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Existence of Measurable Multivalued Mapping

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ABSTRACT

This paper investigates the existence of measurable multivalued mapping by using the concept of Lusin properties and p-continuous mapping.

Keywords: p - continuous mapping, Lusin properties.

1. INTRODUCTION

CASTAING [3] JACOBS [6], HIMMELBERG [5], ROCKAFELLER [11] and many others tried to highlight the different aspects of measurability of multifunctions. They have tried to show the existence of measurable multivalued function depend up on

(i) - Nature of the domain of multifunction as the measurable space

(ii) - Nature of the co-domain of multifunction which is in general may be any topological space.

(iii) - Nature of the multivalued function.

Based on these condition, different form of measurable multivalued mapping have been defined by various authors such as CASTAING [3], HIMMELBERG [5] and MICHAEL [10] etc. MICHAEL also tried to show the different ways of defining the continuity of multivalued mapping. JACOBS [6] has investigated a relationship between continuity and measurability of multivalued mapping under different situation. In his paper JACOBS [6], has taken the topological space as polish space as well as applied the different form of Lusin - properties to establish the existence of measurable multivalued function. Motivated by his investigation, we have tried to generalise the existence of measurable multivalued function by taking the topological space as locally compact metric space with countable base.

2. PRELIMINARIES

2.1 - Let T be any non empty set equipped with σ- algebra M and X be any topological space, then any mapping $\Omega : T \rightarrow X$ which assigns for any $t \in T$, a subset of X is called multifunction.

If for each closed subset B of X, the set

$$\Omega^{-1} (B) = \{ t \in T : \Omega (t) \cap B \neq \phi\} \in M$$

then the multifunction F is said be measurable.

It is said to be weak measurable if

$$\Omega^{-1} (G) = \{t \in T : \Omega(t) \cap G \neq \phi\} \in M \quad (1)$$

for every open subset G of X.

2.2 - Let (X, ρ) be any metrie space, the uniformity on X determined by ρ is $I^{\rho} = (J_{\varepsilon}^{\rho} | \in > 0)$

where $J_{\varepsilon}^{\rho} = \{(x, y) \in X \times X \mid \rho(x, y) < \varepsilon\}$ The uniformity I^{ρ} on X, determines a uniformity $2^{[\rho]}$ on 2^x.

Let $w (J_{\varepsilon}^{\rho}) = \{(A, B) \in 2^x \times 2^x \mid J_{\varepsilon}^{\rho}[A] \supset J_{\varepsilon}^{\rho} [B] \supset A\}$.

Then the uniformity $2^{[\rho]}$ is $\{w(J_{\varepsilon}^{\rho}) \mid \varepsilon > 0\}$

The topology on 2^x determined by $2^{[\rho]}$ is called uniform topology determined by ρ.

2.3 - Let $t_0 \in T$ and let $\tau(t_0)$ denoted the filter base at t_0 consisting of all $s_\varepsilon(t)$, $\in > 0$

where $s_\varepsilon (t_0) = \{t \in T \mid d (t, t_0) < \varepsilon\}$

Then the grill of $\tau(t_0)$ denoted by $\tau''(t_0)$ and consist of all sets $S''(t_0)$ contained in T s.t. $S''(t_0) \cap s_\varepsilon (t_0) \neq \phi$ for every $\varepsilon > 0$

Let Ω: $T \rightarrow 2^x$ be a mapping then pseudo limit superior of Ω as $t \rightarrow t_0$ (Abbr. $\operatorname{plimsup}_{t \to t_0} \Omega(t)$) is defined to be -

$$\bigcap_{s\varepsilon(t_0) \in \tau(t_0)} cl \left[\bigcup_{t \in s\varepsilon(t_0)} \Omega(t) \right]$$

and pseudo-limit inferior of Ω as $t \rightarrow t_0$ (Abbr. p $\operatorname{limsup}_{t \to t_0} \Omega(t)$) defined to be

$$\bigcap_{s''\varepsilon(t_0) \in \tau(t_0)} cl \left[\bigcup_{t \in s''\varepsilon(t_0)} \Omega(t) \right]$$

2.4- semi Let $\Omega : T \to 2^x$ be any multivalued mapping. Then it is called pseudo upper continuous (Abbr : p-usc) at $t_0 \in T$ if $p-\limsup_{t \to t_0} \Omega(t) \subset \Omega(t_0)$.

Similerly it is also called pseudo- lower semi continuous

(Abbr : p-lsc) at $t_0 \in T$ if $\Omega(t_0) \subset p\liminf_{t \to t_0} \Omega(t)$.

Again mapping Ω is called pseudo-continuous (Abbr : p-continuous) at $t_0 \in T$ if Ω is p-usc as well as p-lsc.

Now we state three Lusin properties to be required in our main result.

1. Lusin-c_p-property : For every $\varepsilon > 0$ $\exists$ an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) <$ □ and $\Omega \mid T / E_\varepsilon$ isp-continuous.

2. Lusin-c_u-property : Let 2^x have the uniform topology determined by ρ. For every $\varepsilon > 0$ $\exists$ an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and $\Omega | T / E_\varepsilon$ is continuous.

3 Lusin-c_f property: Let 2^x have the finite topology determined by ρ. For every $\varepsilon > 0$ $\exists$ an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and $\Omega | T / E_\varepsilon$ is continuous.

2.5. A family of mapping $\{f_\alpha : \alpha \in A, f_\alpha : T \to X\}$ is called almost equicontinuous if for every $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and s.t.$\{f_\alpha | T/E_□ : \alpha \in A)$ is an equicontinuous.

We state following lemmas without proof but to be required in our main result-

Lemma-1 : LetΩbe a mapping $\Omega : T \to 2^x$. Then a necessary and sufficient condition that Ω be p-usc at each point ofT is that if $< x_n >$ and $< t_n >$ are sequences in X and T respectively such that $x_n \in \Omega(t_n)$ for every n and such that $x_n \to x$ and $t_n \to t$ as $n \to \infty$, then $x \in \Omega(t)$

Lemma - 2 : A mapping $f_A : X \to R$ defined by $f_A(X) = \rho(x, A)$ is $|f_A(x) - f_A(y)| \leq \rho(x,y)$ $\forall$ x, y $\in$A Where Ais non empty subset of X.

Lemma - 3 : Let X be any separable space; and F be a measurable mapping from T to A (X), then for every $\varepsilon > 0 \exists$ an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and restriction of the mapping $(t, x) \to (x, F(t))$ to $(T/E_\varepsilon) \times X$ is continuous.

Lemma - 4 : Let X be a polish space and mapping $\Omega : T \to 2^x$ has lusin c_u- property then Ω is measurable.

3. MAIN RESULT

Theorem 3.1 : Let X be a locally compact metric space with countable base and T be a metric space. Let a multifunction $F : T \to A(X)$ be a compact values with positive Radon measure. Then we consider the following statement-

(i) Fis measurable;

(ii) Fis C-measurable;

(iii) F is weak measurable;

(iv) Each of mapping $t \to \rho(x, F(t)), x \in X$ is measurable;

(v) $t \to cl(F(t))$ has the Lusin-c_p-property;

(vi) $t \to cl(F(t))$ is measurable;

Then [a] Statement (i) through (iv) are equivalent;

[b] Statement (iii) through (vi) are equivalent;

[C] Statement (i) implies any of the remainning five.

Proof:

Since X is locally compact and has countable base so X is metrizable and σ compact as [2] and that any space which is locally compact, σ-compact and merizable is a polish space.

Hence X is Polish space..............(1)

Moreover X is metric space and every metric space is Housdraff space so any compact subset of X will be closed.

As per Himmelberg [5], X is σ-compact, therefore any multivalued mapping F with compact values will satisfy the equivalence given by (i) through (iv)

Thus [a] holds.

To prove [b] we will take the following cycle order

(iii) ⇒ (iv) ⇒ (v) ⇒(vi) ⇒ (iii)

(iii) ⇒ (iv) are equivalent by [a]

Now we will show (iv) ⇒(v).

Let the mapping $t \to \rho(x, F(t)), x \in X$ is measurable.

Since for each $(t,x) \in T \times X$, $\rho(x, F(t)) = \rho(x, cl(F(t)))$

$\Rightarrow$ the mapping $t \rightarrow \rho(x, F(t))$, is measurable for every $t \in T$.

Then by lemma - 3, for any $\varepsilon > 0$ $\exists$ an open set $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and the mapping

$(t, x) \rightarrow \rho(x, F(t))$; $(t, x) \in (T \setminus E_\varepsilon \times X)$ is continuous.

Suppose $\tau_\varepsilon(t)$ is neighbourhood filter base at $t \in T/E_\varepsilon$ and $\tau''_\varepsilon(t)$ is grill of neighbourhood filterbase of $\tau_\varepsilon(t)$ also $< x_n >$ and $< t_n >$ be sequences in X and $T \setminus E_\varepsilon$ respectively such that $x_n \in cl(F(t_n))$ for each n and $x_n \rightarrow x$ and $t_n \rightarrow t$.

Then by continuity property, $\rho(x_n, cl(F(t_n))) = 0 \rightarrow \rho(x_n, cl(Ft_n))) \Rightarrow x \in cl(F(t))$

By using the lemma (1), the mapping $t \rightarrow cl(F(t)$ is p-usc at each $t \in T \setminus E_\varepsilon$

Again we take $t_0 \in T/E_\varepsilon$ and choose $x_0 \in cl(F(t_0))$ and $S''(t_0) \in \tau_\varepsilon''(t_0)$

then $\rho(x, cl(F(t_0)) \geq \rho(x, cl\, [\cup_{s \in S''(t_0)} cl(F(t_n))]) \ \forall\ (t, x) \in S''(t_0) \times X$

If we select an open sphere $s_{1/n}(t_0) \in \tau_\varepsilon(t_0)$ for n = 1, 2, 3,.........

Then an element $t_n \in S''(t_0) \cap S_{1/n}(t_0)$ for n = 1, 2, 3,.........s.t.

a sequence $< t_n >$ converges to t_0.

Moreover if $< x_n >$ be any sequence in X s.t. $x_n \rightarrow x_0$,

then we have, $\rho(x_0, cl(F(t_0)) = 0 \Rightarrow \lim \rho(x_n, cl(F(t_n))$

$\geq \lim \rho(x_n, cl[\cup_{s \in S''(t_0)} cl(F(s))])$

$= \rho(x_n, cl[\cup_{s \in S''(t_0)} \ cl(F(s))]) > = 0$

$\Rightarrow x_0 \in cl\, [\cup_{s \in S''(t_0)} \ cl(F(s))]$

This proves that $cl(F(t_0)) \subset$p- $\lim\limits_{t \rightarrow t_0} - \inf cl(F(t_0))$

$\Rightarrow$ the mapping $t \rightarrow cl(F(t))$ is p-lsc at each $t \in T \setminus E_\varepsilon$

It proves that the mapping $t \rightarrow cl(F(t))$ is p-usc as well as p-lsc at each $t \in T \setminus E_\varepsilon$

$\Rightarrow$ the mapping $t \rightarrow cl(F(t))$ is p-continuous at every $t \in T \setminus E_\varepsilon$.

Hence the mapping $t \rightarrow cl(F(t))$ has Lusin-c_p property.

we conclude (iv) $\Rightarrow$ (v)

Now we will show (v) $\Rightarrow$ (vi)

If $<T_n>$ be the sequence of compact subsets of T and N be the set of measure zero contained in T s.t. clF(t)/Tn is p - continuous for each n and $\cup_{n=1}^{\infty} T_n = T\backslash N$

Then by applying the lemma (2), the mapping $t \rightarrow cl(F(t))/T_n$ has a closed graph,

$\Rightarrow$ $cl(F(t))/T_n$is measurable for every n.

Now we define a sequence of mapping $F_n^* : T \rightarrow 2^x \cup \{\phi\}$

such that $F_n^*(t) = \begin{cases} clF(t)) & ;\ \text{if } t \in T_n \\ \phi, & ;\ \text{if } t \notin T_n \end{cases}$

Then $F_n^*(t)$ is measurable for each n

Since $clF(t)) = \cup_{n=1}^{\infty} F_n^*(t)$

$\Rightarrow$ the mapping $t \rightarrow cl(Ft))$ is measurable

we conclude (v) $\Rightarrow$(vi)

Again F is compact valued

$\Rightarrow$F(t) will be closed.

$\Rightarrow$ clF(t) = F (t)

Thus F is measurable

Hence (vi) = (i)

since (i) $\Rightarrow$ (iii) has already shown in [a].

Hence [b] is holds.

Condition [c] can be shown by the following scheme-

(i) $\Rightarrow$ (ii); (i) $\Rightarrow$ (iii); (i) $\Rightarrow$ (iv) according as Himmelberg [5].

and (i) $\Rightarrow$(iv); (iv) $\Rightarrow$(v); implies (i) $\Rightarrow$(v)

and (i)$\Rightarrow$ (v); (v) $\Rightarrow$(vi); implies (i) $\Rightarrow$ (vi)

Thus statement (i) implies any of the remainning five.

Hence the condition [c] holds.

Theorem - 3.2 : Let X be locally compact metric space with countable base and T be the metric space with +ve Radon measure and Ω: T $\rightarrow 2^x$ be any multivalued mapping. Then thereexista metric ρ_{∞} on X s.t. the two topologies $\tau(\rho)$ and $\tau(\rho_{\infty})$ coincide and statements-

(i) Ω is measurable

(ii) Ω has the Lusin c_u-property, when 2^x has the uniform topology determined by ρ_{∞} are equivalent.

Proof:

Since X is locally compact metric space with countable base so that by using condition (1) of Theorem 3.1, X will be polish space. By lemma 4, if Ω has the Lusinc_u-property.

$\Rightarrow\Omega$ is measurable.

Thus (ii) $\Rightarrow$ (i) holds

It remains to prove (i) $\Rightarrow$ (ii)

Let Ω is measurable multivalued mapping

By corollary 2.1 of JACOBS [6] if X is taken as compact;

theorem holds.

Moreover, we assume X is not compact

Let X_{∞} denote the one point compactification such as X_{∞}.= XU$\{\infty\}$

$\Rightarrow X_{\infty}$ is metrizable by Dugundiji [4]

$\Rightarrow X_{\infty}$ is complete with respect to metric ρ_{∞} (say) defined

on the topology of X_{∞} and also defining the topology X.

SinceΩ is measurable and let Ω_{∞}: $T\rightarrow A(X_{\infty})$ is s.t. Ω_{∞} (t) is the image of Ω(t) under inclusion mapping i_{∞} : $X\subset X_{\infty}$. Let G_{∞} is be an open subset of X_{∞}

ThenΩ_{∞}^{-1} $(G_{\infty}) = \Omega_{\infty}^{-1}$ $(G_{\infty}\backslash\{\infty\})$

since $G_{\infty}\backslash\{\infty\}$ is open in X and Ω: $T\rightarrow 2^x$ is measurable

$\Rightarrow\Omega_{\infty}^{-1}$ (G_{∞}) is measurable

Therefore by (v) $\Leftrightarrow$(vi) of theorem 3.1

the mapping $t\rightarrow\rho_{\infty}$ (x, $\Omega_{\infty(t)}$ (t)) = ρ_{∞} (x, cl(Ω_{∞}(t)) for each$t \in T$, $x \in X$ are measurable.

For $\varepsilon > 0 \exists$ an open subsct $E_\varepsilon \subset T$ s.t. $\mu(E_\varepsilon) < \varepsilon$and the mapping

$t \to \rho_\infty (x, \Omega_\infty(t)) = \rho_\infty(x, cl(\Omega_\infty(t)); x \in X$

when restricted to T|Eε are equicontinuous.

Let $I^\infty = \{J_\delta^\infty | \delta > 0\}$ where $J_\delta^\infty = \{(x, y) \in X_\infty \times X_\infty \, | \rho_\infty(x, y) < \delta\}$

denote the uniformity which determines the topology of X_∞

Then $I^\infty \cap (X \times X) = \{ J_\delta^\infty \cap (X \times X) | \delta > 0\}$ determines the topology of X .

For $\delta > 0$ $\exists \beta > 0$ s.t., $t, t' \in T| E\varepsilon$, $d(t, t') < \beta \Rightarrow | \rho_\infty (x, \Omega_\infty (t)) - \rho_\infty (x, \Omega_\infty(t')) | < \delta$, $x \in X_\infty$

Thus $t, t' \in T \,|\, E\varepsilon$

$d(t, t') < \beta \Rightarrow [J_\delta^\infty \cap (X \times X)]\ \Omega(t') \supset \Omega (t)$ and $[J_\delta^\infty \cap (X \times X)]\ [\Omega(t)] \supset \Omega (t')$

$\Rightarrow \Omega$ has the Lusinc$_u$ property, when 2^x has the uniform topology determined by ρ_∞

Theorem - 3.3 : Let X be locally compact metric space with countable base and T be a metric space with +ve Radon measure. If a multivalued mapping $\Omega : T \to 2^x$ has the Lusin-c_u property then Ω is measurable.

Proof : The space X is polish by theorem 3.1.

Let N be the set of measure zero contained in T and $< T_n >_{n=1}^{\infty}$be the sequence of compact subsets of T s.t. $\cup_{n=1}^{\infty} T_n = T \backslash N$.

Let 2^x has the uniform topology determined by ρ and mapping $\Omega \backslash Tn : Tn \to 2^x$ are continuous.

$\Rightarrow$mapping F **|** Tn are p-usc

Result follows by condition (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (i) of theorem 3.1

Theorem - 3.4 : Let X be locally compact metric space with countable base and T be metric space with positive Radon measure and a multivalued mapping Ω: $T \to 2^x$ be measurable.

Let f be a continuous mapping f : TxX$\to$Y where Y is Hausdorff space and f_y be a measurable mapping $f_y : T \to Y$ s.t. $f_y (t) \in f (t, \Omega, (t))$ for every $t \in T$. Then there exist a mapping.

f_y: $T \to X$ s.t. $f_x(t) \in \Omega(t)$ and $f_y, (t) = f (t, f_x,(t))$ for each $t \in T$.

Proof : The space X is polish by theorem 3.1

We define a mapping $F : T \to 2^x$ by $F(t) = \{x \in \Omega(t) | f(t, u) = f_y(t)\}$

Since f is continuous, so F(t) is closed valued.

Thus F is measurable.

For each $\varepsilon > 0$ $\exists$ an open set $E\varepsilon \subset T$ s.t. $\mu(E\varepsilon) < \varepsilon$ is satisfying

(a) $\Omega(T|E\varepsilon)$ is p-continuous by theorem 3.1

(b) $f_y(T|E\varepsilon)$ is continuous by proposition of Bourbaki [2].

It follows by lemma 1, F is p-usc on $T|E\varepsilon$

$\therefore$ $F(T|E\varepsilon)$ is measurable and $\mu(E\varepsilon)<\varepsilon$ by lemma 3.2 of CASTING [3].

Since $\varepsilon > 0$ is arbitrary, so that F is measurable.

By using theorem 1 of K. Kuratowski & Ryll-Nardzewski [9].

there exist a measurable mapping $f_x.:T \to X$ s.t. $f_x(t) \in \Omega(t)$ and $f_y(t) = f(t, f_x(t))$ for eacht$\in$ T

Hence the theorem.

Corollary 3.5 - Let X be locally compact metric with countable base and (Y, δ) be any metric space, also T be the metric space with positive Radon measure and $\Omega : T \to 2^x$ be a measurable mapping. Again let $f : T \times X \to Y$ be a mapping s.t. -

(i) the mapping $t \to f(t, x)$, $x \in X$ are measurable

(ii) the mapping $t \to f(t, x)$ $t \in T$ are locally uniformly continuous

let $f_y : T \to Y$ be a measurable mapping s.t. $f_y(t) \in f(t, \Omega(t))$ for every $t \in T$. Then there exist a measurable mapping $f_x : T \to X$ s.t. $f_x(t) \in \Omega(t)$ and $f_y(t) = f(t, f_x(t))$ for every $t \in T$.

Proof : Since by condition 1 of theorem 3.1, the space X is polish so that result follows by theorem 2.5 of JACOBS [6].

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