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The σ-sibling numbers

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Abstract

In this article, through the divisor function $\sigma(n)$, $n \in N$, we define a new class of numbers, which we call the class of σ-sibling numbers of n. We study some properties of these classes. In particular, we show that the set of these classes has cardinality $\aleph_0$ and containing a new subclass of prime numbers denoted by p^*.

Keywords: Divisor function $\sigma(n)$, prime number, σ-sibling number.

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1. INTRODUCTION

The peculiar and irregular distribution of the infinity of prime numbers demonstrated by Euclid in his Book IX of the Elements has since generated many problems still unsolved, called conjectures. Next, we mentioned some open problems or conjectures in mathematics. For more details, check, [1], [2], or [3].

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Goldbach's (strong) conjecture. In a letter dated June 7, 1742, addressed to Euler. Christian Goldbach claims to have observed that every even number greater than two could write as the sum of two primes, considered today how one of the arduous conjectures to solve.

Twin cousins conjecture. Two prime numbers p and q are called twins if $|p - q| = 2$. Let $TC(N)$ be the set of pairs twin primes numbers (p, q). The conjecture states that this set $TC(N)$ possesses infinite elements.

Legendre conjecture. This conjecture tells us that between the squares of two consecutive integers, there is at least one prime, i.e., for every natural number n, there exists at least one prime number p such that $n^2 < p < (n + 1)^2$.

Through the sum of the divisors of a number, $n \in N$, some problems have been left open. A number n is perfect if the sum of all its divisors is $2n$. The perfects numbers conjecture states that there are infinitely many perfects numbers. A pair (m, n) of naturals number is said friendly if the sum of the proper divisors of m (without including the number m) is equal to n, and reciprocally. The conjectured about friendly numbers is that these are infinite.

Conjectures are subjects of study daily, and some have received other approaches. For instance, [4] shows a topological approximation to Collatz's conjecture. As we have seen, all these conjectures mentioned have been based on certain classes of numbers. The numbers have generated very special studies, as by example, in the local spectral theory of operators, we have four types of numbers; nullity, codimension, ascent, and descent. These numbers have allowed recent theoretical developments, see for example, [5] and [6]. So it is of interest to define and study new types of numbers.

In this article, we define through the function sum of the divisors of a number $n \in \mathbb{N}$, that is $\sigma(n)$, a new class of numbers and discover some properties of them. Precisely in section two, we present the class of the σ-sibling numbers of $n \in \mathbb{N}$ and see that it is finite; it also is has been characterization this class of numbers. In section three, we found that there is an infinity of $n \in \mathbb{N}$ that has σ-sibling numbers. Finally, in section four, we define and study a subclass of prime numbers that belong the class of the σ-sibling numbers, and with this subclass, we set up a new problem in the number theory. Thus, we are on the way to defining this new class of numbers.

2. THE σ-SIBLING NUMBERS

The function $\sigma(n)$, [7], is a positive arithmetic function and denotes the sum all positive divisors of n, this is:

$$\sigma(n) = \sum_{d|n} d.$$

This function $\sigma(n)$, it's used to define various number types such as the friendly numbers, perfect numbers, among others. It's well know that:

i) $\sigma(n)$ is multiplicative, that is, if $\gcd(n, m) := (n, m) = 1$, then

$$\sigma(nm) = \sigma(n)\sigma(m).$$

ii) $\sigma(n)$, it's represented as prime factors:

$$\sigma(n) = \prod_{p|n} \frac{p^{\theta+1} - 1}{p - 1},$$

where θ is the number of times that a prime factor to appear in n. Note that n is a prime number if and only if $\sigma(n) = n + 1$.

On the other hand, the derivative of an arithmetic function $f(n)$, it's defined as $f'(n) = f(n)log(n)$ [7]. Now, let $n, m \in N$. Then,

$$\frac{\sigma'(n)}{\sigma(m)} - \frac{\sigma'(m)}{\sigma(n)} = \log\left(\frac{n}{m}\right) \Leftrightarrow$$

$$\sigma^2(n)\log(n) - \sigma^2(m)\log(m) = \sigma(n)\sigma(m)\log(n) - \sigma(n)\sigma(m)\log(m) \Leftrightarrow$$

$$(\sigma(n)\log(n) + \sigma(m)\log(m))(\sigma(n) - \sigma(m)) = 0 \Leftrightarrow$$

$$\sigma(n) = \sigma(m).$$

So, $\forall n \in \mathbb{N}$, we consider the following set

$$\Delta(n) := \{m \in \mathbb{N} : \sigma(n) = \sigma(m)\}.$$

We define a new class of numbers, which we will call the class of **σ-sibling numbers** of $n \in \mathbb{N}$, whose elements belong to $\Delta(n) - \{n\}$. The set $\Delta(n)$ is a finite set for each $n \in \mathbb{N}$, i.e., the amount of *σ-sibling numbers* of n is finite. Indeed.

Theorem 2.1. *Let $n \in \mathbb{N}$. Then $\Delta(n) - \{n\}$ is a finite set.*

Proof. Note that $\sigma(n)$ is finite. Now, suppose there are infinitely many numbers m such that $\sigma(n) = \sigma(m)$. Let us take m_0, a $\sigma-$sibling number of n such that $m_0 > \sigma(n)$. Then $\sigma(m_0) > \sigma(n)$, this is not possibly. $\square$

The σ-sibling numbers of a prime number are to its left side, in fact:

Theorem 2.2. *Let p be a prime number. If $n \in \Delta(p) - \{p\}$, then $n < p$.*

Proof. Let $n \in \Delta(p)$ with $n \neq p$, whereby $\sigma(p) = \sigma(n)$, then $\sigma(n) = n + 1$ implies that $n = p$. Hence, in case that $p < n$, result $\sigma(n) > n + 1 > p + 1 = \sigma(p)$, and this is not possible. We conclude that $n < p$. $\square$

Corollary 2.3. *Let p be a prime number and $n \in \Delta(p) - \{p\}$. Then* $\gcd(n, p) = 1$.

The σ-sibling relationship among two numbers is visible under the prime factorization of the natural numbers, in fact:

Theorem 2.4. *Let $n, m \in N$ be expressed at prime factors that are not repeated: $n = p_1p_2...p_k$, $m = q_1q_2...q_s$. Then, $(p_1 + 1)...(p_k + 1) = (q_1 + 1)...(q_s + 1)$ if and only if $m \in \Delta(n)$.*

Proof. **Necessity**: The function σ is multiplicative, so by the expressions for n and m we have that $\sigma(n) = \sigma(p_1)\sigma(p_2)...\sigma(p_k)$ and $\sigma(m) = \sigma(q_1)\sigma(q_2)...\sigma(q_s)$. Thus, $\sigma(n) = (p_1 + 1)(p_2 + 1)...(p_k + 1)$ and $\sigma(m) = (q_1 + 1)(q_2 + 1)...(q_s + 1)$. Since $\sigma(n) = \sigma(m)$ then $(p_1 + 1)(p_2 + 1)...(p_k + 1) = (q_1 + 1)(q_2 + 1)...(q_s + 1)$.
Sufficiency: Since $\sigma(n)$ is multiplicative then $\sigma(n) = (p_1 + 1)(p_2 + 1)...(p_k + 1)$ and $\sigma(m) = (q_1 + 1)(q_2 + 1)...(q_s + 1)$, by hypothesis result that $\sigma(n) = \sigma(m)$. Therefore, $m \in \Delta(n)$. $\square$

By $D_p(n)$, we denote the set of prime factors of n. Then we get the following theorem in which we consider that h_1 and h_2 are such that their decomposition into prime factors does not present repeated primes.

Theorem 2.5. *Let p_1, p_2 be prime numbers, such that $h_1 \in \Delta(p_1)$ and $h_2 \in \Delta(p_2)$. Then $p_2 \notin D_p(h_1)$, if $p_1 \in D_p(h_2)$.*

Proof. By hypothesis exists $q_1, q_2, ..., q_i$ with $q_i \in D_p(h_2)$ such that $h_2 = q_1q_2...q_ip_1$. We assume that $p_2 \in D_p(h_1)$, then there exists $r_1, r_2, ..., r_j$ with $r_j \in D_p(h_1)$ such that $h_1 = r_1r_2...r_jp_2$. Since $h_1 \in \Delta(p_1)$ and $h_2 \in \Delta(p_2)$, so,

$$\begin{aligned}\sigma(p_1) = \sigma(h_1) &= \sigma(r_1)\sigma(r_2)...\sigma(r_j)\sigma(p_2) \\ &= \sigma(r_1)\sigma(r_2)...\sigma(r_i)\sigma(h_2) \\ &= \sigma(r_1)\sigma(r_2)...\sigma(r_j)\sigma(q_1)\sigma(q_2)...\sigma(q_i)\sigma(p_1),\end{aligned}$$

which implies that

$$\sigma(r_1)\sigma(r_2)...\sigma(r_j)\sigma(q_1)\sigma(q_2)...\sigma(q_i) = 1,$$

but this is absurd. Hence, $p_2 \notin D_p(h_1)$. □

Example 2.6. Consider the prime numbers $p_1 = 11$ and $p_2 = 71$. Note that $h_1 = 6 \in \Delta(11)$ and $h_2 = 55 \in \Delta(71)$. Finally, see that $11 \in D_p(55)$, but $71 \notin D_p(6)$.

3. THE CARDINALITY OF σ-SIBLING NUMBERS' SET

In this section, we focus on proving there exist infinities integers that it has a σ-sibling number, thus, $\aleph_0$ is the cardinality of σ-sibling numbers' set. A consequence of the following theorem makes this fact possible.

Theorem 3.1. *Let p and q be prime numbers and $n \in \mathbb{N}$ such that $(q, n) = 1$. Then, $n \in \Delta(p)$ if and only if $qn \in \Delta(pq)$.*

Proof. **Sufficiency**: If $n \in \Delta(p)$ then $\sigma(p) = \sigma(n)$. Also, $\sigma(pq) = \sigma(p)\sigma(q) = \sigma(n)\sigma(q)$, but $(q, n) = 1$, thus $\sigma(n)\sigma(q) = \sigma(qn)$ and so $\sigma(qp) = \sigma(qn)$. Hence, $qn \in \Delta(pq)$.
Necessity: Given that $\sigma(q) \neq 0$ we conclude that $n \in \Delta(p)$. □

Note that with the σ-sibling numbers of a prime number p multiplying them by a prime number q result a σ-sibling number of pq.

Example 3.2. For the prime number 587, we have that:

$$\Delta(587) = \{260, 332, 485, 533, 587\}.$$

Let's take the prime number $q = 593$ greater than $p = 587$, then $pq = \mathbf{348091}$. Now, the following numbers: $154180 = 260 \cdot \mathbf{593}, 196876 = 332 \cdot \mathbf{593}, 287605 = 485 \cdot \mathbf{593}, 316069 = 533 \cdot \mathbf{593}$, they belong to $\Delta(348091)$, truly.

$$\Delta(\mathit{348091}) = \{149676, \mathit{154180}, 156740, 161540, 162272, 167008, 171872, 174148, 182908, 189010, 191090, 194030, 194668, \mathit{196876}, 197308, 197788, 203762, 210002, 230278, 230822, 231278, 231422, 232846, 261951, 270205, \mathit{287605}, 288605, 291055, \mathit{316069}, 317629, 321317, 324853, 329851, 341291, 345031, 347791, \mathit{348091}\}.$$

Suppose that n is a σ-sibling number to some prime p. Note that $(q, n) = 1$ is verified infinite times since there are infinities prime numbers q, then by Theorem 3.1 varying q, we get the next corollary.

Corollary 3.3. *There are infinities $m \in \mathbb{N}$ and infinities semi-prime numbers $s \in \mathbb{N}$, such that $\sigma(s) = \sigma(m)$.*

Corollary 3.4. *There are infinities $n \in \mathbb{N}$, such that $\Delta(n) - \{n\} \neq \emptyset$.*

The following theorem indicates that from a finite quantity of $\sigma-$sibling numbers which are primes, it is possible to obtain other σ-sibling numbers.

Theorem 3.5. *Let $p_1, p_2, ..., p_n$ be prime numbers such that $p_1 \in \Delta(h_1)$, $p_2 \in \Delta(h_2)$,..., $p_n \in \Delta(h_n)$. Then $\Delta(p_1p_2...p_n) - \{p_1p_2...p_n\} \neq \emptyset$.*

Proof. Note that $h_i \neq p_i$, $i = 1, 2, 3, ..., n$. Now, without loss of generality we assume that $p_1 < p_2 < ... < p_n$, thus by the Theorem 2.2 we have that $h_1 < p_2 < ... < p_n$, consequently $(h_1, p_i) = 1$ for every $i = 1, 2, 3, ..., n$. Hence,

$$\sigma(p_1p_2...p_n) = \sigma(p_1)\sigma(p_2)...\sigma(p_n) = \sigma(h_1)\sigma(p_2)...\sigma(p_n) = \sigma(h_1p_2...p_n).$$

But, $m = h_1p_2...p_n \neq p_1p_2...p_n$. Therefore, $\Delta(p_1p_2...p_n) - \{p_1p_2...p_n\} \neq \emptyset$. □

Example 3.6. Let's see the following table, where p, q, r are prime numbers.

p	q	r	pqr	n	$\sigma(pqr) = \sigma(n)$
11	17	23	4301	5029	5184
41	53	59	128207	135307	136080
71	89	113	714047	736999	738720
11	41	1589	716639	917369	919296

4. THE PRIME NUMBERS p^*

In this section, we consider that the numbers p and q are prime numbers, and we study numbers of the form $p^* = pq + p + q$ which are sometimes prime numbers, in this case, we consider his σ-sibling number which is the semi-prime $h = pq$.

Theorem 4.1. *The number p^* is a prime number if and only if $h \in \Delta(p^*)$.*

Proof. **Sufficiency**: The decomposition in prime factors of p^* is $pq + p + q$ and for h is pq. Now, $p^* + 1 = pq + p + q + 1 = (p + 1)(q + 1)$, thus $\sigma(h) = \sigma(pq) = \sigma(p)\sigma(q) = (p + 1)(q + 1) = p^* + 1 = \sigma(p^*)$. Therefore, $h \in \Delta(p^*)$.
Necessity: We have that $\sigma(p^*) = \sigma(h)$, so $\sigma(p^*) = \sigma(pq) = \sigma(p)\sigma(q) = (p + 1)(q + 1) = p^* + 1$. Hence, p^* is a prime number. □

In the next theorem, we see that if a prime number has a σ-sibling number $h = pq$, then it is of the form p^*.

Theorem 4.2. *Let r be a prime number and $h = pq$ a semi-prime number. Then, $r = p^*$ if and only if $h \in \Delta(r)$.*

Proof. **Sufficiency**: It's followed by the Theorem 4.1.
Necessity: Since r is a prime number, then $\sigma(r) = r + 1$. Also, $\sigma(r) = \sigma(h) = pq + p + q + 1$. Therefore, $r = pq + p + q = p^*$. □

Example 4.3. Next, let's see a table with some prime numbers p^*.

p	q	$h = pq$	p^*	$\sigma(h) = \sigma(p^*)$
2	3	6	11	12
2	127	254	383	384
3	5	15	23	24
3	149	447	599	600
43	2297	98771	101111	101112

Theorem 4.4. *Let $h = pq$ be a number semi-prime less than a prime number r. Then, $r = p^*$ if and only if $\sigma(h) \equiv 1 \pmod{r}$.*

Proof. Clearly, $r = p^* = pq + p + q$ implies that $r + 1 = \sigma(r) = \sigma(h)$, therefore, $\sigma(h) \equiv 1 \pmod{r}$.
Inversely, we assume that $\sigma(h) \equiv 1 \pmod{r}$, i.e., $\sigma(h) - 1 = kr$, $k \in \mathbb{N}$. As, $\sigma(h) = pq + p + q + 1$ result that $pq + p + q = kr$.
On another hand, $k_0 := p + q < pq < r$ and $h = pq = kr - k_0$.
If we assume that $k > 1$, then $kr - k_0 > kr - r \geq r$ which implies that $h > r$, this contradicts the hypothesis. Therefore, $k = 1$. We conclude that $r = p^*$. □

Corollary 4.5. *Let $h = pq$ be a semi-prime number belong to $\{1, 2, ..., p^* - 1\}$. Then, p^* is a prime number if and only if $\sigma(h) \equiv 1 \pmod{p^*}$.*

Proof. Arguing as in the proof of the Theorem 4.4 and considering that $p + q \leq 0$ is not true, is it obtained the proof. □

Since $\sigma(p) \equiv 1 \pmod{p}$, where p is a prime number, we get the following corollary.

Corollary 4.6. *Let $h = pq$ be a semi-prime number belong to $\{1, 2, ..., p^* - 1\}$. Then, p^* is a prime number if and only if $\sigma(h) \equiv \sigma(p^*) \pmod{p^*}$.*

The Euler totient function φ also provides a characterization for the numbers p^*, let's see.

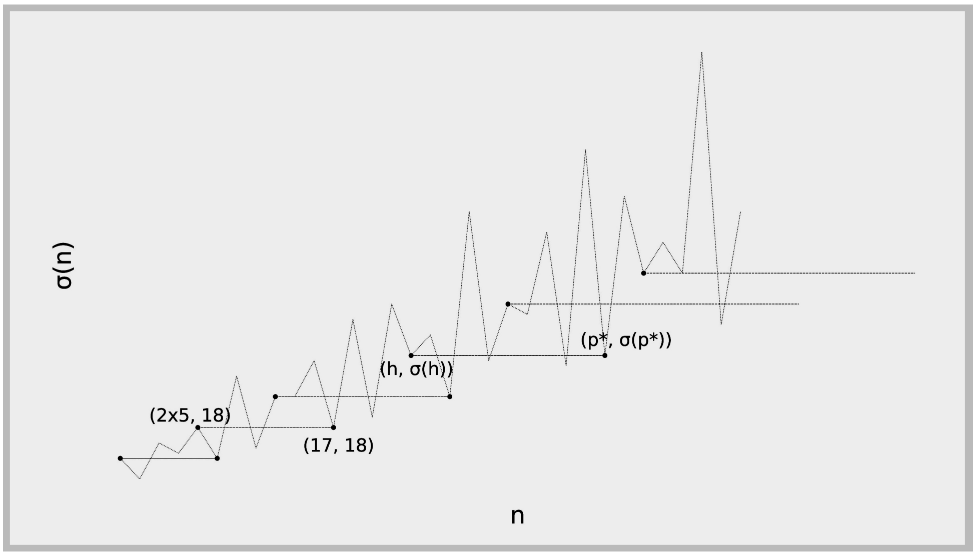
Theorem 4.7. *Let $h = pq$ be a semi-prime number. Then, $\varphi(p^*) + \varphi(h) = 2h$ if and only if $\sigma(p^*) = \sigma(h)$.*

Proof. **Sufficiency**: Clearly $\varphi(h) = pq - p - q + 1$ and replacing in $\varphi(p^*) + \varphi(h) = 2h$, we get that $\varphi(p^*) = p^* - 1$, for that p^* is prime. Hence, by Theorem 4.1 result that $\sigma(p^*) = \sigma(h)$.
Necessity: By Theorem 4.2 we have that $p^* = pq + p + q$, then $\varphi(p^*) = pq + p + q - 1$. Since $\varphi(h) = pq - p - q + 1$, result that $\varphi(p^*) + \varphi(h) = 2h$. □

Example 4.8. Let's consider the prime numbers, $11, 17, 23, 41, 53$. Then:

p	q	h	p^*	$\varphi(h)$	$\varphi(p^*)$	$\sigma(h) = \sigma(p^*)$	$\varphi(p^*) + \varphi(h) = 2h$
2	3	6	11	2	10	12	12
2	5	10	17	4	16	18	20
3	5	15	23	8	22	24	30
2	13	26	41	12	40	42	52
2	17	34	53	16	52	54	68

Next, see us a part of the graph of $\sigma(n)$ with some primes p^*, in the ordered pair, the left side is the semi-prime h, which is a σ-sibling number of p^*.



On another hand, we present a link where we can see Python codes used to determine numbers σ-sibling, and p^* too.

https://drive.google.com/file/d/1EiB1m0xW3S_QuDQSELwCwKEck9jAUFOo/view?usp=sharing

So far, the largest prime number p^* we have found is 62441279. Finally,

We conjecture that exists infinities numbers p^*.

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REFERENCES

[1] Paul Erdös and János Surányi. Topics in the theory of numbers. *AMC*, 10:12, 2003.

[2] Gove Effinger and Gary L Mullen. Elementary Number Theory. CRC Press, 2021.

[3] D Burton. Elementary number theory, 7th edn New York. *NY: McGraw-Hill. [Google Scholar]*, 2011.

[4] F. Mejias, J. Vielma, and A. Guale. A Topological Path to the Collatz Conjecture: A historical point of view. *Advances in Dynamical Systems and Applications (ADSA)*, 15(2):143–151, 2020.

[5] E. Aponte, N. Jayanthi, P. Vasanthakumar, and D. Quiroz. On the property (Bv). *Advances in Dynamical Systems and Applications (ADSA)*, 16(2):565–578, 2021.

[6] E. Aponte, J. Macías, J. Sanabria, and J. Soto. B-Fredholm Spectra of Drazin Invertible Operators and Applications. *Axioms*, 10(2):111, 2021.

[7] Tom M Apostol. Introduction to Analytic Number Theory. Springer Science & Business Media, 1998.