$$\begin{aligned} \|\varphi_0\|_{W^{m,p'}(\Omega)} &\leq \|\varphi\|_{W^{m,p'}(\Omega)} + \int_\Omega |\varphi| dx \|\varphi_1\|_{W^{m,p'}(\Omega)} \\ &\leq \|\varphi\|_{W^{m,p'}(\Omega)} + |\Omega|^{1/p} \|\varphi\|_{L^{p'}(\Omega)} \|\varphi_1\|_{W^{m,p'}(\Omega)} \\ &\leq (1 + |\Omega|^{1/p} \|\varphi_1\|_{W^{m,p'}(\Omega)}) \|\varphi\|_{W^{m,p'}(\Omega)} \\ &= C_1 \|\varphi\|_{W^{m,p'}(\Omega)}. \end{aligned} \tag{6.3}$$

For any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$,

$$|\langle f, \operatorname{div} \boldsymbol{v} \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| = |\langle \boldsymbol{\nabla} f, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}| \le \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)}.$$

By Theorem 6.1, there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = \varphi_0$ in Ω and

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)} \le C\|\varphi_0\|_{W_0^{m,p'}(\Omega)} \le CC_1\|\varphi\|_{W_0^{m,p'}(\Omega)}.$$

By (6.2), we have

$$\begin{aligned}
\langle f, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= \langle f, \varphi_0 \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_\Omega \varphi dx \langle f_1, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\
&= \langle f, \operatorname{div} \boldsymbol{v} \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_\Omega \varphi dx \langle f_1, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\
&= -\langle \boldsymbol{\nabla} f, \boldsymbol{v} \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \int_\Omega \varphi dx \langle f_1, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.
\end{aligned}$$

Thereby,

$$\begin{aligned}
|\langle f, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| &\le \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{m+1,p'}(\Omega)} \\
&\quad + |\Omega|^{1/p} \|\varphi\|_{L^{p'}(\Omega)} |\langle f, \varphi_1 \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \\
&\le CC_1 \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} \|\varphi\|_{W_0^{m,p'}(\Omega)} \\
&\quad + |\Omega|^{1/p} \|\varphi\|_{W_0^{m,p'}(\Omega)} |\langle f, \varphi_1 \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \\
&\le C_2(\|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p}(\Omega)} + 1) \|\varphi\|_{W_0^{m,p'}(\Omega)}.
\end{aligned}$$

Hence (6.2) holds. □

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