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The Formulas for Solution of One Class of Linear Differential Equations of the Second and Third Order with the Variable Coefficients

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Abstract

Exact solutions for linear and nonlinear differential equations play an important role in theoretical and practical research. In particular many works have been devoted to finding a formula for solving second order linear differential equations with variable coefficients. In this paper we obtained the formula for the common solution of the linear differential equations of the second and third order with the variable coefficients in the more common case. We also obtained the new formula for the solution of the Cauchy problem for the linear differential equations of the second and third order with the variable coefficients. Examples illustrating the application of the obtained formula for solving second and third order linear differential equations are given.

Keywords: The linear differential equations, the second order, the third order, the variable coefficients, the new formula for the common solution, Cauchy problem, examples.

We consider the linear differential equations

$$y'' + p(t)y' + q(t)y = f(t), \quad (1)$$

$$y''' + a_1(t)y'' + a_2(t)y' + a_3(t)y = f(t), \quad (2)$$

where $t \epsilon I, I = [t_1, t_2)$ or $I = [t_1, t_2]$ or $I = (t_1, t_2)$, $t_1 < t_2$, p(t), q(t), $a_1(t), a_2(t), a_3(t)$ and $f(t)$ are known continuous functions on I.

Many works [1-8] are dedicated to the determination of the common solutions of the linear and nonlinear ordinary differential equations. Exact solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. These solutions can be used to verify the consistencies end estimate errors of various numerical, asymptotic, and approximate analytical methods. In particular, formulas for solving various linear and nonlinear differential equations are obtained in [1]. In [2-3] are dedicated to the determination of the solutions of the first order and the second order ordinary differential equations. But in common case any formulas for the decision of the linear differential equations of the second order haven't obtained. It is well known that if $p(t) = p_0 = const,\ q(t) = q_0 = const,$ then depending on the sign of discriminant $D = p_0^2 - 4q_0$ the common solution of the equation (1) will be written by three formulas. In [5] and [6] the general formula of the solution for linear ordinary differential equations of the second order with variable coefficients generalizing respectively $D = p_0^2 - 4q_0 > 0$ and $D = p_0^2 - 4q_0 < 0$ are received. In this theme the equation (1) is investigated in the other cases. In [8], formulas for solving a class of third-order linear differential equations with variable coefficients are obtained. This work is based on article [7]. Depending on the correlation between *p(t)* and *q(t)* formulas for the determination of the common solution of this equation were obtained. It is shown in the examples that the obtained formulas generalize many known formulas obtained in [1], [5] and [6]. In this paper, new formulas are obtained for determining the general solution of equations (1) and (2). It is shown by examples that the obtained formulas generalize many well-known formulas obtained in [1].

Theorem 1. Let

$$q(t) = a(t)[p(t) - a(t)] - l^2(t) + a'(t),\ t \in I, \tag{3}$$

$$l(t) = r\exp\left\{\int [2a(t) - p(t)]dt\right\}, \tag{4}$$

where $a(t),\ a'(t),\ p(t),\ f(t) \in C(I),\ r \in R,\ r \neq 0,\ a'(t)$ is the derivative of the function $a(t)$. Then the common solution of the equation (1) will be written in the next form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_3(t),\ t \in I\ , \tag{5}$$

where c_1 and c_2 are arbitrary constants,

$$y_1(t) = \exp\left[-\int_{t_0}^{t}(a(s) + l(s))ds\right], \tag{6}$$

$$y_2(t) = \exp\left[-\int_{t_0}^{t}(a(s) - l(s))ds\right], \tag{7}$$

$$y_3(t)=\int_{t_0}^{t} f(s)(2l(s)y_1(s)y_2(s))^{-1}\left[y_1(s)y_2(t)-y_1(t)y_2(s)\right]ds, \tag{8}$$

$$\Delta_0(t)=\det\begin{pmatrix} y_1(t) & y_2(t)\\ y_1^{'}(t) & y_2^{'}(t)\end{pmatrix}=2l(t)e^{-2\int_{t_0}^{t}a(s)ds}, \quad t\in I.$$

Proof. We show that $L[y_1]=0, L[y_2]=0, L[y_3]=f(t), t\in I.$

At first we prove $L[y_1]=0.$ In fact if we differentiate (4) and (6), we will obtain

$$l^{'}(t)=(2a(t)-p(t))l(t), \tag{9}$$

$$y_1^{'}(t)=-(a(t)+l(t))y_1(t), \tag{10}$$

$$y_1^{''}(t)=y_1(t)[a^2(t)+l^2(t)+2a(t)l(t)-a'(t)-l^{'}(t)]. \tag{11}$$

Then taking into account (10), (11) and (9) we have

$$L[y_1]=y_1(t)[a^2(t)+l^2(t)+2a(t)l(t)-a'(t)-(2a(t)-p(t))l(t)-p(t)a(t)-p(t)l(t)+q(t)]=0, t\in I.$$

We show that $L[y_2]=0.$ If we differentiate (7) we will have

$$y_2^{'}(t)=-(a(t)-l(t))y_2(t), \tag{12}$$

$$y_2^{''}(t)=y_2(t)[a^2(t)+l^2(t)-2a(t)l(t)-a'(t)+l^{'}(t)]. \tag{13}$$

Then taking into account (12), (13) and (9) we obtain

$$L[y_2]=y_2(t)[a^2(t)+l^2(t)-2a(t)l(t)-a'(t)+(2a(t)-p(t))l(t)-p(t)a(t)+p(t)l(t)+q(t)]=0, t\in I.$$

We are going to prove $L[y_3]=f(t), t\in I$. Differentiating (8) we have

$$y_3^{'}(t)=\int_{t_0}^{t} f(s)(2l(s)y_1(s)y_2(s))^{-1}\left[(l(t)-a(t))y_1(s)y_2(t)+(l(t)+a(t))y_1(t)y_2(s)\right]ds, \tag{14}$$

$$y_3^{''}(t)=\int_{t_0}^{t} f(s)(2l(s)y_1(s)y_2(s))^{-1}\{[a^2(t)+l^2(t)-2a(t)l(t)-a'(t)+l^{'}(t)]y_1(s)y_2(t)-$$

$$[a^2(t)+l^2(t)+2a(t)l(t)-a'(t)-l^{'}(t)]y_1(t)y_2(s)\}ds+f(t). \tag{15}$$

Taking into account (14), (15) and (9) we obtain

$$L[y_3]=\int_{t_0}^{t} f(s)(2l(s)y_1(s)y_2(s))^{-1}\{y_1(s)y_2(t)[a^2(t)+l^2(t)-2a(t)l(t)-a'(t)+(2a(t)-$$

$$p(t))l(t)-a(t)p(t)+l(t)p(t)+q(t)]-y_1(t)y_2(s)[a^2(t)+l^2(t)+2a(t)l(t)-a'(t)-(2a(t)-$$

$$-p(t))l(t)-a(t)p(t)-l(t)p(t)+q(t)]\}ds+f(t)=f(t), t\in I.$$

From (6), (7), (10) and (12) we have

$$\Delta_0(t) = 2l(t)e^{-2\int_{t_0}^{t} a(s)ds}, \quad t \in I.$$

Theorem 1 has been proved.

Corollary. Let $t_0 \in I$ and suppose that the conditions of theorem 1 hold. Then solution of the equation (1) with the initial condition $y(t_0) = m, y'(t_0) = n,$ will be written in the next form

$$y(t) = \frac{y_1(t)}{2l(t_0)}[m(l(t_0) - a(t_0)) - n] + \frac{y_2(t)}{2l(t_0)}[n + m(l(t_0) + a(t_0))] + y_3(t), \ t \in I, \qquad (16)$$

where the functions $y_1(t), y_2(t)$ and $y_3(t)$ are defined by the formulas (6), (7) and (8).

Theorem 2. Let $t_0 \epsilon I,$ the functions $a_1(t), a_2(t)$ and $a_3(t)$ are represented as

$$a_1(t) = \propto(t) + p(t),$$

$$a_2(t) = p'(t) + \propto(t)p(t) + a(t)[p(t) - a(t)] - l^2(t) + a'(t),$$

$$a_3(t) = a''(t) + a'(t)[p(t) - 2a(t) + \propto(t)] + a(t)[p'(t) + \propto(t)p(t) -$$

$$- \propto(t)a(t)] - l^2(t)[4a(t) - 2p(t) + \propto(t)],$$

where $t \epsilon I$ and $a(t), a'(t), a''(t), p(t), p'(t), f(t), \propto(t) \epsilon C(I),$ the function $l(t)$ is defined by the formula (4). Then the general solution of the differential equation (2) is written as

$$y(t) = y_0(t) + \sum_{i=1}^{3} c_i y_i(t), t \epsilon G, \qquad (17)$$

where c_1, c_2 and c_3 - arbitrary constants,

$$y_0(t) = \int_{t_0}^{t} [\int_{t_0}^{s} e^{-\int_{\nu}^{s} \alpha(\nu)d\nu} f(\nu)d\nu](2l(s)y_1(s)y_2(s))^{-1}\left[y_1(s)y_2(t) - y_1(t)y_2(s)\right]ds,$$

$$y_1(t) = \exp\left[-\int_{t_0}^{t}(a(s) + l(s))ds\right], \ y_2(t) = \exp\left[-\int_{t_0}^{t}(a(s) - l(s))ds\right],$$

$$y_3(t) = \int_{t_0}^{t} e^{-\int_{t_0}^{s} \alpha(\nu)d\nu} (2l(s)y_1(s)y_2(s))^{-1}\left[y_1(s)y_2(t) - y_1(t)y_2(s)\right]ds.$$

Proof. In this case, by virtue of the condition of Theorem 2, the differential equation (2) can be written as

$$\left(\frac{d}{dt}+\propto(t)\right)[y''+p(t)y'+q(t)y]=f(t),\ \ t\,\epsilon(t_1,t_2), \tag{18}$$

where the function q(t) is defined by the formulas (3) and (4).

From (18) we have

$$y''+p(t)y'+q(t)y=$$
$$=e^{-\int_{t_0}^{t}\propto(s)ds}\left[c_3+\int_{t_0}^{t}e^{\int_{t_0}^{s}\propto(s)ds}f(s)ds\right], \tag{19}$$

where $t\,\epsilon\,I, c_3$- an arbitrary constant. Taking into account (3), (4) and by virtue of Theorem 1, the general solution of the differential equation (19) is written as (17). Theorem 2 is proved.

Theorem 3. Let $t_0\epsilon\,I$, the functions $a_1(t), a_2(t)$ and $a_3(t)$ are represented as

$$a_1(t)=\propto(t)+p(t),$$

$$a_2(t)=2\propto'(t)+\propto(t)p(t)+a(t)[p(t)-a(t)]-l^2(t)+a'(t),$$

$$a_3(t)=\propto''(t)+\propto'(t)p(t)+\propto(t)\{a(t)[p(t)-a(t)]-l^2(t)+a'(t)\},$$

where $t\,\epsilon\,I$ and $f(t),\propto(t),\propto''(t), a(t), a'(t)$, p(t)$\epsilon\,C(I)$, the function $l(t)$ is defined by the formula(4). Then the general solution of the differential equation (2) is written as

$$y(t)=y_0(t)+\sum_{i=1}^{3}c_i y_i(t), t\,\epsilon\,G, \tag{20}$$

where c_1, c_2 and c_3 - arbitrary constants,

$$y_0(t)=\int_{t_0}^{t}e^{-\int_s^t\alpha(\nu)d\nu}z_0(s)ds,\ y_1(t)=e^{-\int_{t_0}^t\alpha(\nu)d\nu},$$

$$y_2(t)=\int_{t_0}^{t}e^{-\int_s^t\alpha(\nu)d\nu}z_1(s)ds,\ y_3(t)=\int_{t_0}^{t}e^{-\int_s^t\alpha(\nu)d\nu}z_2(s)ds,$$

$$z_1(t)=\exp\left[-\int_{t_0}^{t}(a(s)+l(s))ds\right],\ z_2(t)=\exp\left[-\int_{t_0}^{t}(a(s)-l(s))ds\right], \tag{21}$$

$$z_0(t) = \int_{t_0}^{t} f(s)(2l(s)z_1(s)z_2(s))^{-1}\left[z_1(s)z_2(t) - z_1(t)z_2(s)\right]ds. \qquad (22)$$

Proof. By virtue of the condition of Theorem 3, the differential equation (3) can be written as

$$\left[\frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)\right][y'(t) + \propto(t)y(t)] = f(t), t \in G, \qquad (23)$$

where $q(t)$ defined by the formula (3) and (4). Introducing the notation

$$z(t) = y'(t) + \propto(t)y(t), \qquad (24)$$

the differential equation (23) is written as

$$z'' + p(t)z' + q(t)z = f(t), \qquad t \in G. \qquad (25)$$

Taking into account (3), (4) and by virtue of Theorem 1, the general solution of equation (25) is written as

$$z(t) = z_0(t) + c_2 z_1(t) + c_3 z_2(t), t \in G, \qquad (26)$$

where c_2 and c_3 - arbitrary constants, functions $z_1(t)$, $z_2(t)$, $z_0(t)$ defined by the formula (21) and (22). Then, by virtue of (26), from (24) we get

$$y(t) = y_0(t) + c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

Theorem 3 is proved.

We present the formulas from [1], which are special cases of the results of theorems 1-3.

Example 1. Consider equation 2.1.9.84 in [1], i.e. consider equation (1) for $p(t) = 0, q(t) = -\frac{f''(t)}{2f(t)} + \left(\frac{f'(t)}{2f(t)}\right)^2 - \frac{b}{4f^2(t)}, b > 0.$ Then all the conditions of Theorem 1 are satisfied when $a(t) = -\frac{f'(t)}{2f(t)}, l(t) = \frac{\sqrt{b}}{2f(t)}$.

Example 2. Consider equation 3.1.9.44 in [1], i.e. consider equation (2) for $a_1(t) = a - f$, $a_2(t) = b - af$, $a_3(t) = -bf$, where a,b – arbitrary constants, f- an arbitrary continuous function on I and $a^2 - 4b > 0$. Then all the conditions of Theorem 2 are satisfied when $a(t) = \frac{a}{2}$, p(t)= a, $\propto(t) = -f$, $l(t) = \frac{\sqrt{a^2 - 4b}}{2}$.

Example 3. Consider equation 3.1.9.76 in [1], i.e. consider equation (2) for $a_1(t) = \frac{f(t)}{t}$, $a_2(t) = -\frac{2}{t^2}$, $a_3(t) = \frac{2[2 - f(t)]}{t^3}$, where f(t)- an arbitrary continuous function on I. Then all the conditions of Theorem 3 are satisfied

when $\propto(t)=\frac{1}{t}$, $p(t)=\frac{f(t)-1}{t}, l(t)=\frac{e^{-\int\frac{f(t)}{t}dt}}{2t\int\frac{e^{-\int\frac{f(t)}{t}dt}}{t}dt.}, a(t)=-\frac{1}{t}-l(t)$.

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