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Singularities of Algebraic Curve $f(y) = \lambda f(x)$ and Invariants Associated with Each Singular Point Over Finite Field

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Abstract

Let $K = F_q$ be a finite field of order q, (q is a power of a prime p), and $\mathcal{K}$ be an algebraically closed field extension of k. Let $f(t)$ be a monic polynomial of degree n in $\mathcal{K}[t]$. In this paper, we give an algorithm to identify the singularities of the projective curve of the affine curve $H_\lambda; f(y) - \lambda f(x) = 0$ for which $\lambda \neq 0$ in K. The curve H_λ is a general form of the Holm Curve was introduced by ALEANDAR HOLM [6]. As result, we determine types and multiplicities of the singular points, and calculate Milnor number associated with each singularity.

Keywords: Algebraic Curve, Singular Points, Finite Field.

INTRODUCTION

Consider the Holm Curve

$$by(y^2 - 1) = ax(x^2 - 1)$$

was introduced by ALEXANDER HOLM [6], where $a, b \in K,\ ab \neq 0, a \neq \pm b$. If we put $\lambda = \frac{a}{b}$, the Holm's curve becomes

$$y(y^2 - 1) = \lambda x(x^2 - 1)$$

$$y^3 - y = \lambda(x^3 - x)$$

where $\lambda \neq 0, \pm 1$. Suppose $f(t) = t^3 - t,$ then we can write Holm's Curve as follow;

$$f(y) - \lambda f(x) = 0 \text{ of degree3.}$$

Let $\mathcal{K}$ be an algebraically closed field. Our goal to study the singularities of the curve H_λ: $f(y) - \lambda f(x) = 0$ for $\lambda \in \mathcal{K}^*$, and other topics related to for $f(t) \in \mathcal{K}[t]$, a monic polynomail of degree $n \geq 2$.The projective plane model for H_λ is given by

$$\mathcal{H}_\lambda: z^n \left[f\left(\frac{y}{z}\right) - \lambda f\left(\frac{x}{z}\right)\right] \in \mathcal{K}[x, y, z]$$

then $F(x, y, z) = z^n \left[f\left(\frac{y}{z}\right) - \lambda f\left(\frac{x}{z}\right)\right]$ is a homogeneous polynomial of degree n.The sengularities of both projective curve $\mathcal{H}_\lambda$ and affine curve H_λ are given respectively as follow:

$$\mathcal{H}_\lambda = \{(x; y; z) \in \mathcal{K}^3: F(x, y, z) = 0\}$$

$$H_\lambda = \{(x; y; 1) \in \mathcal{K}^3: F(x, y, 1) = 0\}$$

1 SINGULAR POINTS

Theorem 1.1 *Let H_λ be the projective plane model of the affine curve H_λ. Then, H_λ has no singularity at infinity.*

Proof. Let $f(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$ of degree $n \geq 2$ where $a_{n-1}, a_{n-2}, \ldots, a_1, a_0 \in \mathcal{K}$, and let $\lambda \in \mathcal{K}^*$. The curve $\mathcal{H}_\lambda$ is defined by the equation

$$F(x, y, z) = z^n f\left(\frac{y}{z}\right) - \lambda z^n f\left(\frac{x}{z}\right) = 0$$

The partial derivatives F_x, F_y, F_z are given by

$$F_x = -\lambda z^{n-1} f'\left(\frac{x}{z}\right)$$

$$F_y = z^{n-1} f'\left(\frac{y}{z}\right)$$

$$F_z = nz^{n-1}\left[f\left(\frac{y}{z}\right) - \lambda f\left(\frac{x}{z}\right)\right] + z^n\left[-\frac{y}{z^2} f'\left(\frac{y}{z}\right) + \lambda \frac{x}{z^2} f'\left(\frac{x}{z}\right)\right]$$

Using the explicit expression for $f(t)$ we find

$$F_x = -\lambda[nx^{n-1} + (n-1)a_{n-1}zx^{n-2} + \ldots + 2a_2 z^{n-2}x + a_1 z^{n-1}]$$

$$F_y = ny^{n-1} + (n-1)a_{n-1}zy^{n-1} + \ldots + 2a_2 z^{n-2}y + a_1 z^{n-1}$$

$$F_z = a_{n-1}y^{n-1} + 2a_{n-2}zy^{n-2} + \ldots. + (n-1)z^{n-2}y + na_0 z^{n-1}$$

$$\lambda[a_{n-1}x^{n-1} + 2a_{n-2}zx^{n-2} + \ldots. + (n-1)z^{n-2}x + na_0 z^{n-1}]$$

to find the singularities of the curve $\mathcal{H}_\lambda$ we solve the system

$$F = F_x = F_y = F_z = 0$$

to study the singularity at infinity, we put $z = 0$ and the system becomes

$$x = y = z = 0$$

but the point $(0; 0; 0)$ does not exist in the projective plane $\mathbb{P}^2$ hence, $\mathcal{H}_\lambda$ has no singularity at infinity. Next, for $\lambda \in \mathcal{K}^*$ we study affine singularity for the curve $\mathcal{H}_\lambda$. For this purpose, we let $z = 1$ and consider the curve H_λ using the polynomial

$$F(x, y, 1) = f(y) - \lambda f(x)$$

as an abuse of notation, we write

$$F(x, y) = f(y) - \lambda f(x)$$

the singular points on H_λ are obtained by solving the system

$$f'(x) = 0$$

$$f'(y) = 0$$

$$f(y) - \lambda f(x) = 0$$

Let S_λ be the set of singular points on the affine curve H_λ. Since there is no singularity at infinity, S_λ is the set of singular points on $\mathcal{H}_\lambda$.Let R denote the set of roots of $f(t)$ and R' that of $f'(t)$. Let

$$T = (R \cap R') \times (R \cap R')$$

and let

$$T_\lambda = \left\{(\alpha, \beta) \in (R' - R) \times (R' - R): \frac{f(\beta)}{f(\alpha)} = \lambda\right\}$$

Then we have the following theorem;

Theorem 1.2 *For every* $\lambda \in \mathcal{K}^*$, $S_\lambda = T_\lambda \cup T$

Proof. Let $\lambda \in \mathcal{K}^*$ and let $(\alpha, \beta) \in T$ then

$$f'(\alpha) = 0$$

$$f'(\beta) = 0$$

$$f(\beta) - \lambda f(\alpha) = 0$$

therefore $(\alpha, \beta) \in S_\lambda$ hence $T \subset S_\lambda$.Let $(\alpha, \beta) \in T_\lambda$ then

$$f'(\alpha) = 0$$

$$f'(\beta) = 0$$

$$f(\alpha) \neq 0$$

$$f(\beta) \neq 0$$

$$f(\beta) = \lambda f(\alpha)$$

therefore $(\alpha, \beta) \in S_\lambda$ hence $T_\lambda \subset S_\lambda$.Conversely, suppose $(\alpha, \beta) \in S_\lambda$ then

$$f'(\alpha) = 0$$

$$f'(\beta) = 0$$

$$f(\beta) = \lambda f(\alpha)$$

If $f(\alpha) = f(\beta) = 0$, then $(\alpha, \beta) \in T$ and if $f(\alpha) \neq 0$ and $f(\beta) \neq 0$ then, $(\alpha, \beta) \in T_\lambda$. Hence $S_\lambda \subset T_\lambda \cup T$. Explicitly, Theorem 1.2. says that if α, β are common roots of $f(t)$ and $f'(t)$ then (α, β) and (β, α) are singular points on H_λ for any $\lambda \in \mathcal{K}^*$. Moreover, if α, β are roots of $f'(t)$ but not roots of $f(t)$ then (α, β) is a singular point on H_λ for $\lambda = \frac{f(\beta)}{f(\alpha)}$. Every singular point on H_λ is obtained in this fashion.

Theorem 1.3 *Let D be the discriminant of $f(t)$*

1. *If $D = 0$, then for every $\lambda \in K^*$, H_λ is a singular curve*
2. *If $D \neq 0$ and $\lambda \notin \left\{\frac{f(\beta)}{f(\alpha)} : \alpha, \beta \in R'\right\}$ then H_λ is a non-singular curve*

Proof.

1. If $D = 0$ then $T \neq \varnothing$. Let $\alpha, \beta \in T$ ($\alpha = \beta$ is allowed), then for any $\lambda \in \mathcal{K}^*$

$$f'(\alpha) = 0$$

$$f'(\beta) = 0$$

$$f(\beta) = \lambda f(\alpha) = 0$$

hence $(\alpha, \beta) \in S_\lambda$ and H_λ is a singular curve.

2. If $D \neq 0$ then $T = \varnothing$. Suppose $\lambda \notin \left\{\frac{f(\beta)}{f(\alpha)} : \alpha, \beta \in R'\right\}$. If (α, β) were a singular point on H_λ then $\alpha, \beta \in R'$ and $\frac{f(\beta)}{f(\alpha)} = \lambda$ which is a contradiction, hence H_λ is non-singular.

Example 1 *Let $f(t) = t^3 + at + b$, where $a, b \in \mathcal{K}^*$, an algebraically closed field and let $\lambda \in \mathcal{K}^*$.Consider the projective curve*

$$\mathcal{H}_\lambda : y^3 + ayz^2 + bz^3 = \lambda(x^3 + axz^2 + bz^3)$$

Let $F = y^3 + ayz^2 + bz^3 - \lambda(x^3 + axz^2 + bz^3)$, then the curve $\mathcal{H}_\lambda$ is defined in $\mathbb{P}^2$, by

$$F(x, y, z) = 0$$

by Theorem 1.1 $\mathcal{H}_\lambda$ has no singularity at infinity.let H_λ be the affine curve whose projective plane is $\mathcal{H}_\lambda$. The singularities will be on the curve H_λ.

$$f'(t) = 3t^2 + a$$

the set R' of roots of $f'(t)$ is

$$R' = \left\{\pm\sqrt{\frac{-a}{3}}\right\}$$

Let

$$D = -4a^3 - 27b^2$$

be the discriminant of $f(t)$.

1.

Case 1 $D = 0$:

$$f\left(\sqrt{\frac{-a}{3}}\right) = \left(\sqrt{\frac{-a}{3}}\right)^3 + a\sqrt{\frac{-a}{3}} + b = b - \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}} = b - \frac{2\sqrt{3}}{9}\left(\frac{27}{4}b^2\right)^{1/2} = 0$$

$$f\left(-\sqrt{\frac{-a}{3}}\right) = \left(-\sqrt{\frac{-a}{3}}\right)^3 + a\left(-\sqrt{\frac{-a}{3}}\right) + b = b + \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}}$$

$$= b + \frac{2\sqrt{3}}{9}\left(\frac{27}{4}b^2\right)^{1/2} = 2b \neq 0$$

Therefore,

$$R' \cap R = \left\{\sqrt{\frac{-a}{3}}\right\}$$

$$T = \left\{\left(\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right)\right\}$$

Next,

$$T_\lambda = \varnothing, \text{if} \lambda \neq 1$$

$$T_1 = \left\{\left(-\sqrt{\frac{-a}{3}}, -\sqrt{\frac{-a}{3}}\right)\right\}$$

we conclude, in this case that

$$S_\lambda = \left\{\left(\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right)\right\}, \quad \text{if} \lambda \neq 1$$

$$S_1 = \left\{\left(\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right), \left(-\sqrt{\frac{-a}{3}}, -\sqrt{\frac{-a}{3}}\right)\right\}$$

Case 2 $D \neq 0$, *with* $ab \neq 0$,*In this case*

$$T = \varnothing$$

Let

$$\lambda_1 = \frac{b - \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}}}{b + \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}}} = \frac{f\left(\sqrt{\frac{-a}{3}}\right)}{f\left(-\sqrt{\frac{-a}{3}}\right)} \neq 0$$

$$\lambda_2 = \frac{b + \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}}}{b - \frac{2}{9}\sqrt{3}(-a)^{\frac{3}{2}}} = \frac{f\left(-\sqrt{\frac{-a}{3}}\right)}{f\left(\sqrt{\frac{-a}{3}}\right)} = \frac{1}{\lambda_1} \neq 0$$

since $ab \neq 0, \lambda_1 \neq \lambda_2$Then, we can conclude that

$$S_{\lambda_1} = T_{\lambda_1} = \left\{\left(-\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right)\right\}$$

$$S_{\lambda_2} = T_{\lambda_2} = \left\{\left(\sqrt{\frac{-a}{3}}, -\sqrt{\frac{-a}{3}}\right)\right\}$$

$$S_\lambda = \varnothing \text{if} \lambda \neq \lambda_1, \lambda_2$$

Case 3 $D \neq 0, \ a = 0$

$$f(t) = t^3 + b, \ b \neq 0$$

$$f'(t) = 0$$

$$R' = \{0\}$$

In this case

$$T = \varnothing$$

$$S_1 = T_1 = \{(0,0)\}$$

$$S_\lambda = \varnothing, \ \text{if} \lambda \neq 1$$

Case 4 $D \neq 0,\ b = 0$

$$f(t) = t^3 + at,\ a \neq 0$$

$$f'(t) = 3t^2 + a$$

$$R = \{0, \pm\sqrt{-a}\}$$

$$R' = \left\{\pm\sqrt{\frac{-a}{3}}\right\}$$

in this case

$$T = \varnothing$$

$$S_1 = T_1 = \left\{\left(\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right), \left(-\sqrt{\frac{-a}{3}}, -\sqrt{\frac{-a}{3}}\right)\right\}$$

$$S_{-1} = T_{-1} = \left\{\left(\sqrt{\frac{-a}{3}}, -\sqrt{\frac{-a}{3}}\right), \left(-\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right)\right\}$$

$$S_\lambda = \varnothing, \text{if} \lambda \neq \pm 1$$

2 TYPES OF SINGULARITIES

The singular points that are under study are all affine points on H_λ. Let the equation of the curve be given by

$$F(x, y) = f(y) - \lambda f(x) = 0$$

Proposition 2.1 *Let* (α, β) *be a singular point on the curve* H_λ, *then*

1. *If* $F_{xx}(\alpha).F_{yy}(\beta) \neq 0$. *then the singular point* (α, β) *is a node.*
2. *If one of* $F_{xx}(\alpha)$,$F_{yy}(\beta)$ *is zero. then* (α, β) *is a cusp.*
3. *If both* $F_{xx}(\alpha), F_{yy}(\beta)$ *are zero, then* (α, β) *is a triple point.*

Proof. Assume (α, β) is a singular point on the curve

$$F(\alpha, \beta) = F_x(\alpha, \beta) = F_y(\alpha, \beta) = 0$$

Explicitly, in terms of $f(t)$

$$f'(\alpha) = 0$$

$$f'(\beta) = 0$$

$$f(\beta) - \lambda f(\alpha) = 0$$

Because the mixed derivatives are all zeros, the Taylor expansion of $F(x,y)$ at (α,β) is given by

$$F(x,y)=\frac{1}{2!}\left(-\lambda F_{xx}(\alpha)(x-\alpha)^2+F_{yy}(\beta)(y-\beta)^2\right)+\frac{1}{3!}\left(-\lambda F_{xxx}(\alpha)(x-\alpha)^3+F_{yyy}(\beta)(y-\beta)^3\right)+\ldots.$$

we move the singular point (α,β) to the origin using the following substitutions

$$x=\alpha+X \qquad y=\beta+Y$$

F will be replaced by G such that $G(X,Y)=F(\alpha+X,\beta+Y)$, then the Taylor series for G is

$$G(X,Y)=\frac{1}{2!}\left(-\lambda F_{xx}(\alpha)X^2+F_{yy}(\beta)Y^2\right)+\frac{1}{3!}\left(-\lambda F_{xxx}(\alpha)X^3+F_{yyy}(\beta)Y^3\right)+\ldots.$$

1. If $F_{xx}(\alpha).F_{yy}(\beta)\neq 0$, then the tangents $Y=\pm\sqrt{\frac{\lambda F_{xx}(\alpha)}{F_{yy}(\beta)}}X$ are distinct and, hence, the singular point $(0,0)$ is a node, thus (α,β) is a node.
2. If one of $F_{xx}(\alpha),F_{yy}(\beta)$ is zero. then $(0,0)$ is a cusp and has the tangent $X=0$ if $F_{yy}(\beta)=0$,or $Y=0$ if $F_{xx}(\alpha)=0$.thus (α,β) is a cusp.
3. If both $F_{xx}(\alpha),F_{yy}(\beta)$ are zero, then $(0,0)$ is a triple point, thus (α,β) is a triple point.In terms of the polynomial $f(t)$ itself we have the following:

Corollary 2.1 *Let* (α,β) *be a singular point*

1. If $f''(\alpha)f''(\beta)\neq 0$ then (α,β) is a node
2. If one of $f''(\alpha)$, $f''(\beta)$ is zero, then (α,β) is a cusp
3. If $f''(\alpha)$, $f''(\beta)$ are both zero, then (α,β) is a triple point

3 THE MULTIPLICITIES OF THE SINGULAR POINTS

Let (α,β) be a singular point. Moving the singularity to the origin and using the variables x,y, as a change of notation, give

$$G(x,y)=F(\alpha+x,\beta+y)=f(\beta+y)-\lambda f(\alpha+x)$$

$$G(0,0)=f(\beta)-\lambda f(\alpha)=0$$

$$G_x=-\lambda f'(\alpha+x)$$

$$G_x(0,0)=f'(\alpha)=0$$

$$G_y = f'(\alpha + y)$$

$$G_y(0,0) = f'(\beta) = 0$$

The Taylor series in terms of the polynomial $f(t)$ is given by

$$G(x,y) = \frac{1}{2!}(-\lambda f''(\alpha)x^2 + f''(\beta)y^2) + \frac{1}{3!}(-\lambda f'''(\alpha)x^3 + f'''(\beta)y^3) + \ldots + \frac{1}{n!}\left(-\lambda f^{(n)}(\alpha)x^n + f^{(n)}(\beta)y^n\right)$$

which we write as

$$G =_{i=2}^{n} G_i$$

where G_i is the form of degree i

$$G_i = \frac{1}{i!}\left(-\lambda f^{(i)}(\alpha)xi + f^{(i)}(\beta)y^i\right), \quad i = 2,3,\ldots,n$$

Let

$$m_{(0,0)} = \inf\{i: G_i \neq 0\}$$

then $m_{(0,0)}$ is the multiplicity of $(0,0)$ on $G = 0$.Let $m_{(\alpha,\beta)}(F)$ be the multiplicity of (α,β) on the curve $F(x,y) = 0$, then

$$m_{(\alpha,\beta)}(F) = m_{(0,0)}$$

From Theorem 2.2 we have

$$S_\lambda = T_\lambda \cup T$$

For $(\alpha,\beta) \in T$ we have

$$f(\alpha) = f(\beta) = 0$$

$$f'(\alpha) = f'(\beta) = 0$$

Let k_α and k_β be non-negative integers, $0 \leq k_\alpha, k_\beta \leq n$, defined by

$$f^{(k_\alpha)}(\alpha) \neq 0 \text{and} f^{(j)}(\alpha) = 0 \text{for} 0 \leq j \leq k_\alpha - 1$$

$$f^{(k_\beta)}(\beta) \neq 0 \text{and} f^{(j)}(\beta) = 0 \text{for} 0 \leq j \leq k_\beta - 1$$

Then

$$k_\alpha \geq 2$$

$$k_\beta \geq 2$$

For $(\alpha,\beta) \in T_\lambda$ we have

$$f(\alpha) \neq 0$$

$$f(\beta) \neq 0$$
$$f(\beta) - \lambda f(\alpha) = 0$$
$$f'(\alpha) = f'(\beta) = 0$$

Let l_α and l_β be positive integers, $1 \leq l_\alpha, l_\beta \leq n$ defined by

$$f^{(l_\alpha)}(\alpha) \neq 0 \text{and} f^{(j)}(\alpha) = 0 \text{for} 1 \leq j \leq l_\alpha - 1$$
$$f^{(l_\beta)}(\beta) \neq 0 \text{and} f^{(j)}(\beta) = 0 \text{for} 1 \leq j \leq k_\beta - 1$$

Then,

$$l_\alpha \geq 1$$
$$l_\beta \geq 1$$

Theorem 3.1 *Let* $(\alpha, \beta) \in S_\lambda$ *be a singular point with multiplicity* $m_{(\alpha,\beta)}$. *Assume* $char(\mathcal{K}) = 0$ *or* $char(\mathcal{K}) > m$

1. If $(\alpha, \beta) \in T$ then, $m_{(\alpha,\beta)}(F) = \min(k_\alpha, k_\beta)$
2. If $(\alpha, \beta) \in T_\lambda$ then, $m_{(\alpha,\beta)}(F) = \min(l_\alpha, l_\beta)$

Proof. The power series expansion at the origin for $G(x, y)$ is

$$G(x, y) = \frac{1}{2}(-\lambda f''(\alpha)x^2 + f''(\beta)y^2) + HOT$$

where HOT stands for "higher order terms". In the case $char(\mathcal{K}) > m_{(\alpha,\beta)}$, division by any $j \leq m_{(\alpha,\beta)}$ is defined.

1. Suppose $0 \leq k_\alpha \leq k_\beta$. From the definition of k_α we find

$$G(x, y) = \frac{1}{k_\alpha!}\left(-\lambda f^{(k_\alpha)}(\alpha)x^{k_\alpha} + f^{(k_\alpha)}(\beta)y^{k_\alpha}\right) + HOT$$

with $f^{(k_\alpha)}(\alpha) \neq 0,$ hence the multiplicity of $(0,0)$ is

$$m_{(0,0)} = k_\alpha$$

hence$m_{(\alpha,\beta)}(F) = k_\alpha$

2. The case of $k_\beta \leq k_\alpha$ is handled similarly and we conclude that

$$m_{(\alpha,\beta)}(F) = \min(k_\alpha, k_\beta)$$

is proved in a similar fashion.

4 MILNOR NUMBER

The critical points of $F(x, y)$ are the points where both F_x and F_y vanish, hence the set of critical points is $R' \times R'$. We note that the set of singular points is

$$S_\lambda = T \cup T_\lambda = \{(\alpha, \beta) \in R' \times R' : F(\alpha, \beta) = 0\} \subset R' \times R'$$

F has an isolated critical point at (α, β) if (α, β) is isolated point of $R' \times R'$. We say also (α, β) is an isolated singularity of F if (α, β) is isolated of point of S_λ.We fix $(\alpha, \beta) \in S_\lambda$, a singular point. We assume again that $char(\mathcal{K}) = 0$ or $char(\mathcal{K}) > m_{(\alpha,\beta)}(F)$Moving (α, β) to the origin, as before, we obtain the polynomial

$$G(x, y) = \frac{1}{2}(-\lambda f"(\alpha)x^2 + f"(\beta)y^2) + HOT$$

The following Proposition is clear;

Proposition 4.1 *Let $\mathfrak{M}$ be the maximal ideal in $\mathcal{K}[[x, y]]$ then $G(x, y) \in \mathfrak{M}^2$*

Proof. We introduce the Jacobian ideal ([...]

$$J(G) = \langle G_x, G_y \rangle = \langle -\lambda f'(\alpha + x), f'(\beta + y) \rangle = \langle f'(\alpha + x), f'(\beta + y) \rangle$$

which is the ideal in $K[[x, y]]$ generated by the partials. F has an isolated critical point at (α, β). if the Milnor algebra is

$$\mathcal{K}[[x, y]]/J(G) = \mathcal{K}[[x, y]]/\langle G_x, G_y \rangle = \mathcal{K}[[x, y]]/\langle f'(\alpha + x), f'(\beta + y) \rangle$$

We also introduce the Tjurina ideal

$$T(G) = \langle G, G_x, G_y \rangle$$

which is the ideal in $K[[x, y]]$ generated by G and the partials. We have

$$J(G) \subset T(G)$$

The Tjurina algebra is

$$\begin{aligned} \mathcal{K}[[x, y]]/T(G) &= \mathcal{K}[[x, y]]/\langle G, G_x, G_y \rangle \\ &= \mathcal{K}[[x, y]]/\langle f(\beta + y) - \lambda f(\alpha + x), f'(\alpha + x), f'(\beta + y) \rangle \end{aligned}$$

Let

$$\mu(G) = \dim_{\mathcal{K}}\big(\mathcal{K}[[x,y]]/J(G)\big)$$

$$\tau(G) = \dim_{\mathcal{K}}\big(\mathcal{K}[[x,y]]/T(G)\big)$$

$\mu(G)$ is called the *Milnor number* of F at the singular point $(0,0)$.We define the *Milnor number* of (α,β) to be $\mu(G)$ also. $\tau(G)$ is the *Tjurina number* of F at the singular point $(0,0)$. We define the *Tjurina number* of (α,β) to be $\tau(G)$ also. Since $J(G) \subset T(G)$

$$\tau(G) \leq \mu(G)$$

we now calculate the Milnor number of $(\alpha,\beta) \in S_\lambda = T \cup T_\lambda$

Theorem 4.1 *Let* $(\alpha,\beta) \in S_\lambda$ *be a singular point, assume* $char(\mathcal{K}) = 0$ *or* $char(\mathcal{K}) > m_{(\alpha,\beta)}(F)$

1. If $(\alpha,\beta) \in T$, then $\mu(G) = (k_\alpha - 1)\big(k_\beta - 1\big)$
2. If $(\alpha,\beta) \in T_\lambda$, then $\mu(G) = (l_\alpha - 1)\big(l_\beta - 1\big)$

Proof.

1. Suppose $(\alpha,\beta) \in T$ then, the Milnor number

$$\mu(G) = \dim_{\mathcal{K}}\big(\mathcal{K}[[x,y]]/J(G)\big) = \dim_{\mathcal{K}}\big(\mathcal{K}[[x,y]]/\langle G_x, G_y\rangle\big) = I\big(G_x, G_y\big)$$

where $I\big(G_x, G_y\big)$ is the *intersection number* of G_x and G_y at $(0,0)$ (see [Fulton]). With the notation in the proof of Theorem 3.1

$$G(x,y) = \frac{1}{k_\alpha!}\big(-\lambda f^{(k_\alpha)}(\alpha)x^{k_\alpha} + f^{(k_\alpha)}(\beta)y^{k_\alpha}\big) + HOT$$

$$G_x = \frac{-\lambda}{(k_\alpha - 1)!} f^{(k_\alpha)}(\alpha)x^{k_\alpha - 1} + HOT\text{in}x$$

$$G_y = \frac{1}{(k_\beta - 1)!} f^{(k_\beta)}(\beta)y^{k_\beta - 1} + HOT\text{in}y$$

The multiplicity of $(0,0)$ on G_x is $(k_\alpha - 1)$ and the tangent line there is $x = 0$ with multiplicity $(k_\alpha - 1)$. The multiplicity of $(0,0)$ on G_y is $\big(k_\beta - 1\big)$ and the tangent line there is $y = 0$ with multiplicity $\big(k_\beta - 1\big)$.Since G_x and G_y have no common tangent at $(0,0)$, it follows that

$$I\big(G_x, G_y\big) = (k_\alpha - 1)(k_\beta - 1)$$

and therefore the Milnor number at (α, β) is$\mu(G) = (k_\alpha - 1)(k_\beta - 1)$

2. Suppose $(\alpha, \beta) \in T_\lambda$ then a similar argument gives

$$\mu(G) = (l_\alpha - 1)(l_\beta - 1)$$

The following corollary is immediate

Corollary 4.1

1. $\tau(G) < \infty$
2. G has an isolated singularity at $(0,0)$ hence F has an isolated singularity at (α, β) ([Hefez-Rodrigues-Salomao])

Proof. We have defined the Milnor number of an isolated singularity of F, now the total *Milnor number* of F is given as follow

$$\dim_{\mathcal{K}}\left(\mathcal{K}[[x,y]]/J(G)\right) =_{(\alpha,\beta)\in R'\times R'} \mu\left(G,(\alpha,\beta)\right)$$

similarly, the total *Tjurina number of* F:

$$\dim_{\mathcal{K}}\mathcal{K}[[x,y]]/T(G) =_{(\alpha,\beta)\in S_\lambda} \tau\left(G,(\alpha,\beta)\right)$$

For $(\alpha, \beta) \in R' \times R'$, let h_α and h_β be positive integers defined by

$$f^{(i)}(\alpha) = 0, \quad 1 \le i \le h_\alpha - 1 \text{and} f^{(h_\alpha)}(\alpha) \ne 0$$

$$f^{(i)}(\beta) = 0, \quad 1 \le i \le h_\beta - 1 \text{and} f^{(h_\beta)}(\beta) \ne 0$$

As a consequence of Bezout's theorem, we have

Theorem 4.2 *Assume* $char(\mathcal{K}) > max\{h_\alpha, h_\beta : (\alpha, \beta) \in R' \times R'\}$ *then the*

$$_{(\alpha,\beta)\in R'\times R'}(h_\alpha - 1)\left(h_\beta - 1\right) = (n-1)^2$$

Proof. For each $(\alpha, \beta) \in R' \times R'$ we obtain, as before, after moving (α, β) to $(0,0)$

$$I\left((\alpha,\beta), F_x, F_y\right) = I\left((0,0), G_x, G_y\right)$$

Bezout's theorem applied to G_x and G_y gives

$$_{(\alpha,\beta)\in R'\times R'} I\left((\alpha,\beta),F_x,F_y\right)=(n-1)^2$$

but

$$G_x=\frac{-\lambda}{(h_\alpha-1)!}f^{(h_\alpha)}(\alpha)x^{h_\alpha-1}+HOT\text{in}x$$

$$G_y=\frac{1}{(h_\beta-1)!}f^{(h_\beta)}(\beta)y^{h_\beta-1}+HOT\text{in}y$$

Hence, the multiplicity of $(0,0)$ on G_x is $(h_\alpha-1)$ and on G_y it is $\left(h_\beta-1\right)$. The tangents at $(0,0)$ to the two curves are distinct, hence

$$I\left((0,0),G_x,G_y\right)=(h_\alpha-1)\left(h_\beta-1\right)$$

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